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A lower bound for the second moment of Schoenberg operator

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Abstract

In this paper we represent a new lower bound for the second moment for Schoenberg variation-diminishing spline operator. We apply this estimate for $f \in C^2[0,1]$ and generalize the results obtained earlier by Gonska, Pitul and Rasa.

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1 Main result

We start with the definition of variation-diminishing operator, introduced by I.Schoenberg. For the case of equidistant knots we denote it by $S_{n,k}$. Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}$, $n \geq 1$, $k \geq 1$ with equidistant "interior knots", namely

$$\Delta_n : x_{-k} = \cdots = x_0 = 0 < x_1 < x_2 < \cdots < x_n = \cdots = x_{n+k} = 1$$

and $x_i = \dfrac{i}{n}$ for $0 \le i \le n$. For a bounded real-valued function f defined over the interval $[0,1]$ the variation-diminishing spline operator of degree k w.r.t. Δ_n is given by

$$S_{n,k}(f,x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) \dot{N}_{j,k}(x) \tag{1}$$

for $0 \le x < 1$ and

$$S_{n,k}(f,1) = \lim_{y \to 1,\, y<1} S_{n,k}(f,y)$$

with the nodes (Greville abscissas)

$$\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k},\ -k \le j \le n-1, \tag{2}$$

and the normalized $B-$ splines as fundamental functions

$$N_{j,k}(x) := (x_{j+k+1} - x_j)\dot{[}x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

The first quantitative variant of Voronovskaja's Theorem for a broad class of linear positive operators L was obtained very recently by H.Gonska, P.Pitul and I.Rasa in [3](see the proof of Theorem 6.2). We cite this result in the following

Theorem A. *Let $L : C[0,1] \to C[0,1]$ be a positive, linear operator reproducing linear functions. If $f \in C^2[0,1]$ and $x \in [0,1]$ then*

$$\left| L(f;x) - f(x) - \frac{1}{2} \cdot f''(x) \cdot L((e_1 - x)^2; x) \right| \tag{3}$$

$$\le \frac{1}{2} \cdot L((e_1 - x)^2; x) \cdot \tilde{\omega} \left(f'', \frac{1}{3} \cdot \sqrt{\frac{L((e_1 - x)^4; x)}{L((e_1 - x)^2; x)}} \right).$$

Here $e_n : x \in [0,1] \to x^n$, $n = 0, 1, \ldots$ are the monomial functions and $\tilde{\omega}(f, \cdot)$ denotes the least concave majorant of $\omega(f, \cdot)$ given by

$$\tilde{\omega}(f, \varepsilon) = \sup_{0 \le x \le \varepsilon \le y \le 1,\, x \ne y} \frac{(\varepsilon - x)\omega(f, y) + (y - \varepsilon)\omega(f, x)}{y - x},$$

for $0 \le \varepsilon \le 1$.

Here we point out that to estimate the argument of $\tilde{\omega}(f'', \cdot)$ we need a "good" upper bound for $L((e_1 - x)^4; x)$ and a "good" lower bound for the second moment $L((e_1 - x)^2; x)$. If f is convex function it is known that

$$S_{n,k}(f, x) \le B_k(f, x),$$

where $B_k(f, x)$ is the Bernstein operator of degree k. Therefore from the well-known representation of the fourth moment of $B_k(f, x)$ we have

$$(4) \quad S_{n,k}((e_1 - x)^4, x) \le B_k((e_1 - x)^4, x) = \frac{x(1-x)}{k^2} \cdot \left[3(1 - \frac{2}{k}) x(1-x) + \frac{1}{k} \right] \cdot$$

Next we are going to prove a lower bound for the second moment of $S_{n,k}$. We can not use the estimate established in Theorem 12 in [2] because it is valid only when $2 \le k \le n-1$. Here we point out that the new lower bound for the second moment of $S_{n,k}$ is valid for all $k \ge 2, n \ge 2$. In Theorem 3 in [1] it was proved that

$$(5) \qquad S_{n,k}((e_1 - x)^2, x) = S_{n,k}(g_2, x),$$

where the function g_2 is given by

$$g_2(y) = \begin{cases} \frac{1}{k-1} \cdot (-y^2 + \frac{y}{3}\sqrt{\frac{8k}{n} \cdot y + \frac{1}{n^2}}), & 0 \le y \le \min\left\{\frac{k+1}{2n}, \frac{n-1}{2k}\right\}, \\ \frac{1}{k-1} \cdot (y - y^2 - \frac{n^2-1}{6nk}), & \frac{n-1}{2k} \le y \le \frac{1}{2}, \\ \frac{1}{k-1} \cdot \frac{(k+1)(k-1)}{12n^2}, & \frac{k+1}{2n} \le y \le \frac{1}{2}, \\ g_2(1-y), & \frac{1}{2} \le y \le 1. \end{cases}$$

The function $g_2(y)$ is not concave and our goal is to bound it from below by an appropriate concave function $h_2(y)$. If this is possible it is easy to calculate

$$S_{n,k}(g_2, x) \ge S_{n,k}(h_2, x) \ge B_k(h_2, x)$$

and consequently

$$\frac{1}{S_{n,k}((e_1 - x)^2, x)} = \frac{1}{S_{n,k}(g_2, x)} \le \frac{1}{B_k(h_2, x)}. \tag{6}$$

To define the function h_2 we observe that

$$g_2'(0) = \frac{1}{3n(k-1)}$$

for all $n, k \ge 2$. If

$$h_2(y) = \frac{1}{3n(k-1)} y(1-y),\ y \in [0,1] \tag{7}$$

we verify that

$$g_2(y) \ge h_2(y).$$

Further we compute

$$\begin{aligned} B_k(h_2, x) &= \frac{1}{3n(k-1)} \cdot \left[x - (x^2 + \frac{x(1-x)}{k})\right] \\ &= \frac{x(1-x)(1-\frac{1}{k})}{3n(k-1)}. \end{aligned} \tag{8}$$

The last estimate is our lower bound for the second moment valid for all $n \geq 2, k \geq 2$. Thus we obtain

$$\frac{S_{n,k}((e_1-x)^4,x)}{S_{n,k}((e_1-x)^2,x)} \leq \frac{x(1-x)}{k^2} \cdot \frac{3n(k-1)}{x(1-x)(1-\frac{1}{k})} \cdot \left[3(1-\frac{2}{k})x(1-x)+\frac{1}{k}\right] =$$

$$3\frac{n}{k} \cdot \left[3(1-\frac{2}{k})x(1-x)+\frac{1}{k}\right] := \Delta_{n,k}(x). \tag{9}$$

When $\frac{k}{n} \to \infty$ (the polynomial case) then $\lim_{\frac{k}{n}\to\infty} \Delta_{n,k}(x) = 0$. We apply Theorem A to arrive at

Theorem 1 *For $f \in C^2[0,1]$ we have*

$$|S_{n,k}(f,x) - f(x) - \frac{1}{2}S_{n,k}((e_1-x)^2,x)f''(x)|$$

$$\leq \frac{1}{2}S_{n,k}((e_1-x)^2,x) \cdot \tilde{\omega}\left(f'', \frac{1}{3} \cdot \sqrt{\Delta_{n,k}(x)}\right), \tag{10}$$

where $\Delta_{n,k}(x)$ is defined in (8).

Corollary 1 *If we set $n = 1$ in* (9) *and* (10) *we get exactly the result of Gonska in* [4].

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References

[1] L.Beutel, H.H.Gonska, D.Kacso, G.Tachev, *On variation-diminishing Schoenberg operators: new quantitative statements*, Multivariate Approximation and Interpoltaion with Applications (ed. by M.Gasca), Monogr. Academia Ciencas de Zaragoza 20 (2002), 9-58.

[2] L.Beutel, H.H.Gonska, D.Kacso, G.Tachev, *On the Second Moments of Variation-Diminishing Splines*, J. of Concr. and Appl. Math., vol.2,1 (2004), 91-117.

[3] H.Gonska,P.Pitul and I.Rasa, *On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators*, "Numerical Analysis and Approximation Theory" -Proceedings of the Int. Conf. 2006 in Cluj-Napoca, (2006), 55-80.

[4] H.Gonska, *On the degree of Approximation in Voronovskaja's theorem*, Studia Univ. "Babes-Bolyai", Mathematica, vol.LII,3(2007), 103-115.

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