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Quantitative estimates for positive linear operators in weighted spaces

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Abstract

We give some quantitative estimates for positive linear operators in weighted spaces by introducing a new modulus of continuity and then apply these results to the Bernstein-Chlodowsky polynomials.

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1 Introduction

Let $\mathbb{R}_+ = [0, \infty)$ and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an unbounded strictly increasing continuous function such there exist $M > 0$ and $\alpha \in (0, 1]$ with the property

$$|x - y| \le M|\varphi(x) - \varphi(y)|^{\alpha}, \text{ for every } x, y \ge 0. \tag{1}$$

Let $\rho(x) = 1 + \varphi^2(x)$ be a weight function and let $B_\rho(\mathbb{R}_+)$ be the space defined by

$$B_\rho(\mathbb{R}_+) = \left\{ f : \mathbb{R}_+ \to \mathbb{R} \mid \|f\|_\rho = \sup_{x\ge 0} \frac{f(x)}{\rho(x)} < +\infty \right\}.$$

We define also the spaces

$$C_\rho(\mathbb{R}_+) = \{\, f \in B_\rho(\mathbb{R}_+),\ f \text{ is continuous} \,\},$$
$$C^k_\rho(\mathbb{R}_+) = \left\{\, f \in C_\rho(\mathbb{R}_+),\ \lim_{x\to+\infty} \frac{f(x)}{\rho(x)} = K_f < +\infty \right\},$$
$$U_\rho(\mathbb{R}_+) = \{\, f \in C_\rho(\mathbb{R}_+), f/\rho \text{ is uniformly continuous} \,\}.$$

We have the inclusions $C^k_\rho(\mathbb{R}_+) \subset U_\rho(\mathbb{R}_+) \subset C_\rho(\mathbb{R}_+) \subset B_\rho(\mathbb{R}_+)$.

We consider $(A_n)_{n\geq 1}$ a sequence of positive linear operators acting from $C_\rho(\mathbb{R}_+)$ to $B_\rho(\mathbb{R}_+)$. In [1] is given the following

Theorem 1 *If $A_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)$ is a sequence of linear operators such that*

$$\lim_{n\to\infty} \left\| A_n\varphi^i - \varphi^i \right\|_\rho = 0, \quad i = 0, 1, 2, \tag{2}$$

then for any function $f \in C^k_\rho(\mathbb{R}_+)$ we have

$$\lim_{n\to\infty} \| A_n f - f \|_\rho = 0.$$

Remark 1 *The conditions (2) can be replaced with:*

$$\lim_{n\to\infty} \left\| A_n\rho^{i/2} - \rho^{i/2} \right\|_\rho = 0, \quad i = 0, 1, 2$$

and the theorem remains valid. (see [2])

Remark 2 *Taking $f^*(x) = \varphi^2(x)\cos \pi x$, we notice that $f^* \in U_\rho(\mathbb{R}_+)$. But it was proved in [1] that there is a sequence A_n of positive linear operators such that $\lim_{n\to\infty} \| A_n f^* - f^* \|_\rho \geq 1$. So, the space $C^k_\rho(\mathbb{R}_+)$, from Theorem 1 cannot be replaced by $U_\rho(\mathbb{R}_+)$.*

In [2] it was introduced a weighted modulus of continuity to estimate the rate of approximation in these spaces: for every $\delta \geq 0$ and for every $f \in C_\rho(\mathbb{R}_+)$

$$\Omega_\rho(f,\delta) = \sup_{\substack{x,y\in\mathbb{R}_+\\ |\rho(x)-\rho(y)|\leq\delta}} \frac{|f(x)-f(y)|}{[|\rho(x)-\rho(y)|+1]\rho(x)},$$

where ρ was defined as a continuously differentiable function on $\mathbb{R}_+$ with $\rho(0) = 1$ and $\inf_{x\geq 0} \rho'(x) \geq 1$. For this modulus, it was proved

Theorem 2 *Let A_n be a sequence of positive linear operators such that*

$$\left\|A_n\rho^0 - \rho^0\right\|_\rho = \alpha_n,$$
$$\left\|A_n\rho - \rho\right\|_\rho = \beta_n,$$
$$\left\|A_n\rho^2 - \rho^2\right\|_{\rho^2} = \gamma_n,$$

where α_n, β_n and γ_n tend to zero as n goes to the infinity. Then

$$\left\|A_n f - f\right\|_{\rho^4} \leq 16 \cdot \Omega_\rho\left(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right) + \left\|f\right\|_\rho \alpha_n$$

for all $f \in C_\rho^k(\mathbb{R}_+)$ and n large enough.

We want to improve this result and give an application.

2 A new weighted modulus of continuity

For each $f \in C_\rho(\mathbb{R}_+)$ and for every $\delta \geq 0$ we introduce

$$\omega_\varphi(f,\delta) = \sup_{\substack{x,y\geq 0\\ |\varphi(x)-\varphi(y)|\leq\delta}} \frac{|f(x)-f(y)|}{\rho(x)+\rho(y)}.$$

Remark 3 *Because of the symmetry we have*

$$\omega_\varphi(f,\delta) = \sup_{\substack{y\geq x\geq 0 \\ |\varphi(y)-\varphi(x)|\leq\delta}} \frac{|f(x)-f(y)|}{\rho(x)+\rho(y)}.$$

We observe that $\omega_\varphi(f,0)=0$ *and* $\omega_\varphi(f,\cdot)$ *is a nonnegative, increasing function for all* $f\in C_\rho(\mathbb{R}_+)$*. Moreover,* $\omega_\varphi(f,\cdot)$ *is bounded, which follows from the inequality* $|f(y)-f(x)|\leq \|f\|_\rho\,(\rho(y)+\rho(x))$*.*

Remark 4 *If* $\varphi(x)=x$*, then* ω_φ *is equivalent with* Ω *defined by*

$$\Omega(f,\delta) = \sup_{x\geq 0,\ |h|\leq\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)},$$

in the sense that

$$\omega_\varphi(f,\delta)\leq\Omega(f,\delta)\leq 3\cdot\omega_\varphi(f,\delta),$$

the first inequality being true for $\delta\leq\frac{1}{\sqrt{2}}$ *and the second for all* $\delta\geq 0$*. Indeed,* $\omega_\varphi(f,\delta)\leq\Omega(f,\delta)$ *is equivalent with the inequality*

$$1+x^2+h^2+x^2h^2\leq 1+x^2+1+x^2+2xh+h^2,\ \forall\, x\geq 0$$

or $x^2(1-h^2)+2xh+1\geq 0, \forall\, x\geq 0$*, which is true if* $2h^2-1\leq 0$*.*
The inequality $\Omega(f,\delta)\leq 3\cdot\omega_\varphi(f,\delta)$ *is equivalent with*

$$1+x^2+1+x^2+2xh+h^2\leq 3(1+x^2+h^2+x^2h^2),\ \forall\, x\geq 0$$

or $x^2+2h^2+2x^2h^2+(xh-1)^2\geq 0$*, which is true for all* $h,x\in\mathbb{R}$*.*

Lemma 1 $\lim\limits_{\delta\searrow 0}\omega_\varphi(f,\delta)=0$*, for every* $f\in U_\rho(\mathbb{R}_+)$*.*

Proof. Let $y \geq x \geq 0$ such that $0 \leq \varphi(y) - \varphi(x) \leq \delta$. Then

$$\begin{aligned}\frac{|f(x)-f(y)|}{\rho(x)+\rho(y)} &\leq \left|\frac{f(x)}{\rho(x)} - \frac{f(y)}{\rho(y)}\right| \cdot \frac{\rho(x)}{\rho(x)+\rho(y)} + \frac{|f(y)|}{\rho(y)} \cdot \frac{|\rho(x)-\rho(y)|}{\rho(x)+\rho(y)} \\ &\leq \omega\left(\frac{f}{\rho}, |x-y|\right) \cdot \frac{1}{2} + \|f\|_\rho \cdot \frac{|\varphi(x)-\varphi(y)| \cdot [\varphi(x)+\varphi(y)]}{2+\varphi^2(x)+\varphi^2(y)} \\ &\leq \frac{1}{2} \cdot \omega\left(\frac{f}{\rho}, M|\varphi(x)-\varphi(y)|^\alpha\right) + \|f\|_\rho \cdot \frac{|\varphi(x)-\varphi(y)|}{2} \\ &\leq \frac{M+1}{2} \cdot \omega\left(\frac{f}{\rho}, \delta^\alpha\right) + \|f\|_\rho \cdot \frac{\delta}{2},\end{aligned}$$

where $\omega(f, \delta)$ is the usual modulus of continuity. We obtain

$$\omega_\varphi(f, \delta) \leq \frac{M+1}{2} \cdot \omega\left(\frac{f}{\rho}, \delta^\alpha\right) + \|f\|_\rho \cdot \frac{\delta}{2}$$

The right-hand side tend to zero when δ tend to zero, because f/ρ is uniformly continuous, so the lemma is proved.

Lemma 2 *For every $\delta \geq 0$ and $\lambda \geq 0$ we have*

$$\omega_\varphi(f, \lambda\delta) \leq (2+\lambda) \cdot \omega_\varphi(f, \delta).$$

Proof. We prove that $\omega_\varphi(f, m\delta) \leq (m+1) \cdot \omega_\varphi(f, \delta)$, for every nonnegative integer m. The property for $\lambda \in \mathbb{R}_+$ can be easily obtained by using the inequalities $[\lambda] \leq \lambda \leq [\lambda]+1$, where $[\lambda]$ denotes the greatest integer less or equal to λ.

For $m = 0$ and $m = 1$ the inequality is obvious. For $m \geq 2$, let $y > x \geq 0$ such that $\varphi(y) - \varphi(x) \leq m\delta$. We construct, inductively, the sequence of points $x = x_0 < x_1 < \cdots < x_m = y$ such that for each $k \in \{1, \ldots, m\}$,

$$\varphi(x_k) - \varphi(x_{k-1}) = c = \frac{\varphi(y)-\varphi(x)}{m} \leq \delta.$$

For simplifying the computations we set $a_k = \varphi(x_k) \geq 0$. We have

$$\sum_{k=1}^{m} a_k^2 + a_{k-1}^2 - 2\sum_{k=1}^{m} a_k a_{k-1} = \sum_{k=1}^{m} (a_k - a_{k-1})^2 = mc^2,$$

and

$$\sum_{k=1}^{m} a_k^2 + a_k a_{k-1} + a_{k-1}^2 = \frac{1}{c}\sum_{k=1}^{m} a_k^3 - a_{k-1}^3 = \frac{a_m^3 - a_0^3}{c} = m(a_m^2 + a_m a_0 + a_0^2).$$

We deduce

$$\begin{aligned} 3\sum_{k=1}^{m} a_k^2 + a_{k-1}^2 &= 2\sum_{k=1}^{m} a_k^2 + a_k a_{k-1} + a_{k-1}^2 + \sum_{k=1}^{m} a_k^2 - 2a_k a_{k-1} + a_{k-1}^2 \\ &= 2m(a_m^2 + a_m a_0 + a_0^2) + mc^2 \\ &\leq 3m(a_m^2 + a_0^2) + (a_m - a_0)^2 \\ &\leq 3(m+1)(a_m^2 + a_0^2). \end{aligned}$$

Using this, we have

$$\begin{aligned} \frac{|f(y) - f(x)|}{\rho(y) + \rho(x)} &\leq \sum_{k=1}^{m} \frac{|f(x_k) - f(x_{k-1})|}{\rho(x_k) + \rho(x_{k-1})} \cdot \frac{2 + \varphi^2(x_k) + \varphi^2(x_{k-1})}{\rho(y) + \rho(x)} \\ &\leq \omega_\varphi(f,\delta) \sum_{k=1}^{m} \frac{2 + a_k^2 + a_{k-1}^2}{2 + a_m^2 + a_0^2} \\ &\leq (m+1)\omega_\varphi(f,\delta). \end{aligned}$$

The supremum being the least upper bound, we obtain

$$\omega_\varphi(f, m\delta) \leq (m+1)\omega_\varphi(f,\delta).$$

Lemma 3 *For every $f \in C_\rho(\mathbb{R}_+)$, for $\delta > 0$ and for all $x, y \geq 0$*

$$|f(y) - f(x)| \leq (\rho(y) + \rho(x)) \left(2 + \frac{|\varphi(y) - \varphi(x)|}{\delta}\right) \omega_\varphi(f,\delta).$$

Proof. From the definition of the modulus we deduce

$$|f(y) - f(x)| \leq [\rho(y) + \rho(x)] \cdot \omega_\varphi(f, |\varphi(y) - \varphi(x)|).$$

If $|\varphi(y) - \varphi(x)| \leq \delta$ then by the monotony of the modulus we have

$$\omega_\varphi(f, |\varphi(y) - \varphi(x)|) \leq \omega_\varphi(f, \delta).$$

If $|\varphi(y) - \varphi(x)| \geq \delta$ then by the previous lemma

$$\begin{aligned}\omega_\varphi(f, |\varphi(y) - \varphi(x)|) &= \omega_\varphi\left(f, \frac{|\varphi(y) - \varphi(x)|}{\delta} \cdot \delta\right) \\ &\leq \left(2 + \frac{|\varphi(y) - \varphi(x)|}{\delta}\right) \omega_\varphi(f, \delta).\end{aligned}$$

Theorem 3 *Let $A_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)$ be a sequence of positive linear operators with*

$$\begin{aligned}\left\|A_n\varphi^0 - \varphi^0\right\|_{\rho^0} &= a_n, \\ \left\|A_n\varphi - \varphi\right\|_{\rho^{\frac{1}{2}}} &= b_n, \\ \left\|A_n\varphi^2 - \varphi^2\right\|_{\rho} &= c_n, \\ \left\|A_n\varphi^3 - \varphi^3\right\|_{\rho^{\frac{3}{2}}} &= d_n,\end{aligned}$$

where a_n, b_n, c_n and d_n tend to zero as n goes to the infinity. Then

$$\|A_n f - f\|_{\rho^{\frac{3}{2}}} \leq (7 + 4a_n + 2c_n) \cdot \omega_\varphi(f, \delta_n) + \|f\|_\rho a_n$$

for all $f \in C_\rho(\mathbb{R}_+)$, where

$$\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n.$$

Proof. By the previous lemma and by the fact that

$$[\rho(x) + \rho(y)]|\varphi(y) - \varphi(x)| \leq \left[2\rho(x) + |\varphi^2(y) - \varphi^2(x)|\right] |\varphi(y) - \varphi(x)|$$

we obtain

$$
\begin{aligned}
&|A_n f(x) - f(x)| \le |f(x)| \cdot |A_n \varphi^0(x) - \varphi^0(x)| + A_n(|f(y) - f(x)|, x) \\
&\le \|f\|_\rho \, a_n \rho(x) + \omega_\varphi(f, \delta_n) \cdot \Bigg[2A_n\rho(x) + 2\rho(x) A_n \varphi^0(x) \\
&\quad + \frac{2\rho(x) A_n(|\varphi(y) - \varphi(x)|, x) + A_n([\varphi(y) + \varphi(x)][\varphi(y) - \varphi(x)]^2, x)}{\delta_n} \Bigg]
\end{aligned} \tag{3}
$$

Applying Cauchy-Schwarz inequality we have

$$
A_n(|\varphi(y) - \varphi(x)|, x) \le \left(A_n([\varphi(y) - \varphi(x)]^2, x)\right)^{\frac{1}{2}} \cdot \left(A_n \varphi^0(x)\right)^{\frac{1}{2}}
$$

and using

$$
\begin{aligned}
&A_n([\varphi(y) - \varphi(x)]^2, x) \\
&= A_n\varphi^2(x) - \varphi^2(x) - 2\varphi(x)[A_n\varphi(x) - \varphi(x)] + \varphi^2(x)[A_n\varphi^0(x) - \varphi^0(x)] \\
&\le \rho(x) c_n + 2\rho^{\frac{1}{2}}(x)\varphi(x) b_n + a_n \varphi^2(x),
\end{aligned}
$$

we obtain

$$
A_n(|\varphi(y) - \varphi(x)|, x) \le \rho^{\frac{1}{2}}(x) \cdot \sqrt{(a_n + 2b_n + c_n)(1 + a_n)}.
$$

Because

$$
\begin{aligned}
&A_n(\varphi(y)[\varphi(y) - \varphi(x)]^2, x) \\
&= A_n\varphi^3(x) - \varphi^3(x) - 2\varphi(x)[A_n\varphi^2(x) - \varphi^2(x)] + \varphi^2(x)[A_n\varphi(x) - \varphi(x)] \\
&\le \rho^{\frac{3}{2}}(x) d_n + 2\rho(x)\varphi(x) c_n + b_n \varphi^2(x)\rho^{\frac{1}{2}}(x),
\end{aligned}
$$

we obtain

$$
A_n([\varphi(y) + \varphi(x)][\varphi(y) - \varphi(x)]^2, x) \le \rho^{\frac{3}{2}}(x) \cdot (d_n + 2c_n + b_n + a_n + 2b_n + c_n) \,.
$$

Choosing $\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n$

$$|A_n f(x) - f(x)| \leq \left[2\rho(x)(2a_n + c_n + 3) + \rho^{\frac{3}{2}}(x)\right] \omega_\varphi(f, \delta_n) + \|f\|_\rho a_n \rho(x).$$

So,

$$\|A_n f - f\|_{\rho^{\frac{3}{2}}} \leq (7 + 4a_n + 2c_n) \cdot \omega_\varphi(f, \delta_n) + \|f\|_\rho a_n.$$

Remark 5 *In the conditions of the Theorem 3, using Lemma 1 we have*

$$\lim_{n\to\infty} \|A_n f - f\|_{\rho^{\frac{3}{2}}} = 0,$$

for all $f \in U^k_{\rho^{\frac{3}{2}}}(\mathbb{R}_+)$.

Corollary 1 *Let* $A_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)$ *be a sequence of positive linear operators with*

$$\left\|A_n\varphi^0 - \varphi^0\right\|_{\rho^0} = a_n,$$
$$\|A_n\varphi - \varphi\|_{\rho^{\frac{1}{2}}} = b_n,$$
$$\left\|A_n\varphi^2 - \varphi^2\right\|_{\rho} = c_n,$$
$$\left\|A_n\varphi^3 - \varphi^3\right\|_{\rho^{\frac{3}{2}}} = d_n,$$

where a_n, b_n, c_n *and* d_n *tend to zero as* n *goes to the infinity. Let* η_n *be a sequence of real numbers such that*

$$\lim_{n\to\infty} \eta_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \rho^{\frac{1}{2}}(\eta_n)\delta_n = 0,$$

where $\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n$. *Then for every* $f \in C_\rho(\mathbb{R}_+)$

$$\sup_{0\leq x\leq\eta_n} \frac{|A_n f(x) - f(x)|}{\rho(x)} \leq (7 + 4a_n + 2c_n) \cdot \omega_\varphi\left(f, \rho^{\frac{1}{2}}(\eta_n)\delta_n\right) + \|f\|_\rho a_n.$$

Proof. Replacing δ_n from (3) with $\rho^{\frac{1}{2}}(\eta_n)\delta_n$ we obtain the result.

3 Application

We want to apply the result obtained in the Corollary 1 for the weight $\rho(x) = 1 + x^2$ and the Bernstein-Chlodowsky operators defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n} b_n\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k},$$

for $0 \le x \le b_n$ and $B_n f(x) = f(x)$, for $x > b_n$, where b_n is a sequence of positive numbers such that

$$\lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

The condition (1) over $\varphi(x) = x$ is verified for $\alpha = 1$ and $M = 1$.

Theorem 4 *If $B_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)$ is the sequence of Bernstein-Chlodowsky operators, then for all $f \in C_\rho(\mathbb{R}_+)$*

$$(4) \quad \|B_n f - f\|_\rho \le \left(7 + \frac{b_n}{n}\right) \cdot \omega_\varphi \left(f, \sqrt{1 + b_n^2} \left(\sqrt{\frac{2b_n}{n}} + 3\frac{b_n}{n} + \frac{b_n^2}{2n^2}\right)\right).$$

Proof. We have

$$B_n e_0(x) = 1,$$

$$B_n e_1(x) = x,$$

$$B_n e_2(x) = x^2 + \frac{x(b_n - x)}{n},$$

$$B_n e_3(x) = x^3 + \frac{x(b_n - x)[(3n-2)x + b_n]}{n^2}$$

We obtain

$$
\begin{aligned}
a_n &= \|B_n e_0 - e_0\|_{\rho^0} = 0, \\
b_n &= \|B_n e_1 - e_1\|_{\rho^{\frac{1}{2}}} = 0, \\
c_n &= \|B_n e_2 - e_2\|_{\rho} = \sup_{0\le x\le b_n} \frac{x(b_n - x)}{n(1+x^2)} = \frac{b_n^2}{2n\left(\sqrt{1+b_n^2}+1\right)} \le \frac{b_n}{2n}, \\
d_n &= \|B_n e_3 - e_3\|_{\rho^{\frac{3}{2}}} \le \sup_{0\le x\le b_n} \frac{x(b_n - x)}{n(1+x^2)} \cdot \sup_{0\le x\le b_n} \frac{(3n-2)x + b_n}{n\sqrt{1+x^2}} \\
&\le \frac{b_n}{2n}\left(3 + \frac{b_n}{n}\right).
\end{aligned}
$$

Setting $\eta_n = b_n$ in the Corollary 1, and considering

$$2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n \le \sqrt{\frac{2b_n}{n}} + 3\frac{b_n}{n} + \frac{b_n^2}{2n^2},$$

we obtain the estimation from the theorem.

Remark 6 *In order to obtain*

$$\lim_{n\to\infty} \|B_n f - f\|_{\rho} = 0,$$

in the relation (4) from Theorem 4, we must have $f \in U_\rho(\mathbb{R}_+)$ *and*

$$\lim_{n\to\infty} \frac{b_n^3}{n} = 0.$$

References

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