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Data dependence for some integral equation via weakly Picard operators

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Abstract

In this paper we study data dependence for the following integral equation:

$$u(x) = h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s_1, \cdots, \theta_m s_m))ds,$$

$$x \in \prod_{i=1}^{m} [0, b_i], \theta_i \in (0, 1), (\forall) i = \overline{1, m}$$

by using c-WPOs technique.

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1 Introduction

Let (X, d) be a metric space and $A : X \to X$ an operator.We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ the fixed points set of A.

$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ the family of the nonempty invariant subsets of A.

$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in N$.

Definition 1.*[1] An operator A is weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n \in N}$$

converges , for all $x \in X$ and the limit (which depend on x) is a fixed point of A.

Definition 2.*[1] If the operator A is WPO and $F_A = \{x^*\}$ then by definition A is Picard operator.*

Definition 3.*[1] If A is WPO, then we consider the operator*

$$A^\infty : X \to X, A^\infty(x) = \lim_{n \to \infty} A^n(x).$$

We remark that $A^\infty(X) = F_A$.

Definition 4.*[1] Let be A an WPO and $c > 0$.The operator A is c-WPO if*

$$d(x, A^\infty(x)) \leq c \cdot d(x, A(x)).$$

We have the following characterization of the WPOs:

Theorem 1.*[1]Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that

(a) $X_\lambda \in I(A)$

(b) $A \mid X_\lambda : X_\lambda \to X_\lambda$ *is a Picard (c-Picard) operator, for all* $\lambda \in \Lambda$.

For the class of c-WPOs we have the following data dependence result:

Theorem 2. *[1] Let* (X, d) *be a metric space and* $A_i : X \to X, i = \overline{1,2}$ *an operator. We suppose that:*

(i) *the operator* A_i *is* $c_i - WPO$, $i = \overline{1,2}$.

(ii) *there exists* $\eta > o$ *such that*

$$d(A_1(x), A_2(x)) \leq \eta, (\forall) x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \; max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional.

We have:

Lemma 1. *[1],[3] Let* $(X, d, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator such that:*

a)A is monotone increasing.

b)A is WPO.

Then the operator A^∞ *is monotone increasing.*

Lemma 2. *[1],[3] Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ such that :*

(i)$A \leq B \leq C$.

(ii) the operators A,B,C are W.P.O s.

(iii) the operator B is monotone increasing.

Then

$$x \leq y \leq z \Longrightarrow A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z).$$

2 Main results

Data dependence for functional integral equations was studied [1], [2], [3]. In what follow we consider the integral equation

$$u(x) = h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s, \cdots, \theta_m s))ds, \tag{1}$$

where

$$x, s \in D = \prod_{i=1}^{m} [0, b_i], \theta_i \in (0, 1), (\forall) i = \overline{1, m}.$$

Let $(X, \|\cdot\|, \leq)$ be an ordered Banach space.

Theorem 3. *We suppose that:*

(i)$h \in C(D \times X)$ and $K \in C(D \times D \times X)$.

(ii)$h(0, \alpha) = \alpha, (\forall)\alpha \in X$.

(iii)there exists $L_K > 0$ such that

$$\|K(x, s, u_1) - K(x, s, u_2)\| \leq L_K \|u_1 - u_2\|,$$

for all $x, s \in D$ *and* $u_1, u_2 \in X$.

In these conditions the equation(1) has in C(D,X) an infinity of solutions.

Moreover if

(iv) $h(x,\cdot)$ *and* $K(x,s,\cdot)$ *are monotone increasing for all* $x, s \in D$

then if u *and* v *are solutions of the equation (1) such that* $u(0) \leq v(0)$ *we have* $u \leq v$.

Proof. Consider the operator

$$A : (C(D,X), \|\cdot\|_\tau) \to (C(D,X), \|\cdot\|_\tau),$$

$$A(u)(x) := h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s, \cdots, \theta_m s))ds.$$

Here $\|u\|_\tau = \max_{x \in D} |u(x)| e^{-\tau \sum_{i=1}^{m} x_i}$.

Let $\lambda \in X$ and $X_\lambda = \{u \in C(D,X) \mid u(0) = \lambda\}$.Then

$$C(D,X) = \bigcup_{\lambda \in X} X_\lambda.$$

is a partition of $C(D,X)$ and $X_\lambda \in I(A)$, for all $\lambda \in X$.

For all $u, v \in X_\lambda$, we have have

$$\|A(u)(x) - A(v)(x)\| \leq \frac{L_K}{\tau^m \theta_1 \cdots \theta_m} e^{\tau \sum_{i=1}^{m} x_i}.$$

So the restriction of the operator A on X_λ is a c-Picard operator with $c = (1 - \frac{L_K}{\tau^m \theta_1 \cdots \theta_m})^{-1}$,for a suitable choices of τ such that $\frac{L_K}{\tau^m \theta_1 \cdots \theta_m} < 1$.
If $u \in X$ then we denote by $\widetilde{u}$ the constant operator

$$\widetilde{u} : C(D,X) \to C(D,X)$$

defined by

$$\widetilde{u}(t) = u.$$

If $u, v \in C(D, X)$ is the solutions of (1) with $u(0) \le v(0)$ then $\widetilde{u(0)} \in X_{u(0)}, \widetilde{v(0)} \in X_{v(0)}$.

By lema 1 we have that

$$\widetilde{u(0)} \le \widetilde{v(0)} \Longrightarrow A^{\infty}(\widetilde{u(0)}) \le A^{\infty}(\widetilde{v(0)}).$$

But

$$u = A^{\infty}(\widetilde{u(0)}), v = A^{\infty}(\widetilde{v(0)}).$$

So, $u \le v$.

Theorem 4. *Let $h_i \in C(D \times X)$ and $K_i \in C(D \times D \times X)$, $\imath = \overline{1,3}$ satisfy the conditions (i)(ii)(iii) from the Theorem 3. We suppose that*

(a) $h_2(x, \cdot)$ and $K_2(x, s, \cdot)$ are monotone increasing , for all $x, s \in D$.

(b) $h_1 \le h_2 \le h_3$ and $K_1 \le K_2 \le K_3$.

Let u_i be a solution of the equation (1) corresponding to h_i and K_i.

Then

$$u_1(0) \le u_2(0) \le u_3(0) \quad \textit{imply} \quad u_1 \le u_2 \le u_3.$$

Proof. The proof follows from Lemma 2.

For studding of data dependence we consider the following equations:

$$u(x) = h_1(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K_1(x, s, u(\theta_1 s_1, \cdots, \theta_m s_m))ds \tag{2}$$

$$(3) \qquad u(x) = h_2(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K_2(x, s, u(\theta_1 s_1, \cdots, \theta_m s_m))ds$$

Theorem 5. *We consider (2), (3) under the followings conditions:*

(i) $h_i \in C(D \times X)$ *and* $K_i \in C(D \times D \times X)$, $i = \overline{1,2}$;

(ii) $h_i(0, \alpha) = \alpha, (\forall)\alpha \in X$, $i = \overline{1,2}$;

(iii) there exists $L_{K_i} > 0$, $i = \overline{1,2}$ *such that*

$$|K_i(x, s, u_1) - K_i(x, s, u_2)| \leq L_{K_i}|u_1 - u_2|,$$

for all $x, s \in D$ *and* $u_1, u_2 \in X$;

(iv) there exists $\eta_1, \eta_2 > 0$ *such that*

$$|h_1(x, u) - h_2(x, u)| \leq \eta_1$$

$$|K_1(x, s, u) - K_2(x, s, u)| \leq \eta_2,$$

forall $x, s \in D, u \in X$.

If S_1, S_2 *are the solutions sets of the equations (2),(3), then we have:*

$$H(S_1, S_2) \leq (\eta_1 + \eta_2 \prod_{i=1}^{m} b_i) \max_{i=\overline{1,2}} \left\{ \frac{1}{1 - \dfrac{L_{K_i}}{\tau^m \theta_1 \cdots \theta_m}} \right\},$$

for $\tau > \max\limits_{i=\overline{1,2}} \sqrt[m]{\frac{L_{K_i}}{\theta_1 \cdots \theta_m}}$

Proof. We consider the following operators:

$$A_i : (C(D,X), \|\cdot\|_\tau) \to (C(D,X), \|\cdot\|_\tau)),$$

$$A_i u(x) := h_i(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K_i(x, s, u(\theta_1 s, \cdots, \theta_m s))ds, \ i = \overline{1,2}$$

From:

$$\|A_1(u)(x) - A_2(u)(x)\| \leq \|h_1(x, u(0)) - h_2(x, u(0))\| +$$

$$\int_0^{x_1} \cdots \int_0^{x_m} \|K_1(x, s, u(\theta_1 s, \cdots, \theta_m s)) - K_2(x, s, u(\theta_1 s, \cdots \theta_m s))\| ds \leq$$

$$\leq \eta_1 + \eta_2 \prod_{i=1}^{m} b_i.$$

we have that $\|A(u) - A(v)\|_\tau \leq \eta_1 + \eta_2 \prod_{i=1}^{m} b_i$

From this, using by Theorem 2 we have conclusion.

References

[1] I. A. Rus, *Weakly Picard operators and applications*, Seminar on fixed point theory Cluj-Napoca, vol. 2, 2001,41-57.

[2] I.A.Rus, *Generalized contractions*, Seminar on fixed point theory,No 3,1983,1-130.

[3] I. A. Rus, *Functional-Differential equation of mixed point, via weakly Picard operator*, Seminar on mixed point theory Cluj-Napoca, vol. 3, 2002,335-345.

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