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A complex variable boundary element method for the problem of the free-surface heavy inviscid flow over an obstacle[1]

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Abstract

The object of this paper is to solve the problem of the bidimensional heavy fluid flow over an immersed obstacle situated near the free surface, using the Complex Variable Boundary Element Method (CVBEM). The CVBEM is an advanced mathematical modeling approach and represents a numerical application of Cauchy Integral theorem for complex variable analytic functions.The problem is equivalent with an integro-differential equation with boundary conditions which is solved in this paper using linear boundary elements and is reduced to a system of linear equations in terms of nodal values of the components of the velocity field. After solving the system the velocity is obtained and further the local pressure coefficient.

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1 Introduction

A uniform steady potential plane free surface flow of a heavy inviscid fluid is perturbed by the presence of an arbitrary wing (obstacle) immersed in

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the immediate proximity of the free surface. We intend to use the Complex Variable Boundary Element Method (CVBEM), to determine the perturbation induced by the presence of the obstacle (wing) and the action exerted by the fluid on this obstacle, so to find the fluid velocity field and the local pressure coefficient. We assume that the boundary Γ of the wing is smooth enough to avoid the existence of some angular points (and implicitly of a Kutta type condition). Using dimensionless variables defined with the characteristic length L (the length of the wing) and the characteristic velocity U (the upstream uniform velocity), by splitting the velocity potential Φ into the unperturbed (uniform) stream potential and the perturbation (due to the obstacle) potential we have: $\Phi(x,y) = x + \varphi(x,y)$, (1) where $\varphi(x,y)$ is the perturbation potential which satisfies the Laplace equation $\Delta\varphi(x,y) = 0,\ x \in (-\infty,+\infty),\ y \in (-\infty,0)$ (2). In[1] and [5] the same problem is solved using Schwarz principle, without a free-surface discretization, but without the possibility to obtain the perturbed free-surface.

At the beginning we consider that the free surface can be approximated by the real axis Ox.Linearizing the Bernoulli's integral, the following boundary condition on free surface holds (see[1])

$$(3)\quad \frac{\partial^2\varphi}{\partial x^2} + k_0\frac{\partial\varphi}{\partial y} = 0,\quad (x,y) \in (-\infty,+\infty)\times\{0\},\ k_0 = \frac{1}{Fr^2},\ Fr = \frac{U}{\sqrt{gL}}$$

On the surface of the immersed wing the slip condition becomes

$$(4)\qquad \frac{\partial\varphi}{\partial n}\,|_{\Gamma} = -n_x$$

where $\overline{n}(n_x, n_y)$ is the outward unit normal drawn at Γ while, at far field,

$$(5)\qquad \lim_{x\to\infty}\varphi(x,y) = 0.$$

.

By introducing the stream (perturbation) function $\psi(x,y)$ - the harmonic conjugate of $\varphi(x,y)$, and by using the complex variable $z = x + iy$, the complex (perturbation) potential $f(z) = \varphi(x,y) + i\psi(x,y)$ satisfies the relations

$$\frac{df}{dz} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = w$$

$$Re\frac{d^2 f}{dz^2} = \frac{\partial^2 \varphi}{\partial x^2}, \quad Im\frac{df}{dz} = -\frac{\partial \varphi}{\partial y} \tag{6}$$

where $w = u - iv$ is the complex (perturbation) velocity constructed on the components u and v of the perturbation velocity ($u = \frac{\partial \varphi}{\partial x}$, $v = \frac{\partial \varphi}{\partial y}$). Hence (4) and (5) become:

$$Im\left(i\frac{d^2 f}{dz^2} - k_0\frac{df}{dz}\right) = 0, \tag{7}$$

for $z = x \in R$; $\quad Re\left(\frac{df}{dz}\left(n_x + in_y\right)\right) = -n_x,$ on Γ

By introducing the holomorphic (in the flow domain) function

$$F\left(z\right) = i\frac{d^2 f}{dz^2} - k_0\frac{df}{dz} = i\frac{dw}{dz} - k_0 w \tag{8}$$

we get

$$ImF\left(z\right) = 0, \quad \text{for } z = x \in R\ . \tag{9}$$

As $\lim\limits_{|z|\to\infty} F\left(z\right) = 0$, the use of the Cauchy's formula for the whole domain (the lower half plane without the obstacle domain) allows us to write

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{F\left(\varsigma\right)}{\varsigma - z}d\varsigma = F\left(z\right) - \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{F\left(\varsigma\right)}{\varsigma - z}d\varsigma \tag{10}$$

Using (8) and the extension of Cauchy's formula for the first derivative of the holomorphic function $\dfrac{dw}{d\varsigma}$(or integrating "by parts") we can write:

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{Re\left(F\left(\varsigma\right)\right) + iIm\left(F\left(\varsigma\right)\right)}{\varsigma - z}d\varsigma = i\frac{dw\left(z\right)}{dz} - k_0 w\left(z\right) + \frac{k_0}{2\pi i}\int\limits_{\Gamma}\frac{w\left(\varsigma\right)}{\varsigma - z}d\varsigma$$
$$- \frac{1}{2\pi}\int\limits_{\Gamma}\frac{w\left(\varsigma\right)}{\left(\varsigma - z\right)^2}d\varsigma \tag{11}$$

and further, through (9), we get:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{Re\left(F\left(\varsigma\right)\right)}{\varsigma-z}d\varsigma=-\frac{dw\left(z\right)}{dz}-ik_0w\left(z\right)$$
$$+\frac{1}{2\pi i}\left(\int\limits_{\Gamma}k_0i\frac{w\left(\varsigma\right)}{\varsigma-z}d\varsigma+\int\limits_{\Gamma}\frac{w\left(\varsigma\right)}{\left(\varsigma-z\right)^2}d\varsigma\right)\tag{12}$$

2 Solving the integro-differential equation using linear boundary elements

We solve the integro-differential equation using a complex variable boundary elements method with linear elements. For the term $\frac{dw\left(z\right)}{dz}$ an appropriate finite difference scheme will be used.

The border, Γ, is approximated by a polygonal line made by segments $L_j,\ j=\overline{1,N}$, by choosing a set of control points of affixes $z_i, i=\overline{1,N}$ on it. L_j has the end points of affixes $z_j,\ z_{j+1},\ j=\overline{1,N}$, where $z_{N+1}=z_1$. Using linear boundary elements we have the following linear approximation for $w\left(z\right)$ (see [3], [4]):

$$\widetilde{w}\left(\varsigma\right)=w\left(z_j\right)\frac{\varsigma-z_{j+1}}{z_j-z_{j+1}}+w\left(z_{j+1}\right)\frac{z_j-\varsigma}{z_j-z_{j+1}},\quad j=\overline{1,N}\tag{13}$$

(precisely all the elements with index $N+1$ are seen as having index 1).

For the beginning we consider the first integral from the right side of equation (12). Using relation (13) we deduce the following approximation for it:

$$\int\limits_{\Gamma}k_0i\frac{w\left(\varsigma\right)}{\varsigma-z}d\varsigma=k_0i\sum_{j=1}^{N}\int\limits_{L_j}\left[w\left(z_j\right)\frac{\varsigma-z_{j+1}}{\left(z_j-z_{j+1}\right)\left(\varsigma-z\right)}\right.$$
$$\left.+\,w\left(z_{j+1}\right)\frac{z_j-\varsigma}{\left(z_j-z_{j+1}\right)\left(\varsigma-z\right)}\right]d\varsigma\tag{14}$$

By denoting $w(z_i) = w_i$ we deduce:

(15)
$$\int\limits_{\Gamma} k_0 i \frac{w(\varsigma)}{\varsigma - z} d\varsigma = k_0 i \sum_{j=1}^{N} (w_{j+1} - w_j) +$$

$$+k_0 i \sum_{j=1}^{N} \left[w_{j+1} \left(\frac{z - z_j}{z_{j+1} - z_j} \right) - w_j \left(\frac{z - z_{j+1}}{z_{j+1} - z_j} \right) \right] \ln \left(\frac{z_{j+1} - z}{z_j - z} \right)$$

and further:

(16)
$$\int\limits_{\Gamma} k_0 i \frac{w(\varsigma)}{\varsigma - z} d\varsigma = k_0 i \sum_{j=1}^{N} \left[w_{j+1} \left(\frac{z - z_j}{z_{j+1} - z_j} \right) - w_j \left(\frac{z - z_{j+1}}{z_{j+1} - z_j} \right) \right]$$
$$\cdot \left[\ln(z_{j+1} - z) - \ln(z_j - z) \right]$$

Because the complex function $\ln(z_j - z)$ has multiple possible values we consider a branch cut for $\ln(\xi - z)$, a line from z to$z_1$and so when $\int_\Gamma \frac{d\varsigma}{\varsigma - z} d\varsigma$ is evaluated on Γ_1 at $z_1$we obtain $\ln(z_1 - z)$, but when it is evaluated on Γ_N at z_1 we obtain $\ln(z_1 - z) + 2\pi i$ (17). After some calculus we get:
(18)
$$\int\limits_{\Gamma} k_0 i \frac{w(\varsigma)}{\varsigma - z} d\varsigma = k_0 i \sum_{j=1}^{N} a_j (z_j - z) \ln(z_j - z) + k_0 i \left[w_1 + \frac{w_1 - w_N}{z_1 - z_N} \right] (z - z_1)$$

where

(19)
$$a_j = \frac{w_{j+1} - w_j}{z_{j+1} - z_j} - \frac{w_j - w_{j-1}}{z_j - z_{j-1}}$$

For evaluating the second integral from the right term in relation (12) we use the following theorem demonstrated in paper [3]:
Theorem. *Let Γ be a simple closed contour with finite length L and simply connected interior Ω. Let $h(\varsigma)$ be a continuous function on Γ.Then $\widehat{\omega}(z)$ is analytic in Ω, where $\widehat{\omega}(z)$ is defined by the contour integral*

(20)
$$\widehat{\omega}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{h(\varsigma)}{\varsigma - z} d\varsigma$$

and $\widehat{\omega\prime}(z)$ *is given by the integral*:

(20’)
$$\widehat{\omega\prime}(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{h(\varsigma)}{(\varsigma - z)^2} d\varsigma$$

Because the linear model for the approximation function offers it a global continuity we can apply the above theorem for evaluating the mentioned integral, and we get:

(21)
$$\int\limits_{\Gamma} \frac{w(\varsigma)}{(\varsigma - z)^2} d\varsigma = \left(\sum_{j=1}^{N} a_j (z_j - z)\ln(z_j - z) + 2\pi i\left[w_1 + \frac{w_1 - w_N}{z_1 - z_N}\right](z - z_1)\right)'$$
$$= -\sum_{j=1}^{N} a_j \ln(z_j - z) - \sum_{j=1}^{N} a_j + 2\pi i\left(w_1 + \frac{w_1 - w_N}{z_1 - z_N}\right)$$

Using (19) and (21) equation (12) can be written as:

(22)
$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \frac{Re(F(\varsigma))}{\varsigma - z} d\varsigma + \frac{dw(z)}{dz} + ik_0 w(z)$$

$$= \frac{k_0}{2\pi}\left(\sum_{j=1}^{N} a_j (z_j - z)\ln(z_j - z) + \left[w_1 + \frac{w_1 - w_N}{z_1 - z_N}\right](z - z_1)\right)$$
$$-\frac{1}{2\pi i}\left(\sum_{j=1}^{N} a_j \ln(z_j - z) + \sum_{j=1}^{N} a_j - 2\pi i\left(w_1 + \frac{w_1 - w_N}{z_1 - z_N}\right)\right).$$

Regarding a_j we can write: $a_j = m_j w_{j+1} + n_j w_j + p_j w_{j-1}$, where

(23)
$$m_j = \frac{1}{z_{j+1} - z_j}, \quad n_j = -\frac{z_{j+1} - z_{j-1}}{(z_{j+1} - z_j)(z_j - z_{j-1})}, \quad p_j = \frac{1}{z_j - z_{j-1}}$$

Equation (22) can also be written under the form:

(24)
$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \frac{Re(F(\varsigma))}{\varsigma - z} d\varsigma + \frac{dw(z)}{dz} + ik_0 w(z) = \sum_{j=1}^{N} w_j A_j$$

where using the notation: $c(z,x) = \ln(z-x)(k_0(z-x)+i)$ the coefficients A_j, $j=\overline{1,N}$ are given by the following relations:

$$(25)\quad A_1 = \frac{1}{2\pi}\{m_N[c(z_N,z)+i] + n_1[c(z_1,z)+i] + p_2[c(z_2,z)+i]\} +$$

$$1 + \frac{1}{z_1 - z_N} + \frac{k_0}{2\pi}(z-z_1)\left(1+\frac{1}{z_1-z_N}\right),$$

$$A_j = \frac{1}{2\pi}\{m_{j-1}[c(z_{j-1},z)+i] + n_j[c(z_j,z)+i] + p_{j+1}[c(z_{j+1},z)+i]\},$$

$$A_N = \frac{1}{2\pi}\{m_{N-1}[c(z_{N-1},z)+i] + n_N[c(z_N,z)+i] + p_1[c(z_1,z)+i]\}$$

$$+1+\frac{1}{z_1-z_N} + \frac{k_0}{2\pi}\frac{z_1-z}{z_1-z_N}.$$

Now if we let $z \to z_i \in \Gamma$, $i=\overline{1,N}$, we obtain

$$(26)\qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{Re(F(\varsigma))}{\varsigma - z_i}d\varsigma + \frac{dw(z_i)}{dz} + ik_0 w(z_i) = \sum_{j=1}^{N} w_j A_{ij},$$

where the denotation with two indexes points out that the limits of the involved coefficients are considered. More, by using for the value of the complex velocity derivative at the node i its approximation by a forward finite difference scheme, namely $\dfrac{dw(z_i)}{dz} = \dfrac{w(z_i)-w(z_{i+1})}{z_i - z_{i+1}}$, we obtain:

$$(27)\qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{Re(F(\varsigma))}{\varsigma - z_i}d\varsigma + \frac{w_i - w_{i+1}}{z_i - z_{i+1}} + ik_0 w_i = \sum_{j=1}^{N} w_j A_{ij}$$

where $i,j=\overline{1,N}$, while by index $N+1$ we should understand 1.

Concerning the calculation of the coefficients from the above equation, it is performed by imposing effectively $z \to z_i \in \Gamma$ in the previous expressions of A_j. Except the elements originated from the integrals calculated on the segments whose edges contain the point i (i.e., $\Gamma_{i-1}$and Γ_i) and which become singular, this implies a simple replacement of z with z_i. With regard to the coefficients coming from the singular integral, we have used the equality $\lim\limits_{z\to z_i}(z-z_i)\log(z-z_i)=0$ and their finite parts, in according with the finite part of an integral (see[2]). We get the following expressions:

$$A_{i1}=\frac{1}{2\pi}\left\{m_N\left[c\left(z_N,z_i\right)+i\right]+n_1\left[c\left(z_1,z_i\right)+i\right]+p_2\left[c\left(z_2,z_i\right)+i\right]\right\}$$

$$+1+\frac{1}{z_1-z_N}+\frac{k_0}{2\pi}\left(z_i-z_1\right)\left(1+\frac{1}{z_1-z_N}\right),i\neq N,1,2, \tag{28}$$

$$\begin{aligned}A_{N1}=&\frac{1}{2\pi}\left\{m_Ni+n_1\left[c\left(z_1,z_N\right)+i\right]+p_2\left[c\left(z_2,z_N\right)+i\right]\right\}+1\\&+\frac{1}{z_1-z_N}+\frac{k_0}{2\pi}\left(z_N-z_1\right)\left(1+\frac{1}{z_1-z_N}\right),\end{aligned}$$

$$A_{11}=\frac{1}{2\pi}\left\{m_N\left[c\left(z_N,z_1\right)+i\right]+n_1i+p_2\left[c\left(z_2,z_1\right)+i\right]\right\}+1+\frac{1}{z_1-z_N},$$

$$\begin{aligned}A_{21}=&\frac{1}{2\pi}\left\{m_N\left[c\left(z_N,z_2\right)+i\right]+n_1\left[c\left(z_1,z_2\right)+i\right]+p_2i\right\}+1\\&+\frac{1}{z_1-z_N}+\frac{k_0}{2\pi}\left(z_2-z_1\right)\left(1+\frac{1}{z_1-z_N}\right)\end{aligned}$$

$$\begin{aligned}A_{ij}&=\frac{1}{2\pi}\left\{m_{j-1}\left[c\left(z_{j-1},z_i\right)+i\right]+n_j\left[c\left(z_j,z_i\right)+i\right]+p_{j+1}\left[c\left(z_{j+1},z_i\right)+i\right]\right\},\\ j&=\overline{2,N-1},i\neq j\pm1,j\end{aligned}$$

$$A_{jj}=\frac{1}{2\pi}\left\{m_{j-1}\left[c\left(z_{j-1},z_j\right)+i\right]+n_ji+p_{j+1}\left[c\left(z_{j+1},z_j\right)+i\right]\right\},$$

$$A_{j-1j}=\frac{1}{2\pi}\left\{m_{j-1}i+n_j\left[c\left(z_j,z_{j-1}\right)+i\right]+p_{j+1}\left[c\left(z_{j+1},z_{j-1}\right)+i\right]\right\},$$

$$A_{j+1j}=\frac{1}{2\pi}\left\{m_{j-1}\left[c\left(z_{j-1},z_{j+1}\right)+i\right]+n_j\left[c\left(z_j,z_{j+1}\right)+i\right]+p_{j+1}i\right\},$$

$$\begin{aligned}
A_{iN} &= \frac{1}{2\pi}\left\{m_{N-1}\left[c\left(z_{N-1},z_i\right)+i\right]+n_N\left[c\left(z_N,z_i\right)+i\right]+p_1\left[c\left(z_1,z_i\right)+i\right]\right\}\\
&\quad+1+\frac{1}{z_1-z_N}+\frac{k_0}{2\pi}\frac{z_1-z_i}{z_1-z_N}, i\neq N-1,N,1\\
A_{N-1N} &= \frac{1}{2\pi}\left\{m_{N-1}i+n_N\left[c\left(z_N,z_{N-1}\right)+i\right]+p_1\left[c\left(z_1,z_{N-1}\right)+i\right]\right\}\\
&\quad+1+\frac{1}{z_1-z_N}+\frac{k_0}{2\pi}\frac{z_1-z_{N-1}}{z_1-z_N},\\
A_{NN} &= \frac{1}{2\pi}\left\{m_{N-1}\left[c\left(z_{N-1},z_N\right)+i\right]+n_N i+p_1\left[c\left(z_1,z_N\right)+i\right]\right\}\\
&\quad+1+\frac{1}{z_1-z_N}+\frac{k_0}{2\pi},\\
A_{1N} &= \frac{1}{2\pi}\left\{m_{N-1}\left[c\left(z_{N-1},z_1\right)+i\right]+n_N\left[c\left(z_N,z_1\right)+i\right]+p_1 i\right\}+1+\frac{1}{z_1-z_N}.
\end{aligned}$$

As i takes all the natural values from 1 to N, we obtain a system of N equations, of the form (22), with N unknowns which can be written as:.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{Re\left(F\left(\varsigma\right)\right)}{\varsigma-z_i}d\varsigma=\sum_{j=1}^{N}w_j\widetilde{A}_{ij} \tag{29}$$

where

$$\widetilde{A}_{ij}=A_{ij},\ j\neq i,\ j\neq i+1;\ \widetilde{A}_{ii}=\left(A_{ii}-\frac{1}{z_i-z_{i+1}}-ik_0\right); \tag{30}$$
$$\widetilde{A}_{ii+1}=\left(A_{ii+1}+\frac{1}{z_i-z_{i+1}}\right)$$

For evaluate the integral on the right sight we use linear boundary elements too. As the perturbation vanishes at far field we can accept that

$$\frac{1}{2\pi}\int_{--\infty}^{+\infty}\frac{Re\left(F\left(\varsigma\right)\right)}{\varsigma-z_i}d\varsigma=\frac{1}{2\pi}\int_{a}^{b}\frac{Re\left(F\left(\varsigma\right)\right)}{\varsigma-z_i}d\varsigma \tag{31}$$

Using relation (8), the expression of w and the linear condition (3) we obtain the following condition on the free surface

$$Re\left(F\left(z\right)\right)=-\frac{1}{k_0}\frac{\partial^2 u}{\partial x^2}-k_0u,\ z=x \tag{32}$$

Using M isoparametric linear boundary elements (with $M+1$ equidistant nodes on the free surface:$x_k \in [a,b]$, $k=\overline{0,M}$, $a=x_0$, $x_k = a+k\frac{b-a}{M}$, $k=\overline{1,M}$) we deduce (as we did for ()) that:

(33) $$\frac{1}{2\pi}\int_a^b \frac{Re\left(F\left(\varsigma\right)\right)}{\varsigma - z}d\varsigma = \frac{-k_0}{2\pi}\sum_{k=1}^{M-1} a'_k\left(x_k - z\right)\ln\left(x_k - z\right)$$

$$-\frac{k_0}{2\pi}\left[\frac{u_0-u_1}{x_1-x_0}\left(z-x_0\right)\ln\left(x_0-z\right)+\frac{u_M-u_{M-1}}{x_M-x_{M-1}}\left(z-x_M\right)\ln\left(x_M-z\right)\right]$$
$$+\frac{k_0}{2\pi}u_0\ln\left(x_0-z\right)-\frac{k_0}{2\pi}u_M\ln\left(x_M-z\right)$$

(34) $$a'_k = \frac{u_{k+1}-u_k}{x_{k+1}-x_k} - \frac{u_k-u_{k-1}}{x_k-x_{k-1}}.$$

Regarding a'_k we can write:

(35) $$a'_k = m'_k u_{k+1} + n'_k u_k + p'_k u_{k-1},$$
$$m'_k = \frac{1}{x_{k+1}-x_k}, \quad n'_k = -\frac{x_{k+1}-x_{k-1}}{\left(x_{k+1}-x_k\right)\left(x_k-x_{k-1}\right)}, \quad p'_k = \frac{1}{x_k-x_{k-1}}$$

We obtain:

(36) $$\frac{1}{2\pi}\int_a^b \frac{Re\left(F\left(\varsigma\right)\right)}{\varsigma - z_i}d\varsigma = \sum_{l=0}^{M} B_{il}u_l$$

where

(37) $$B_{il} = -\frac{k_0}{2\pi}\left[\frac{\left(x_{l+1}-z_i\right)}{x_{l+1}-x_l}\ln\left(x_{l+1}-z_i\right)+\frac{\left(x_{l-1}-z_i\right)}{x_l-x_{l-1}}\ln\left(x_{l-1}-z_i\right)\right]$$
$$-\frac{k_0}{2\pi}\frac{\left(x_{l+1}-x_{l-1}\right)\left(x_l-z_i\right)}{\left(x_{l+1}-x_l\right)\left(x_l-x_{l-1}\right)}\ln\left(x_l-z_i\right), l=\overline{1,M-1}$$
$$B_{i0} = -\frac{k_0}{2\pi}\left[\frac{\left(x_1-z_i\right)}{x_1-x_0}\ln\left(x_1-z_i\right)+\frac{\left(z_i-x_0\right)}{x_1-x_0}\ln\left(x_0-z_i\right)-\ln\left(x_0-z_i\right)\right]$$

$$B_{iM} = -\frac{k_0}{2\pi}\left[\frac{(x_{M-1}-z_i)}{x_M - x_{M-1}}\ln\left(x_{M-1}-z_i\right) + \frac{(z_i - x_M)}{x_M - x_{M-1}}\ln\left(x_M - z_i\right)\right.$$
$$\left. + \ln\left(x_M - z_i\right)\right]$$

By denoting with v_n, v_s the normal and the tangential, respectively, components of the perturbation velocity we can write that, on the border, $w = (v_n - iv_s)(n_x + in_y)$ while on Γ, $v_n = -n_x$, so that $w = (-n_x - iv_s)(n_x + in_y)$. With this new remarks and relation (), and denoting (for sake of simplicity) $v_s^i = v_i,\ i = \overline{1, N}$ the system (29) becomes:

$$\sum_{l=0}^{M} B_{il} u_l = \sum_{j=1}^{N}\left(-n_x^j - iv_j\right)\left(n_x^j + in_y^j\right)\widetilde{A}_{ij}, i = \overline{1, N} \tag{38}$$

As the number of unknowns $N + M + 1$ is greater than the number of equations for "closing" the system we should now perform $z \to x_k,\ k = \overline{0, M}$ in the relations (17) and (19). So we get

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{Re\left(F\left(\varsigma\right)\right)}{\varsigma - x_k}d\varsigma + \frac{dw\left(x_k\right)}{dx} + ik_0 w\left(x_k\right) = \sum_{j=1}^{N} w_j \widehat{A}_{kj} \tag{39}$$

where using the notation: $c(z, x) = \ln(z - x)(k_0(z - x) + i)$ the coefficients $\widehat{A}_{kj}$ have the expressions:

(40)

$$\widehat{A}_{kj} = \frac{1}{2\pi}\left\{m_{j-1}\left[c\left(z_{j-1}, x_k\right) + i\right] + n_j\left[c\left(z_j, x_k\right) + i\right] + p_{j+1}\left[c\left(z_{j+1}, x_k\right) + i\right]\right\}.$$

Using a forward finite difference for the complex velocity derivative, for $k = \overline{0, M-1}$,and for $x_k = x_M$ a backward finite difference $\dfrac{dw\left(x_M\right)}{dx} = \dfrac{w\left(x_M\right) - w\left(x_{M-1}\right)}{x_M - x_{M-1}}$ we get:

$$\sum_{l=0}^{M} B'_{kl} u_l + \frac{w\left(x_k\right) - w\left(x_{k+1}\right)}{x_k - x_{k+1}} + ik_0 w\left(x_k\right) = \tag{41}$$
$$\frac{1}{2\pi i}\sum_{j=1}^{N} w_j \widehat{A}_{kj},\ k = \overline{0, M-1}$$

$$\sum_{l=0}^{M} B'_{kl} u_l + \frac{w(x_M) - w(x_{M-1})}{x_M - x_{M-1}} + i k_0 w(x_M) = \frac{1}{2\pi i} \sum_{j=1}^{N} w_j \widehat{A}_{Mj}. \tag{42}$$

where the coefficients B_{lk} and C_{lk} have analogous expressions with those of (36), the only one difference being that now, for all nonsingular integrals (i.e., when x_k is not a node of the element on which the integral is calculated),z_i is replaced with x_k and a natural logarithm of a real number is implied.

Thus,

$$\begin{aligned} B'_{kl} = & -\frac{k_0}{2\pi}\left[\frac{(x_{l+1}-x_k)}{x_{l+1}-x_l}\ln(x_{l+1}-x_k) + \frac{(x_{l-1}-x_k)}{x_l - x_{l-1}}\ln(x_{l-1}-x_k)\right] \\ & -\frac{k_0}{2\pi}\left[\frac{(x_{l+1}-x_{l-1})(x_l - x_k)}{(x_{l+1}-x_l)(x_l - x_{l-1})}\ln(x_l - x_k)\right], l = \overline{1, M-1}, \end{aligned} \tag{43}$$

$$\begin{aligned} B'_{k0} &= -\frac{k_0}{2\pi}\left[\frac{(x_1 - x_k)}{x_1 - x_0}\ln(x_1 - x_k) + \frac{(x_k - x_0)}{x_1 - x_0}\ln(x_0 - x_k) - \ln(x_0 - x_k)\right] \\ B'_{kM} &= -\frac{k_0}{2\pi}\left[\frac{(x_{M-1}-x_k)}{x_M - x_{M-1}}\ln(x_{M-1}-x_k) + \frac{(x_k - x_M)}{x_M - x_{M-1}}\ln(x_M - x_k)\right. \\ &\quad \left. + \ln(x_M - x_k)\right] \end{aligned}$$

for $k \neq l \pm 1,\ k \neq l$ when $l = \overline{1, M-1}$; $k \neq 0, 1$ when $l = 0$; $k \neq M-1, M$ when $l = M$. For the case when singular coefficients arise (for B'_{00} and B'_{MM} we have considered their finite parts) we finally get the following expressions:

(43')

$$\begin{aligned} B'_{l-1l} &= -\frac{k_0}{2\pi}\left[\frac{(x_{l+1}-x_{l-1})}{x_{l+1}-x_l}\ln(x_{l+1}-x_{l-1}) + \frac{(x_{l+1}-x_{l-1})}{(x_{l+1}-x_l)}\ln(x_l - x_{l-1})\right], \\ B'_{l+1l} &= -\frac{k_0}{2\pi}\left[\frac{(x_{l-1}-x_{l+1})}{x_l - x_{l-1}}\ln(x_{l-1}-x_{l+1}) - \frac{(x_{l+1}-x_{l-1})}{(x_l - x_{l-1})}\ln(x_l - x_{l+1})\right], \\ B'_{ll} &= -\frac{k_0}{2\pi}\left[\frac{(x_{l+1}-x_l)}{x_{l+1}-x_l}\ln(x_{l+1}-x_l) + \frac{(x_{l-1}-x_l)}{x_l - x_{l-1}}\ln(x_{l-1}-x_l)\right], \\ B'_{00} &= -\frac{k_0}{2\pi}\ln(x_1 - x_0), B_{10} = 0, B_{M-1M} = 0,\ B_{MM} = \frac{k_0}{2\pi}\ln(x_{M-1}-x_M) \end{aligned}$$

By replacing in (41) and (42) the expression of the complex velocity on the boundary as function of the perturbation velocity components and using the denotation s for the component v of the complex velocity on the free surface (for avoiding any confusion), we can write for $k = \overline{0, M-1}$

$$\sum_{l=0}^{M} B'_{kl} u_l + \frac{u_k - is_k - u_{k+1} + is_{k+1}}{x_k - x_{k+1}} + ik_0 \left(u_k - is_k\right) = \frac{1}{2\pi i} \sum_{j=1}^{N} \left(-n_x^j - iv_j\right) \left(n_x^j + in_y^j\right) \widehat{A}_{kj}, \tag{44}$$

respectively, for $k = M$,

$$\sum_{l=0}^{M} B'_{kl} u_l + \frac{u_M - is_M - u_{M-1} + is_{M-1}}{x_M - x_{M-1}} + ik_0 \left(u_M - is_M\right) = \frac{1}{2\pi i} \sum_{j=1}^{N} \left(-n_x^j - iv_j\right) \left(n_x^j + in_y^j\right) \widehat{A}_{Mj} \tag{45}$$

In this way we have obtained the rest of the $M+1$ equations that ensures the mathematical coherence of our mathematical problem, i.e., the solving of the system for the components of the perturbation velocity on the free surface and on the border (boundary) of the obstacle.

Summarizing, the final system which should be solved is made by equations (38),(44) and (45).

3 Numerical results

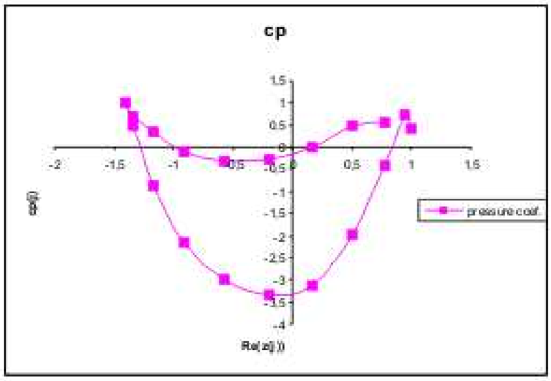
For the outward normal components at the obstacle boundary, at the control points, we have the expressions

$$n_x^j = Im\left(z_j - z_{j+1}\right) |z_j - z_{j+1}|; \quad n_y^j = \frac{-Re\left(z_j - z_{j+1}\right)}{|z_j - z_{j+1}|} \tag{46}$$

and consequently all the coefficients which are present in the system obtained can be expressed as functions of the discretization nodes coordinates, and their calculation can be performed by a computer.

The unknowns are the N components of the tangential perturbation velocity on the obstacle border, evaluated at N nodes of the involved discretization, and the $2(M+1)$ components of the perturbation velocity on the free surface at $M+1$ nodes chosen by its discretization.

Using a MATHCAD code we will find not only the perturbation velocity field but also the local pressure coefficient, i.e., $c_p^j = 1 - \left(v_j + n_y^j\right)^2$ (47). In the figure there are represented the numerical results found for the case of a circular obstacle when we use 20 nodes for the discretization of the obstacle's boundary. We took $[a, b] = [-4, 4]$, and $M = 19$. The distance from the free boundary is taken equal with 2, and for the parameter k_0 we chose the value $k_0 = 3, 5$.



When applying the CVBEM method a better approximation we can obtain by growing the number of nodes used for the discretization of the boundary. Doing so we can find exactly values for the unknown for a grater number of nodes and so we can find a better approach. We must also remember that the computer effort is much greater and not always the improvement is very evident. With this approach further it is possible to determine the shape of the unknown free surface using the velocity field.

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