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Quadrature based two-step iterative methods for non-linear equations[1]

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Abstract

In this paper, we present two-step quadrature based iterative methods for solving non-linear equations. The convergence analysis of the methods is discussed. It is established that the new methods have convergence order five and six Numerical tests show that the new methods are comparable with the well known existing methods and in many cases gives better results. Our results can be considered as an improvement and refinement of the previously known results in the literature.

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Key words: Iterative methods, Three-step methods, Quadrature rule, Predictor-corrector methods, Nonlinear-equations.

1 Introduction

Let us consider a single variable non-linear equation

$$f(x) = 0. \tag{1.1}$$

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Finding zeros of a single variable nonlinear equation (1.1) efficiently, is an interesting very old problem in numerical analysis and has many applications in applied sciences. In recent years, researches have developed many iterative methods for solving equation (1.1). These methods can be classified as one-step, two-step and three-step methods, see [1-12]. These methods have been proposed using Taylor series, decomposition techniques, error analysis and quadrature rules, etc. Abbasbandy [1], Chun [3] and Grau [7] have proposed many two-step and three-step methods.

In this paper, we present two-step quadrature based iterative methods for solving non-linear equations. We prove that the new methods have order of convergence five and six. The methods and their algorithms are described in section 2. The convergence analysis of the methods is discussed in section 3. Finally, in section 4, the methods are tested on numerical examples given in the literature. It was noted that the new methods are comparable with the well known existing methods and in many cases gives better results. Our results can be considered as an improvement and refinement to the previously known results in the literature.

2 The Iterative Method

Weerakoon and Fernando [12], Gyurhan Nedzhibov [11] and M. Frontini and E. Sormani [5-6] have proposed various methods by the approximation of the indefinite integral

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt, \tag{2.1}$$

using Newton Cotes formulae of order zero and one. We approximate, here however the integral (2.1) by rectangular rule at a generic point $\lambda x+(1-\lambda)z_n$ with the end-points x and z_n. We thus have:

$$\int_{z_n}^{x} f'(t)dt = (x - z_n)f'(\lambda x + (1 - \lambda)z_n),$$

this gives

$$-f(z_n) = (x - z_n)(f'(z_n) + \lambda(x - z_n)^2 f''(z_n)). \tag{2.2}$$

From (2.2), we have:

$$x = z_n - \frac{f(z_n)f'(z_n)}{f'^2(z_n) - \lambda f(z_n)f''(z_n)}. \tag{2.3}$$

From (2.3), for $\lambda = 0$, we have Newton's method and for $\lambda = \frac{1}{2}$, we have Halley method.

This formulation allows to suggest many one-step, two-step and three-step methods. We suggest here, however the following two-step methods:

Algorithm 2.1. *For a given initial guess x_0, find the approximate solution by the iterative scheme:*

$$y_n = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - \lambda f(x_n)f''(x_n)}, \tag{2.4}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{2.5}$$

Algorithm 2.1 can further be modified by using an approximation for $f'(y_n)$ with the help of Taylor's expansion.

Let y_n be defined by (2.4). If we use Taylor expansion of $f'(y_n)$:

$$f'(y_n) \simeq f'(x_n) + f''(x_n)(y_n - x_n),$$

(where the higher derivatives are neglected) in combination with Taylor approximation of $f(y_n)$:

$$f(y_n) \simeq f(x_n) + f'(x_n)(y_n - x_n) + \frac{1}{2}f''(x_n)(y_n - x_n)^2,$$

we can remove the second derivative and approximate $f'(y_n)$ as:

$$f'(y_n) \simeq 2\left[\frac{f(y_n) - f(x_n)}{y_n - x_n}\right] - f'(x_n). \tag{2.6}$$

then Algorithm 2.1 can be written in the form of the following algorithm:

Algorithm 2.2. *For a given initial guess x_0, find the approximate solution by the iterative scheme:*

$$y_n = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - \lambda f(x_n)f''(x_n)}, \tag{2.7}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{2\left[\dfrac{f(y_n) - f(x_n)}{y_n - x_n}\right] - f'(x_n)}. \tag{2.8}$$

3 Convergence Analysis

Let us now discuss the convergence analysis of the above mentioned algorithms.

Theorem 3.1. *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to α, then the algorithm 2.1 has sixth order convergence for $\lambda = \frac{1}{2}$.*

Proof: Let α be a simple zero of f and $x_n = \alpha + e_n$.By Taylor's expansion we have:

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6) + O(e_n^7) \tag{3.1}$$

$$f'(x_n) = f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5) + O(e_n^6), \tag{3.2}$$

$$f''(x_n) = f'(\alpha)(2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4) + O(e_n^5) \tag{3.3}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2, 3, \ldots \text{ and } \ e_n = x_n - \alpha. \tag{3.4}$$

Using (3.1), (3.2) and (3.3) in (2.4), we have:

$$y_n = \alpha + (c_2 - 2\lambda c_2)e_n^2 + (2c_3 + 8\lambda c_2^2 - 6\lambda c_3 - 4\lambda^2 c_2^2 - 2c_2^2)e_n^3 + \tag{3.5}$$

$$+(28\lambda^2 c_2^3 + 4c_2^3 + 38\lambda c_2 c_3 + 3c_4 - 26\lambda c_2^3 - 8\lambda^3 c_2^3 - 24\lambda^2 c_2 c_3 -$$

$$-7c_2c_3 - 12\lambda c_4)e_n^4 + (166c_3\lambda c_2^2 - 8c_2^4 + 4c_5 - 48\lambda^2 c_2 c_4 +$$

$$+42\lambda c_3^2 + 68\lambda c_2 c_4 - 132\lambda^2 c_2^4 - 72\lambda^3 c_2^2 c_3 + 20c_3 c_2^2 - 16\lambda^4 c_2^4 +$$

$$+80\lambda^3 c_2^4 - 36\lambda^2 c_3^2 - 10c_2c_4 + 216\lambda^2 c_2^2 c_3 - 20\lambda c_5 + 76\lambda c_2^4 - 6c_3^2)e_n^5 +$$

$$+(5c_6 - 13c_2c_5 - 17c_3c_4 + 106\lambda c_2 c_5 + 144\lambda c_3 c_4 - 30\lambda c_6$$

$$+16c_2^5 + 33c_2c_3^2 + 28c_2^2c_4 - 52c_2^3c_3 - 208c_2^5\lambda -$$

$$-338\lambda c_2 c_3^2 - 280c_2^2\lambda c_4 + 606c_2^3\lambda c_3 - 144\lambda^2 c_3 c_4 +$$

$$+540\lambda^2 c_3^2 c_2 + 404\lambda^2 c_2^2 c_4 - 1236\lambda^2 c_2^3 c_3 - 144\lambda^3 c_2^2 c_4 +$$

$$+856\lambda^3 c_2^3 c_3 - 80\lambda^2 c_2 c_5 - 216\lambda^3 c_2 c_3^2 - 192\lambda^4 c_2^3 c_3 +$$

$$+516\lambda^2 c_2 c_2^5 - 496\lambda^3 c_2^5 + 208\lambda^4 c_2^5 - 32\lambda^5 c_2^5)e_n^6 + O(e_n^7).$$

By Taylor's series, we have

(3.6)

$$f(y_n) = f'(\alpha)(-c_2(2\lambda - 1)e_n^2 + (8\lambda c_2^2 + 2c_3 - 6\lambda c_3 - 4\lambda^2 c_2^2 - 2c_2^2)e_n^3 +$$
$$+(32\lambda^2 c_2^3 - 24\lambda^2 c_2 c_3 - 8\lambda^3 c_2^3 - 7c_2c_3 - 12\lambda c_4 + 38\lambda c_2 c_3 -$$
$$-30\lambda c_2^3 + 3c_4 + 5c_2^3)e_n^4 + (-72\lambda^3 c_2^2 c_3 + 24c_3c_2^2 - 10c_2c_4 -$$
$$-172\lambda^2 c_2^4 - 6c_3^2 + 68\lambda c_2 c_4 - 48\lambda^2 c_2 c_4 - 20\lambda c_5 + 96\lambda^3 c_2^4 -$$
$$-12c_2^4 + 240\lambda^2 c_2^2 c_3 + 42\lambda c_3^2 + 100\lambda c_2^4 + 4c_5 - 16\lambda^4 c_2^4 -$$
$$-36\lambda^2 c_3^2 - 186c_3\lambda c_2^2)e_n^5) + O(e_n^6).$$

By Taylor's series, we have:

$$f'(y_n) = f'(\alpha)(1 + 2c_2(-2\lambda c_2 + c_2)e_n^2 + 2c_2(8\lambda c_2^2 + 2c_3 - 6\lambda c_3 - \tag{3.7}$$

$$-4\lambda^2 c_2^2 - 2c_2^2)e_n^3 + (2c_2(38\lambda c_2 c_3 - 24\lambda^2 c_2 c_3 - 12\lambda c_4 - 7c_2c_3 + 3c_4 -$$

$$-26\lambda c_2^3 - 8\lambda^3 c_2^3 + 28\lambda^2 c_2^3 + 4c_2^3) + 3c_3(-2\lambda c_2 + c_2)^2)e_n^4 +$$

$$+(2(80\lambda^3c_2^4-72\lambda^3c_2^2c_3-10c_2c_4+76\lambda c_2^4-36\lambda^2c_3^2-20\lambda c_5+$$
$$+68\lambda c_2c_4-16\lambda^4c_2^4+4c_5+216\lambda^2c_2^2c_3+42\lambda c_2^3-132\lambda^2c_2^4+$$
$$+20c_3c_2^2-8c_2^4-166c_3\lambda c_2^2-48\lambda^2c_2c_4-6c_3^2)c_2+6(-2\lambda c_2+$$
$$+c_2)(8\lambda c_2^2+2c_3-6\lambda c_3-4\lambda^2c_2^2-2c_2^2)c_3)e_n^5)+O(e_n^5),$$

Using (3.5), (3.6) and (3.7) in (2.5), we have:

$$x_{n+1}:=\alpha+(c_2^3+4\lambda^2c_2^3-4\lambda c_2^3)e_n^4+(24\lambda^2c_2^2c_3-4c_2^4+4c_3c_2^2+ \tag{3.8}$$
$$+16\lambda^3c_2^4-20c_3\lambda c_2^2-40\lambda^2c_2^4+24\lambda c_2^4)e_n^5+(-30\lambda c_6-$$
$$-13c_2c_5-17c_3c_4+106\lambda c_2c_5+144\lambda c_3c_4-192\lambda^4c_2^3c_3-$$
$$-216\lambda^3c_2c_3^2+38c_2^5+41c_2c_3^2+40c_2^2c_4-93c_2^3c_3+972\lambda^2c_2^5-$$
$$-396\lambda c_2^5-864\lambda^3c_2^5+304\lambda^4c_2^5-32\lambda^5c_2^5-386\lambda c_2c_3^2+$$
$$+500\lambda^2c_2^2c_4-1824\lambda^2c_2^3c_3+908\lambda c_2^3c_3-144\lambda^2c_3c_4+$$
$$+612\lambda^2c_3^2c_2-144\lambda^3c_2^2c_4+1120\lambda^3c_2^3c_3-352\lambda c_2^2c_4-$$
$$-80\lambda^2c_2c_5+5c_6)e_n^6+O(e_n^7),$$

thus, for $\lambda=\frac{1}{2}$, we have:

$$x_{n+1}=\alpha+(-7c_2^5-10c_6+33c_2^3c_3-26c_2c_3^2-29c_2^4c_4+20c_2c_5+ \\ +19c_3c_4)e_n^6+O(e_n^7) \tag{3.9}$$

or

$$e_{n+1}=(-7c_2^5-10c_6+33c_2^3c_3-26c_2c_3^2-29c_2^4c_4+20c_2c_5+ \\ +19c_3c_4)e_n^6+O(e_n^7). \tag{3.10}$$

Thus, we observe that the algorithm 2.1 has sixth order convergence for $\lambda=\frac{1}{2}$.

Theorem 3.2. *Let $\alpha\in I$ be a simple zero of sufficiently differentiable function $f:I\subseteq R\to R$ for an open interval I. If x_0 is sufficiently close to α, then the algorithm 2.2 has fifth order convergence for $\lambda=\frac{1}{2}$.*

Proof. Let α be a simple zero of f and $x_n = \alpha + e_n$. By Taylor's expansion, we have:

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6) + O(e_n^7) \tag{3.11}$$

$$f'(x_n) = f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5) + O(e_n^6), \tag{3.12}$$

$$f''(x_n) = f'(\alpha)(2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4) + O(e_n^5). \tag{3.13}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2, 3, \ldots \text{ and } \ e_n = x_n - \alpha. \tag{3.14}$$

Using (3.1), (3.2) and (3.3) in (2.7), we have:

$$\begin{aligned} y_n = \alpha + (c_2 - 2\lambda c_2)e_n^2 + (2c_3 + 8\lambda c_2^2 - 6\lambda c_3 - 4\lambda^2c_2^2 - 2c_2^2)e_n^3 + \\ +(28\lambda^2c_2^3 + 4c_2^3 + 38\lambda c_2c_3 + 3c_4 - 26\lambda c_2^3 - 8\lambda^3c_2^3 - 24\lambda^2c_2c_3 - \\ -7c_2c_3 - 12\lambda c_4)e_n^4 + (-166c_3\lambda c_2^2 - 8c_2^4 + 4c_5 - 48\lambda^2c_2c_4 + \\ +42\lambda c_3^2 + 68\lambda c_2c_4 - 132\lambda^2c_2^4 - 72\lambda^3c_2^2c_3 + 20c_3c_2^2 - 16\lambda^4c_2^4 + \\ +80\lambda^3c_2^4 - 36\lambda^2c_3^2 - 10c_2c_4 + 216\lambda^2c_2^2c_3 - 20\lambda c_5 + 76\lambda c_2^4 - 6c_3^2)e_n^5 + O(e_n^6). \end{aligned} \tag{3.15}$$

By Taylor's series, we have:

$$\begin{aligned} f(y_n) = f'(\alpha)(-c_2(2\lambda - 1)e_n^2 + (8\lambda c_2^2 + 2c_3 - 6\lambda c_3 - 4\lambda^2c_2^2 - 2c_2^2)e_n^3 + \\ +(32\lambda^2c_2^3 - 24\lambda^2c_2c_3 - 8\lambda^3c_2^3 - 7c_2c_3 - 12\lambda c_4 + 38\lambda c_2c_3 - \\ -30\lambda c_2^3 + 3c_4 + 5c_2^3)e_n^4 + (-72\lambda^3c_2^2c_3 + 24c_3c_2^2 - 10c_2c_4 - \\ -172\lambda^2c_2^4 - 6c_3^2 + 68\lambda c_2c_4 - 48\lambda^2c_2c_4 - 20\lambda c_5 + 96\lambda^3c_2^4 - \end{aligned} \tag{3.16}$$

$$-12c_2^4+240\lambda^2c_2^2c_3+42\lambda c_2^3+100\lambda c_2^4+4c_5-16\lambda^4c_2^4-$$
$$-36\lambda^2c_3^2-186c_3\lambda c_2^2)e_n^5)+O(e_n^6).$$

By using (3.1), (3.2), (3.5) and (3.6) in (2.6), we have:

$$\begin{aligned}f^{'}(y_n)=&f^{'}(\alpha)(1+(-4\lambda c_2^2-c_3+2c_2^2)e_n^2+(-4c_2^3-2c_4+16\lambda c_2^3-\\&16\lambda c_2c_3+6c_2c_3-8\lambda^2c_2^3)e_n^3+(-16\lambda^3c_2^4-28\lambda c_2c_4-3c_5-\\&48\lambda^2c_2^2c_3+56\lambda^2c_2^4+84\lambda c_2^2c_3-12\lambda c_3^2+8c_2^4+4c_3^2+8c_2c_4-\\&16c_3c_2^2-52\lambda c_2^4)e_n^4+(6c_6-24c_3c_4-16c_2c_5+252\lambda c_3c_4+\\&168\lambda c_2c_5-60\lambda c_6-432\lambda^3c_2c_3^2-384\lambda^4c_2^3c_3+56c_2c_3^2+\\&46c_2^2c_4-106c_2^3c_3+1248\lambda^2c_2^5-464\lambda c_2^5-1216\lambda^3c_2^5+\\&480\lambda^4c_2^5-64\lambda^5c_2^5+1184\lambda c_2^3c_3+808\lambda^2c_2^2c_4-2664\lambda^2c_2^3c_3-\\&604\lambda c_2c_3^2-288\lambda^2c_3c_4+1080\lambda^2c_3^2c_2-288\lambda^3c_2^2c_4+\\&1856\lambda^3c_2^3c_3-488\lambda c_2^2c_4-160\lambda^2c_2c_5+40c_2^5)e_n^5)+O(e_n^6),\end{aligned}\tag{3.17}$$

Using (3.5), (3.6) and (3.7) in (2.8) , we have:

$$\begin{aligned}x_{n+1}=&\alpha+(2\lambda c_2c_3-c_2c_3+4\lambda^2c_2^3-4\lambda c_2^3+c_2^3)e_n^4+(-2c_2c_4\\&+8c_3c_2^2+36\lambda^2c_2^2c_3-4c_2^4-36\lambda c_2^2c_3+24\lambda c_2^4-2c_3^2\\&+16\lambda^3c_2^4-40\lambda^2c_2^4+6\lambda c_3^2+4\lambda c_2c_4)e_n^5+O(e_n^6),\end{aligned}\tag{3.18}$$

thus, for $\lambda=\frac{1}{2}$, we have:

$$x_{n+1} = \alpha+(-c_3c_2^2+c_3^2)e_n^5+O(e_n^6), \tag{3.19}$$
$$\text{or } e_{n+1} = (-c_3c_2^2+c_3^2)e_n^5+O(e_n^6). \tag{3.20}$$

Thus, we observe that the algorithm 2.2 has fifth order convergence for $\lambda=\frac{1}{2}$.

4 Numerical examples

We consider here some numerical examples to demonstrate the performance of the new developed two-step iterative methods, namely algorithm 2.1 and 2.2.We compare the classical Newton's method (NM), the method of Grau (GM)[7] and the new developed two-step method algorithm 2.1 ($MN6$) and algorithm 2.2 ($MN5$), in this paper. All the computations are performed using Maple 10.0. We take $\in = 10^{-15}$ as tolerance. All the values are computed using 128 significant digits.

The following criteria is used for estimating the zero:

(i) $\delta = |x_{n+1} - x_n| < \in$

(ii) $|f(x_n)| < \in$

The following examples are used for numerical testing:

Examples	**Exact Zero**
$f_1 = x^3 + 4x^2 - 15,$	$\alpha = 1.6319808055660636,$
$f_2 = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$	$\alpha = -1.207647827130919,$
$f_3 = \sin(x) - \frac{1}{2}x,$	$\alpha = 0,$
$f_4 = 10xe^{-x^2} - 1,$	$\alpha = 1.67963061042845,$
$f_5 = \cos(x) - x,$	$\alpha = 0.73908513321516067,$
$f_6 = \sin^2(x) - x^2 + 1,$	$\alpha = 1.4044916482153411,$
$f_7 = e^{-x} + \cos(x),$	$\alpha = 1.74613953040801241765.$

For convergence criteria, it was required that δ, the distance between two consecutive iterates was less than $10^{-15}, n$ represents the number of iterations and $f(x_n)$, the absolute value of the function. The numerical comparison is given in Table 4.1.

	n	$f(x_n)$
$f_1, x_0 = 2$		
NM	6	8.23e-54
GM	3	1.00e-126
M6	3	0
M5	3	0
$f_2, x_0 = -0.2$		
NM	22	2.60e-48
GM	5	1.20e-126
M6	4	1.10e-126
M5	5	1.10e-126
$f_3, x_0 = 0.5$		
NM	15	1.17e-54
GM	4	0
M6	3	0
M5	4	0
$f_4, x_0 = 1.8$		
NM	5	5.94e-29
GM	3	1.00e-127
M6	3	1.00e-127
M5	3	4.00e-128
$f_5, x_0 = 1$		
NM	5	1.51e-41
GM	3	0
M6	3	0
M5	3	0

	n	$f(x_n)$
$f_6, x_0 = 0.2$		
NM	13	2.90e-44
GM	6	1.00e-127
M6	5	2.91e-105
M5	5	1.00e-127
$f_7, x_0 = 2$		
NM	5	2.67e-42
GM	3	4.00e-128
M6	3	4.00e-128
M5	3	4.00e-128

Table 4.1.

5 CONCLUSION

From Table 4.1, we observe that our two-step iterative methods of convergence orders five and six are comparable with the sixth order method of M. Grau, J.L. Diaz-Barrero[7] and in many cases gives better results in terms of the function evaluation $f(x_n)$.The computational efficiency of the methods described in this paper is better than the efficiency of most of the other methods defined in the literature. For algorithm 2.1 with convergence order 6, the computational efficiency is $6^{\frac{1}{5}} = 1.430969$ and for algorithm 2.2 with convergence order 5, the computational efficiency is $5^{\frac{1}{4}} = 1.495349$.

References

[1] S. Abbasbandy, *Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method*, Appl. Math. Comput. 145(2003)887-893.

[2] R. L. Burden, J. D. Faires, *Numerical Analysis*, PWS publishing company, Boston USA, 2001.

[3] C. Chun,*Iterative methods improving Newton's method by the decomposition method*, Comput & Math with Appl. 50(2005)1559-1568.

[4] J.E.Dennis, R.B. Schnable, *Numerical methods of unconstrained optimization and non-linear equations*, Prentice Hall, 1983.

[5] M. Frontini, E. Sormani, *Some variants of Newton's method with third order convergence and multiple roots*, J. Comput. Appl. Math. 156 (2003), 345-354.

[6] M. Frontini, E. Sormani, *Third order methods for quadrature formulae for solving system of nonlinear equations*, Appl. Math. Comput. 149 (2004), 771-782.

[7] M. Grau, J.L. Diaz-Barrero, *An improvement to Ostrowski root-finding method*, Appl. Math. Comput. 173(2006) 450-456.

[8] Jisheng Kou, Yitian Li and Xiuhua Wang,*Third order modification of Newton's method*, Appl. Math. and Comput.,(2006, in press).

[9] M. V. Kanwar, V. K. Kukreja, S. Singh,*On a class of quadratically convergent iteration formulae*, Appl. Math. Comput. 166 (3) (2005), 633-637.

[10] Mamta, V. Kanwar, V. K. Kukreja, S. Singh,*On some third order iterative methods for solving non-linear equations*, Appl. Math. Comput. 171 (2005), 272-280.

[11] Gyurhan Nedzhibov, *On a few iterative methods for solving nonlinear equations*, Application of Mathematics in Engineering and Economics, in: Proceedings of the XXVIII Summer School Sozopol 2002, Heron Press, Sofia, 2002.

[12] S. Weerakoon, T. G. I. Fernando,*A variant of Newton's method with accelerated third order convergence*, Appl. Math. Lett. 13(2000), 87-93.

[13] A.M. Ostrowski,*Solutions of Equations and System of Equations*, Academic Press, New York, 1960.

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