General Mathematics Vol. 16, No. 2 (2007), 29–35

Quasi-Hadamard product of certain classes of uniformly analytic functions [1]

B.A. Frasin

Abstract

In this paper, we establish certain results concerning the quasi-Hadamard product of certain classes of uniformly analytic functions.

2000 Mathematics Subject Classification: 30C45.
Key words: Analytic functions, Quasi-Hadamard product, uniformly analytic functions.

1 Introduction and definitions

Throughout the paper, let the functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_1 > 0, a_n \geq 0), \tag{1.1}$$

[1] *Received 2 July, 2007*
Accepted for publication (in revised form) 3 January, 2008

$$g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n \qquad (b_1>0,\ b_n\geq 0), \tag{1.2}$$

$$f_i(z)=a_{1,i}z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{1,i}>0,\ a_{n,i}\geq 0), \tag{1.3}$$

and

$$g_j(z)=b_{1,j}z-\sum_{n=2}^{\infty}b_{n,j}z^n \qquad (b_{1,j}>0,\ b_{n,j}\geq 0), \tag{1.4}$$

be analytic in the open unit disc $\mathcal{U}=\{z:|z|<1\}$.

Let $ST_0(\alpha,k)$ denote the class of functions $f(z)$ defined by (1.1) and satisfy the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\geq k\left|\frac{zf'(z)}{f(z)}-1\right|+\alpha. \qquad (z\in\mathcal{U}) \tag{1.5}$$

for some k $(0\leq k<\infty)$and α $(0\leq\alpha<1)$. Also denote by $UCT_0(\alpha,k)$ the class of functions $f(z)$ defined by (1.1) and satisfy the condition

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geq k\left|\frac{zf''(z)}{f'(z)}\right|+\alpha. \qquad (z\in\mathcal{U}) \tag{1.6}$$

for some k $(0\leq k<\infty)$ and α $(0\leq\alpha<1)$. The classes $ST_0(\alpha,k)$ and $UCT_0(\alpha,k)$ are of special interest for it contains many well-known classes of analytic functions. For example and when $a_1=1$ the classes $ST_0(\alpha,k)\equiv k{-}S_pT(\alpha)$ and $UCT_0(\alpha,k)\equiv k{-}UCV(\alpha)$ were introduced and studied by Bharati *et al.*[1]. Also, the classes $ST_0(0,k)\equiv k{-}ST$ and $UCT_0(0,k)\equiv k{-}UCV$ are, respectively, the subclasses of $\mathcal{A}$ consisting of functions which are *k- starlike* and *k-uniformly convex* in $\mathcal{U}$ introduced by Kanas and Winsiowska ([3, 4])(see also the work of Kanas and Srivastava [5], Goodman ([9, 10]), Rønning ([12, 13]), Ma and Minda [11] and Gangadharan et al.[8]). For $k=0$, the classes $ST_0(\alpha,0)\equiv ST_0^*(\alpha)$ and

$UCT_0(\alpha, 0) \equiv C_0(\alpha)$ are, respectively, the well-known classes of *starlike* functions of order α $(0 \leq \alpha < 1)$ and *convex* of order α $(0 \leq \alpha < 1)$ in $\mathcal{U}$ (see [14]).

Using similar arguments as given by Bharati *et al.*[1], one can prove the following analogous results for functions in the classes $ST_0(\alpha, k)$ and $UCT_0(\alpha, k)$.

A function $f(z) \in ST_0(\alpha, k)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\alpha)]a_n \leq (1-\alpha)a_1; \tag{1.7}$$

and $f(z) \in UCT_0(\alpha, k)$ if and only if

$$\sum_{n=2}^{\infty}n[n(1+k)-(k+\alpha)]a_n \leq (1-\alpha)a_1. \tag{1.8}$$

We now introduce the following class of analytic functions which plays an important role in the discussion that follows.

A function $f(z) \in ST_m(\alpha, k)$ if and only if

$$\sum_{n=2}^{\infty}n^m[n(1+k)-(k+\alpha)]a_n \leq (1-\alpha)a_1, \tag{1.9}$$

where k $(0 \leq k < \infty)$, α $(0 \leq \alpha < 1)$ and k is any fixed nonnegative real number.

Evidently, $ST_1(\alpha, k) \equiv UCT_0(\alpha, k)$ and, for $m = 0$, $ST_m(\alpha, k)$ is identical to $ST_0(\alpha, k)$. Further, $ST_m(\alpha, k) \subset ST_h(\alpha, k)$ if $m > h \geq 0$, the containment being proper. Whence, for any positive integer m, we have the inclusion relation

$$ST_m(\alpha, k) \subset ST_{m-1}(\alpha, k) \subset \ldots \subset ST_2(\alpha, k) \subset UCT_0(\alpha, k) \subset ST_0(\alpha, k).$$

We note that for every nonnegative real number m, the class $ST_m(\alpha, k)$ is nonempty as the functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty}\frac{(1-\alpha)a_1}{n^m[n(1+k)-(k+\alpha)]}\lambda_n z^n,$$

where $0 \leq k < \infty$, $0 \leq \alpha < 1$, $a_1 > 0$, $\lambda_n \geq 0$ and $\sum\limits_{n=2}^{\infty} \lambda_n \leq 1$, satisfy the inequality (1.9).

Let us define the quasi-Hadamard product of the functions $f(z)$ and $g(z)$ by

$$f * g(z) = a_1 b_1 z - \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.10}$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

In this paper, we establish certain results concerning the quasi-Hadamard product of functions in the classes $ST_m(\alpha, k)$, $ST_0(\alpha, k)$ and $UCT_0(\alpha, k)$ analogous to the results Kumar ([6, 7])(see also [2]).

2 Main Theorem

Theorem. *Let the functions $f_i(z)$ defined by (1.3) be in the class $UCT_0(\alpha, k)$ for every $i = 1, 2, \ldots, r$; and let the functions $g_i(z)$ defined by (1.4) be in the class $ST_0(\alpha, k)$ for every $j = 1, 2, \ldots, s$. Then the quasi-Hadamard product $f_1 * f_2 * \ldots * f_r * g_1 * g_2 * \ldots * g_s(z)$ belongs to the class $ST_{2r+s-1}(\alpha, k)$.*

Proof. Let $h(z) := f_1 * f_2 * \ldots * f_r * g_1 * g_2 * \ldots * g_s(z)$, then

$$h(z) = \left\{ \prod_{i=1}^{r} a_{1,i} \prod_{j=1}^{s} b_{1,j} \right\} z - \sum_{n=2}^{\infty} \left\{ \prod_{i=1}^{r} a_{n,i} \prod_{j=1}^{s} b_{n,j} \right\} z^n. \tag{2.11}$$

We need to show that

$$\sum_{n=2}^{\infty} \left[n^{2r+s-1} \{n(1+k) - (k+\alpha)\} \left\{ \prod_{i=1}^{r} a_{n,i} \prod_{j=1}^{s} b_{n,j} \right\} \right]$$

$$\leq (1-\alpha) \left\{ \prod_{i=1}^{r} a_{1,i} \prod_{j=1}^{s} b_{1,j} \right\}. \tag{2.12}$$

Since $f_i(z) \in UCT_0(\alpha, k)$, we have

$$\sum_{n=2}^{\infty} n[n(1+k)-(k+\alpha)]a_{n,i} \leq (1-\alpha)a_{1,i} \tag{2.13}$$

for every $i = 1, 2, \ldots, r$. Therefore,

$$a_{n,i} \leq \left[\frac{1-\alpha}{n[n(1+k)-(k+\alpha)]}\right] a_{1,i}$$

which implies that

$$a_{n,i} \leq n^{-2} a_{1,i} \tag{2.14}$$

for every $i = 1, 2, \ldots, r$. Similarly, for $g_j(z) \in ST_0(\alpha, k)$, we have

$$\sum_{n=2}^{\infty} [n(1+k)-(k+\alpha)]b_{n,j} \leq (1-\alpha)b_{1,j}. \tag{2.15}$$

for every $j = 1, 2, \ldots, s$. Hence we obtain

$$b_{n,j} \leq n^{-1} b_{1,j} \tag{2.16}$$

for every $j = 1, 2, \ldots, s$.

Using (2.14) for $i = 1, 2, \ldots, r$, (2.16) for $j = 1, 2, \ldots, s-1$, and (2.15) for $j = s$, we obtain

$$\sum_{n=2}^{\infty} \left[n^{2r+s-1}[n(1+k)-(k+\alpha)] \left\{ \prod_{i=1}^{r} a_{n,i} \prod_{j=1}^{s} b_{n,j} \right\} \right]$$

$$\begin{aligned}
&\leq \sum_{n=2}^{\infty} \left[n^{2r+s-1}[n(1+k)-(k+\alpha)]b_{n,s} \left\{ n^{-2r} n^{-(s-1)} \left(\prod_{i=1}^{r} a_{1,i} \prod_{j=1}^{s-1} b_{1,j} \right) \right\} \right] \\
&= \left(\sum_{n=2}^{\infty} [n(1+k)-(k+\alpha)]b_{n,s} \right) \left(\prod_{i=1}^{r} a_{1,i} \prod_{j=1}^{s-1} b_{1,j} \right) \\
&\leq (1-\alpha) \left\{ \prod_{i=1}^{r} a_{1,i} \prod_{j=1}^{s} b_{1,j} \right\}.
\end{aligned}$$

Hence $h(z) \in ST_{2r+s-1}(\alpha, k)$.

Note that we can prove the above theorem by using using (2.14) for $i = 1, 2, \ldots, r-1$, (2.16) for $j = 1, 2, \ldots, s$, and (2.13) for $i = r$.

Taking into account the quasi-Hadamard product functions $f_1(z), f_2(z), \ldots, f_r(z)$ only, in the proof of the above theorem, and using (2.14) for $i = 1, 2, \ldots, r-1$, and (2.13) for $i = r$, we obtain

Corollary 1. *Let the functions $f_i(z)$ defined by (1.3) be in the class $UCT_0(\alpha, k)$ for every $i = 1, 2, \ldots, r$. Then the quasi-Hadamard product $f_1 * f_2 * \ldots * f_r$ belongs to the class $ST_{2r-1}(\alpha, k)$.*

Next, taking into account the quasi-Hadamard product functions $g_1(z)$, $g_2(z), \ldots, g_r(z)$ only, in the proof of the above theorem, and using (2.16) for $j = 1, 2, \ldots, s-1$, and (2.15) for $j = s$, we obtain

Corollary 2. *Let the functions $g_i(z)$ defined by (1.4) be in the class $ST_0(\alpha, k)$ for every $j = 1, 2, \ldots, s$. Then the quasi-Hadamard product $g_1 * g_2 * \ldots * g_s(z)$ belongs to the class $ST_{s-1}(\alpha, k)$.*

References

[1] R. Bharati, R. Parvatham and A. Swaminathan, *On Subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math., 28 (1997), 17-32.

[2] E. W. Darwish, *The Quasi-Hadamard product of certain starlike and convex functions*, Appl. Math. Lett. 20 (2007), 692-695.

[3] S. Kanas and A. Wisniowska, *Conic regions and k- uniform convexity*, J. Comput. Appl. Math. 105 (1999), 327-336.

[4] S. Kanas and A. Wisniowska, *Conic regions and k- starlike functions*, Rev. Roumaine Math. Pures Appl., 45(4)(2000), 647-657.

[5] S. Kanas and H.M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Integral Transform. Spec. Funct. 9, 121-132, (2000).

[6] V. Kumar, *Hadamard product of certain starlike functions*, J. Math. Anal. Appl. 110(1985), 425-428.

[7] V. Kumar, *Quasi-Hadamard product of certain univalent*, J. Math. Anal. Appl. 126(1987), 70-77.

[8] A. Gangadharan, T.N. Shanmugan and H.M. Srivastava, *Generalized Hypergeometric functions associated with k-uniformly convex functions,* Comput. Math. App. 44 (2002), 1515-1526.

[9] A.W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. 56, 87-92, (1991).

[10] A.W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. 155, 364-370, (1991).

[11] W.C. Ma and D. Minda, *Uniformly convex functions*, Ann. Polon. Math. 57 (1992), no.2, 165-175.

[12] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. 118 (1993), no.1, 189-196.

[13] F. Rønning, *On starlike functions associated with parabolic regions*, Ann Univ. Mariae Curie-Sklodowska Sect. A 45 (1991), 117-122.

[14] H. Silverman, *Extreme points of univalent functions with two fixed points*, Trans. Amer. Math. Soc. 219 (1976), 385-397.

Department of Mathematics,
Al al-Bayt University,
P.O. Box: 130095
Mafraq, Jordan.
E-mail address: *bafrasin@yahoo.com.*