**General Mathematics ,Vol. 16, Nr. 1 (2008), 59–72**

# About a class of linear positive operators[1]

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**Abstract**

In this paper we construct a class of linear positive operators $(L_m)_{m\geq 1}$ with the help of some nodes. We study the convergence and we demonstrate the Voronovskaja-type theorem for them. By particularization, we obtain some known operators.

**2000 Mathematics Subject Classification:** 41A10, 41A25, 41A35, 41A36.

**Key words:** Linear positive operators, convergence theorem.

## 1 Introduction

In this section, we recall some notions and operators which we will use in this article.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $p_{m,k}(x)$ the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.1}$$

[1] *Received 9 November 2007*
*Accepted for publication (in revised form) 4 December 2007*

for any $x \in [0,1]$ and any $k \in \{0,1,\dots,m\}$ (see [5] or [21]). For the following construction see [15]. Define the natural number $m_0$ by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.2}$$

For the real number $\beta$, we have that

$$m + \beta \geq \gamma_\beta \tag{1.3}$$

for any natural number $m$, $m \geq m_0$, where

$$\gamma_\beta = m_0 + \beta = \begin{cases} \max\left\{1+\beta, \{\beta\}\right\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.4}$$

For the real numbers $\alpha, \beta$, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta \\ 1 + \dfrac{\alpha-\beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.5}$$

For the real numbers $\alpha$ and $\beta$, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.6}$$

for any natural number $m$, $m \geq m_0$ and for any $k \in \{0,1,\dots,m\}$.

For the real numbers $\alpha$ and $\beta$, $\alpha \geq 0$, $m_0$ and $\mu^{(\alpha,\beta)}$ defined by (1.2)-(1.6), let the operators $P_m^{(\alpha,\beta)} : C\big([0,\mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, defined for any function $f \in C\big([0,\mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.7}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$. These operators are named Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [20]. In [20], the domain of definition of the Stancu operators is $C([0,1])$ and the numbers $\alpha$ and $\beta$ verify the condition $0 \leq \alpha \leq \beta$.

**Remark 1.1.** *For $\alpha = \beta = 0$ we obtain the Bernstein operators.*

**Remark 1.2.** *For $\alpha = 0$, $p \in \mathbb{N}_0$ and choosing $m$ by $m+p$ and $p$ by $m-p$, we obtain the Schurer operators.*

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [4] a sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.8}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \mid f : [0,\infty) \to \mathbb{R},$ $f$ bounded and continuous on $[0,\infty)\}$.

For $m \in \mathbb{N}$ consider the operators $S_m : C_2([0,\infty)) \to C([0,\infty))$ defined for any function $f \in C_2([0,\infty))$ by

$$(S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.9}$$

for any $x \in [0,\infty)$, where $C_2([0,\infty)) = \left\{ f \in C([0,\infty)) : \lim\limits_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}$.

The operators $(S_m)_{m\geq 1}$ are named Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [11].

They were intensively studied by J. Favard in 1944 in [8] and O. Szász in 1950 in [22].

Let for $m \in \mathbb{N}$ the operators $V_m : C_2([0,\infty)) \to C([0,\infty))$ be defined for any function $f \in C_2([0,\infty))$ by

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right), \tag{1.10}$$

for any $x \in [0,\infty)$.

The operators $(V_m)_{m\geq 1}$ are named Baskakov operators and they were introduced in 1957 by V. A. Baskakov in [2].

W. Meyer-König and K. Zeller have introduced in [10] a sequence of linear and positive operators. After a slight adjustment given by E. W. Cheney and A. Sharma in [6], these operators take the form $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(1.11)\qquad (Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right),$$

for any $m \in \mathbb{N}$ and for any $x \in [0,1)$.

These operators are named the Meyer-König and Zeller operators.

Observe that $Z_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$.

In the paper [9], M. Ismail and C. P. May consider the operators $(R_m)_{m\geq 1}$.

For $m \in \mathbb{N}$, $R_m : C([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C([0,\infty))$ by

$$(1.12)\qquad (R_m f)(x) = e^{-\frac{mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m}\right)$$

for any $x \in [0,\infty)$.

We consider $I \subset \mathbb{R}$, $I$ an interval and we shall use the following functions sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on $I$, $B(I) = \{f \,|\, f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f \,|\, f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, consider the function $\psi_x : I \to \mathbb{R}$ defined by $\psi_x(t) = t - x$, for any $t \in I$.

# 2 Preliminaries

The following construction is about the idea from [15]. Let $I, J$ be real intervals with $I \cap J \neq \emptyset$ and $p_m = m$ for any $m \in \mathbb{N}$ (the finite case) or $p_m = \infty$ for any $m \in \mathbb{N}$ (the infinite case). For any $m \in \mathbb{N}$ and $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$, consider the nodes $x_{m,k} \in I$ and the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$, for any $x \in J$. We suppose that for any compact $K \subset I \cap J$ there exists the sequence $(u_m(K))_{m\geq 1}$, depending on $K$ such that

$$(2.1)\qquad \lim_{m\to\infty} u_m(K) = 0$$

uniformly on $K$ and

$$\left|\sum_{k=0}^{p_m}\varphi_{m,k}(x)-1\right|\leq u_m(K) \tag{2.2}$$

for any $x\in K$, any $m\in\mathbb{N}$ and we note $u(K)=\sup\{u_m(K): m\in K\}$.

**Remark 2.1.** *From* (2.1) *it result that* $\lim\limits_{m\to\infty}\sum\limits_{k=0}^{p_m}\varphi_{m,k}(x)=1$, *for any* $x\in J$.

Let a fixed function $w: I\to(0,\infty)$, called the weight function and the set functions

$$E_w(I)=\{f|f: I\to\mathbb{R}\text{ such that } wf \text{ is bounded on } I\}. \tag{2.3}$$

For $f\in E_w(I)$ there exists a positive constant $M(f)$, depending on $f$, such that $w(x)|f(x)|\leq M(f)$ for any $x\in I$. Then, for $m\in\mathbb{N}$ and $x\in J$, and taking in the end (2.2) into account, we have

$$\left|\sum_{k=0}^{p_m}\varphi_{m,k}(x)f(x_{m,k})\right|\leq\sum_{k=0}^{p_m}\varphi_{m,k}(x)|f(x_{m,k})|\leq\frac{M(f)}{w(x)}\sum_{k=0}^{p_m}\varphi_{m,k}(x)\leq$$
$$\leq\frac{M(f)}{w(x)}(1+u_m(K))\leq\frac{M(f)}{w(x)}(1+u(K)),$$

from where it results that the sum $\sum\limits_{k=0}^{p_m}\varphi_{m,k}(x)f(x_{m,k})$ exists.

We consider the operators $(L_m)_{m\geq1}$ defined by

$$(L_mf)(x)=\sum_{k=0}^{p_m}\varphi_{m,k}(x)f(x_{m,k}) \tag{2.4}$$

for any $f\in E_w(I)$, $x\in J$ and $m\in\mathbb{N}$.

**Proposition 2.1.** *The operators* $(L_m)_{m\geq1}$ *are linear and positive on* $E_w(I)$.

**Proof.** The proof follows immediately.

# 3 Main results

In the following, let $s$ be fixed natural number, $s$ even. For any $x \in I \cap J$ we suppose that $\psi_x^i \in E_w(I)$, where $i \in \{0, 1, \ldots, s+2\}$. For $m \in \mathbb{N}$ and $i \in \{0, 1, \ldots, s+2\}$ define

$$(T_i L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{p_m} (x_{m,k} - x)^i \varphi_{m,k}(x) \tag{3.1}$$

for any $x \in I \cap J$.

**Theorem 3.1.** *Let $x \in I \cap J$ and we suppose that there exist $\alpha_{s+2} \geq 0$ and $m(s) \in \mathbb{N}$ such that $\frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$. If $\gamma \in \mathbb{R}$ verify $\gamma < s + 2 - \alpha_{s+2}$ and $\delta > 0$, then*

$$\lim_{m \to \infty} m^\gamma \sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^s \varphi_{m,k}(x) = 0. \tag{3.2}$$

*If for the compact interval $K \subset I \cap J$ exist $m(s) \in \mathbb{N}$ and the constant $k_{s+2}(K) \in \mathbb{R}$, depending on $K$, such that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$ we have*

$$\frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} \leq k_{s+2}(K), \tag{3.3}$$

*then the convergence given in (3.2) is uniform on $K$.*

**Proof.** We have

$$\sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^s \varphi_{m,k}(x) \leq \frac{1}{\delta^2} \sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^{s+2} \varphi_{m,k}(x) \leq$$
$$\leq \sum_{k=0}^{p_m} (x_{m,k} - x)^{s+2} \varphi_{m,k}(x) = \frac{1}{\delta^2 m^{s+2}} (T_{s+2} L_m)(x),$$

so

$$m^\gamma \sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^s \varphi_{m,k}(x) \leq \frac{1}{\delta^2 m^{s+2-\gamma}} (T_{s+2} L_m)(x). \tag{3.4}$$

But
$$\frac{1}{\delta^2 m^{s+2-\gamma}}(T_{s+2}L_m)(x) = \frac{1}{\delta^2 m^{s+2-\alpha_{s+2}-\gamma}} \cdot \frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}}$$
and because $\gamma < s+2-\alpha_{s+2}$, we get $s+2-\alpha_{s+2}-\gamma > 0$. Because $\frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, it results that
$$\lim_{m\to\infty} \frac{1}{\delta^2 m^{s+2-\alpha_{s+2}-\gamma}} \cdot \frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}} = 0.$$
Considering the limit compute above, the fact that $s$ is even and (3.4), we obtain (3.2).

**Remark 3.1.** *In* Theorem 3.1 *we choose the smallest* $\alpha_{s+2}$ *and the bigger* $\gamma$, *if they exists.*

In the following, we suppose that exists $M > 0$ such that the inequality

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) \leq M \tag{3.5}$$

holds for any $x \in J$ and any $m \in \mathbb{N}$.

**Theorem 3.2.** *If* $f \in E_w(I)$ *is a* $s$ *times differentiable function at* $x \in I \cap J$ *(if* $s = 0$ *we consider that* $f$ *is continuous on* $I \cap J$*) and we suppose that exists* $\alpha_{s+2} \geq 0$ *and* $m(s) \in \mathbb{N}$ *such that* $\frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}}$ *is bounded for any* $m \in \mathbb{N}$, $m \geq m(s)$, *then for any* $\gamma$ *which verify*

$$\gamma < s + 2 - \alpha_{s+2} \tag{3.6}$$

*we have*

$$\lim_{m\to\infty} m^{\gamma} \left[ (L_m f)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!} (T_i L_m)(x) f^{(i)}(x) \right] = 0. \tag{3.7}$$

*If* $f \in E_w(I)$ *is a* $s$ *times differentiable function on* $I$ *and for the compact interval* $K \subset I \cap J$ *exist* $m(s) \in \mathbb{N}$ *and the constant* $k_{s+2}(K) \in \mathbb{R}$, *depending on* $K$, *such that for any* $m \in \mathbb{N}$, $m \geq m(s)$ *and* $x \in K$ *we have*

$$\frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}} \leq k_{s+2}(K), \tag{3.8}$$

*then the convergence given in* (3.7) *is uniform on* $K$.

**Proof.** According to Taylor's formula for the function $f$ arround $x$, we have

$$f(t)=\sum_{i=0}^{s}\frac{(t-x)^i}{i!}f^{(i)}(x)+(t-x)^s\mu(t-x) \tag{3.9}$$

where $\mu$ is a bounded function and $\lim_{t\to x}\mu(t-x)=0$. Then exists a neighborhood $V=[-a,a]$ of the point 0 such that for any $\epsilon>0$, exists $\delta_\epsilon>0$, for any $h\in V$ with $|h|<\delta_\epsilon$, we have

$$|\mu(h)|<\epsilon. \tag{3.10}$$

If we replace $t$ with $x_{m,k}$ in (3.9), multiply by $\varphi_{m,k}(x)$ and sum after $k$, when $k\in\{0,1,\dots,p_m\}$, we obtain

$$\begin{aligned}(L_mf)(x)&=\sum_{k=0}^{p_m}\sum_{i=0}^{s}\frac{(x_{m,k}-x)^i}{i!}\varphi_{m,k}(x)f^{(i)}(x)+\\&+\sum_{k=0}^{p_m}(x_{m,k}-x)^s\varphi_{m,k}(x)\mu(x_{m,k}-x)=\\&=\sum_{i=0}^{s}\frac{1}{m^ii!}\left[m^i\sum_{k=0}^{p_m}(x_{m,k}-x)^i\varphi_{m,k}(x)\right]f^{(i)}(x)+\\&+\sum_{k=0}^{p_m}(x_{m,k}-x)^s\varphi_{m,k}(x)\mu(x_{m,k}-x),\end{aligned}$$

or

$$(L_mf)(x)-\sum_{i=0}^{s}\frac{1}{m^ii!}(T_iL_m)(x)f^{(i)}(x)=\sum_{k=0}^{p_m}(x_{m,k}-x)^s\varphi_{m,k}(x)\mu(x_{m,k}-x)$$

and thus

$$m^\gamma\left[(L_mf)(x)-\sum_{i=0}^{s}\frac{1}{m^ii!}(T_iL_m)(x)f^{(i)}(x)\right]=(R_mf)(x), \tag{3.11}$$

where

$$(R_mf)(x)=m^\gamma\sum_{k=0}^{p_m}(x_{m,k}-x)^s\varphi_{m,k}(x)\mu(x_{m,k}-x). \tag{3.12}$$

Consider $\delta_\epsilon$ from (3.10), $I_m = \{0, 1, \dots, p_m\} \cap \mathbb{N}$, $I_{m,1} = \{k \in I_m : |x_{m,k} - x| < \delta_\epsilon\}$ and $I_{m,2} = \{k \in I_m : |x_{m,k} - x| \geq \delta_\epsilon\}$. Then

$$|(R_m f)(x)| \leq m^\gamma \sum_{k=0}^{p_m} (x_{m,k} - x)^s \varphi_{m,k}(x) |\mu(x_{m,k} - x)| =$$
$$= m^\gamma \sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) |\mu(x_{m,k} - x)| +$$
$$+ m^\gamma \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x) |\mu(x_{m,k} - x)|$$

and taking (3.10) into account, and considering the fact that $\mu$ is bounded, so $\sup_{t \in V} |\mu(t)| = \eta$, we have

$$|(R_m f)(x)| \leq m^\gamma \epsilon \sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) + \tag{3.13}$$
$$+ m^\gamma \eta \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x).$$

But $(x_{m,k} - x)^s \leq (2a)^s$, so

$$\sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) \leq (2a)^s \sum_{k \in I_{m,1}} \varphi_{m,k}(x) \leq (2a)^s \sum_{k=0}^{p_m} \varphi_{m,k}(x). \tag{3.14}$$

Taking (3.5) and (3.14) into account, we have that

$$m^\gamma \epsilon \sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) \leq m^\gamma (2a)^s M. \tag{3.15}$$

From (3.6), we have that $\gamma < s + 2 - \alpha_{s+2}$ and then from Theorem 3.1 we obtain $\lim_{m \to \infty} m^\gamma \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x) = 0$, thus for $\epsilon$ from (3.10), there exists $m(\epsilon) \in \mathbb{N}$, for any $m \in \mathbb{N}$, $m \geq m(\epsilon)$, we have

$$m^\gamma \eta \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x) < \epsilon. \tag{3.16}$$

Choose $\epsilon = \frac{1}{m[m^\gamma (2a)^s M+1]}$ and there exists $m(\epsilon) \in \mathbb{N}$, for any $m \in \mathbb{N}$, $m \geq m(\epsilon)$, from (3.13)-(3.16) it results that $|(R_m f)(x)| < \frac{1}{m}$, and so

$$\lim_{m\to\infty} (R_m f)(x) = 0. \tag{3.17}$$

From (3.11) and (3.13), (3.7) follows. For the second afirmation from Theorem 3.2, we apply in the proof above the Theorem 3.1.

For $s = 0$, respectively $s = 2$ in Theorem 3.2 we obtain the Corollary 3.1.

**Corollary 3.1.** *If $f \in E_w(I)$ is a $s$ times differentiable function at $x \in I \cap J$ and we suppose that exist $\alpha_{s+2} \geq 0$ and $m(s) \in \mathbb{N}$ such that $\frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, then for any $\gamma$ which verify*

$$\gamma < s + 2 - \alpha_{s+2}, \tag{3.18}$$

*we have*

$$\lim_{m\to\infty} m^\gamma [(L_m f)(x) - (T_0 L_m)(x) f(x)] = 0 \tag{3.19}$$

*if $s = 0$, and*

$$\lim_{m\to\infty} m^\gamma \left[ (L_m f)(x) - (T_0 L_m)(x) f(x) - \frac{1}{m}(T_1 L_m)(x) f^{(1)}(x) - \right.$$
$$\left. - \frac{1}{2m^2}(T_2 L_m)(x) f^{(2)}(x) \right] = 0, \tag{3.20}$$

*if $s = 2$.*
*If $f \in E_w(I)$ is a $s$ times differentiable function on $I$ and for the compact $K \subset I \cap J$ exist $m(s) \in \mathbb{N}$ and the constant $k_{s+2}(K) \in \mathbb{R}$, depending on $K$ such that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$ we have*

$$\frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}} \leq k_{s+2}(K), \tag{3.21}$$

*where $s \in \{0, 2\}$, then the convergences given in (3.19) and (3.20) are uniform on $K$.*

**Remark 3.2.** *The relation* (3.20) *from* Corollary 3.1 *is a Voronovskaja-type identity.*

In the following, in every application we have $\sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1$, so $(T_0 L_m)(x) = 1$ for any $x \in J$, $m \in \mathbb{N}$, $u_m(K) = 0$ for any $K \subset I \cap J$ and $m \in \mathbb{N}$, $\alpha_2 = 1$, $\alpha_4 = 2$, $\gamma = 0$ if $s = 0$ and $\gamma = 1$ if $s = 2$.

In the following, by particularization of the sequence $x_{m,k}$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ and applying Corollary 3.1, we can obtain convergence theorems and Voronovskaja-type theorems for the operators from the first section of this paper. Because every application is a simple substitute in the Corollary 3.1, we won't replace anything.

**Application 3.1.** We study a particular case of the Stancu operators. Let $\alpha = 10$ and $\beta = -\frac{1}{2}$. We obtain $I = [0, 22]$, $K = J = [0, 1]$ and for any $f \in C([0, 22])$, $x \in [0, 1]$ and $m \in \mathbb{N}$

$$(P_m^{(10,-1/2)} f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{2k + 20}{2m - 1}\right),$$

where $\varphi_{m,k}(x) = p_{m,k}(x)$ and $x_{m,k} = \frac{2k+20}{2m-1}$, $k \in \{0, 1, \ldots, m\}$. We obtain $(T_1 P_m^{(10,-1/2)})(x) = \frac{m(20+x)}{2m-1}$, $(T_2 P_m^{(10,-1/2)})(x) = m^2 \frac{4mx(1-x)+(20+x)^2}{(2m-1)^2}$ for any $m \in \mathbb{N}$ and $x \in [0, 1]$, $k_2(K) = \frac{5}{4}$, $k_4(K) = \frac{19}{16}$ (see [19]).

For the Bleimann-Butzer-Hahn operators and for the Meyer-König and Zeller operators we only give the convergence theorems.

**Application 3.2.** We consider $I = J = [0, \infty)$, $E_w(I) = C_B([0, \infty))$, $w(x) = 1$ for any $x \in [0, \infty)$, $K = [0, b]$, $b > 0$, $p_m = m$, $x_{m,k} = \frac{k}{m+1-k}$, $\varphi_{m,k}(x) = \frac{1}{(1+x)^m} \binom{m}{k} x^k$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$, $x \in [0, \infty)$ and in this case we obtain the Bleimann-Butzer-Hahn operators. We have $(T_1 L_m)(x) = -mx \left(\frac{x}{1+x}\right)^m$, $x \in K$ and $k_2(K) = 4b(1 + b)^2$ for $m \geq 24(1 + b)$ (see [17]).

**Application 3.3.** If $I = J = [0, 1]$, $w(x) = 1$ for any $x \in [0, 1]$, $E_w(I) = B([0, 1])$, $K = [0, 1]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m+k}$, $(\varphi_{m,k})(x) = \binom{m+k}{k}(1 - x)^{m+1} x^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \in [0, 1]$, we obtain the Meyer-König and Zeller operators and we have $(T_1 Z_m)(x) = 0$, $m \in \mathbb{N}$, $x \in [0, 1]$, and $k_2(K) = 2$ (see [16]).

**Application 3.4.** If $I = J = [0,\infty)$, $w(x) = \frac{1}{1+x^2}$ for any $x \in [0,\infty)$, $E_w(I) = C_2([0,\infty))$, $K = [0,b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \in [0,\infty)$, we obtain the Mirakjan-Favard-Szász operators. We have $(T_1S_m)(x) = 0$, $(T_2S_m)(x) = mx$, $m \in \mathbb{N}$, $x \in [0,\infty)$, $k_2(K) = b$ and $k_4(K) = 3b^2 + b$ (see [16]).

**Application 3.5.** Let $I = J = [0,\infty)$, $w(x) = \frac{1}{1+x^2}$ for any $x \in [0,\infty)$, $E_w(I) = C_2([0,\infty))$, $K = [0,b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = (1+x)^{-m}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $x \in [0,\infty)$. In this case we obtain the Baskakov operators and we have $(T_1V_m)(x) = 0$, $(T_2V_m)(x) = mx(1+x)$, $m \in \mathbb{N}$, $x \in [0,\infty)$, $k_2(K) = b(1+b)$ and $k_4(K) = 9b^4 + 10b^3 + 10b^2 + b$ (see [16]).

**Application 3.6.** If $I = J = [0,\infty)$, $w(x) = 1$ for any $x \in [0,\infty)$, $E_w(I) = C([0,\infty))$, $K = [0,b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = \frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k e^{-\frac{(k+m)x}{1+x}}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \in [0,\infty)$, we obtain the Ismail-May operators. We have $(T_1R_m)(x) = 0$, $(T_2R_m)(x) = mx(1+x)^2$, $m \in \mathbb{N}$, $x \in [0,\infty)$, $k_2(K) = 1 + b(1+b)^2$ and $k_4(K) = 1 + b^2(1+b)^4$ (see [18]).

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