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New results in discrete asymptotic analysis

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Abstract

We present the order of magnitude of the sequence $\left(\Omega_{n,r}^{[\alpha]}_{[\beta]}\right)_n$ of general term $\Omega_{n,r}^{[\alpha]}_{[\beta]} = \dfrac{\alpha(\alpha+r)(\alpha+2r)\ldots(\alpha+(n-1)r)}{\beta(\beta+r)(\beta+2r)\ldots(\beta+(n-1)r)}$, where $r > 1$ and $0 < \alpha < \beta \leq r$ are fixed.

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1 Introduction

The asymptotic analysis, usually considered in connection with the functions of real or complex variable, can be also applied to the study of the functions of natural variable n, for $n \to \infty$.

This study forms the discrete asymptotic analysis. The principal purposes of this analysis are to obtain, for a given sequence:

(α) the order of magnitude;

(β) the convergence of the given sequence or of a derived one;

(γ) the first iterated limit (respecting an auxiliary scale of sequences);

(γ') a two sided estimation for the convergence; it must permit to find again the limit of (γ);

(δ) (if possible) the asymptotic expansion (respecting the given scale of sequences).

Some examples are classic.

For (α). The harmonic sum $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n$ has the order of magnitude of $\ln n + \gamma$, namely

$$H_n = \ln n + \gamma + \varepsilon_n,$$

where $\gamma = \lim_{n\to\infty}(H_n - \ln n) = 0,577\ldots$ is the famous constant of *Euler* and $\varepsilon_n \to 0$.

The factorial's magnitude is described by the formula of *Stirling* $n! \approx n^n \mathrm{e}^{-n}\sqrt{2\pi n}$, having the precise signification that

$$\lim_{n\to\infty} \frac{n!}{n^n \mathrm{e}^{-n}\sqrt{2\pi n}} = 1.$$

The number $\pi(n)$ of primes not exceeding a given number n is $\pi(n) \approx n/(\ln n)$ (i.e. $\lim_{n\to\infty}(\pi(n))/(n/(\ln n)) = 1$). This formula has a rich history related to *Legendre*, *Gauss*, *Tchebycheff*, *Hadamard*, *De La Vallée*, *Poussin* and others.

For (β). The sequence $(e_n)_n$ of general term $e_n = (1 + 1/n)^n$ defines by its limit, the famous constant e of *Napier* and *Euler*.

For $(H_n)_n$ the limit is ∞, but $\gamma_n = H_n - \ln n$ is an convergent sequence related to H_n; its limit defines the constant of *Euler*.

If we consider $S_n = \log_2 3 + \log_3 4 + \ldots + \log_n(n+1)$ (a sum of *L. Panaitopol*), then the sequence $x_n = S_n - (n-1) - \ln(\ln n)$ is convergent to $x = \gamma + \lim\limits_{n\to\infty} \left(\sum_{k=2}^{n} \frac{1}{k \ln k} - \ln\ln n \right)$.

For (γ). Two examples of first iterated limits are the following

$$\lim_{n\to\infty} n \left(\mathrm{e} - \left(1 + \frac{1}{n}\right)^n \right) = \frac{\mathrm{e}}{2};$$

$$\lim_{n\to\infty} n \left(\gamma_n - \gamma\right) = \frac{1}{2}.$$

For (γ'). The corresponding two sided estimations are

$$\frac{\mathrm{e}}{2n+2} < \mathrm{e} - \left(1 + \frac{1}{n}\right)^n < \frac{\mathrm{e}}{2n+1},$$

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}.$$

For (δ). As examples of asymptotic developments (expansions) we can consider the corresponding for H_n and $n!$ and others.

Many examples are known.

In the following we will use some of standard notations.

• $a_n = O(b_n)$ if there are two constants $M > 0$ and $c > 0$ such that $|a_n| < M|b_n|$ for any $n \in \mathbb{N}$, $n > c$.

• $a_n = o(b_n)$ if $\lim\limits_{n\to\infty} a_n/b_n = 0$.

• $O(1)$ is a notation for a sequence which is bounded.

• $o(1)$ is a notation for a sequence which tends to zero, where $n \to \infty$.

• the sequences $(a_n)_n$ and $(b_n)_n$ are called asymptotic equivalent and we write $a_n \sim b_n$ if $\lim\limits_{n\to\infty} a_n/b_n = 1$.

2 The starting example

Let

(1) $$\Omega_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}$$

be. This expression has a certain importance, because it appears often in many concrete questions of analysis:

– It is related to the bigger binomial coefficient (the middle term) of $(1+1)^n$, namely $\binom{2n}{n} = 4^n \Omega_n$.

– It is related to the *Mac Laurin* expansions of $(1+x)^{1/2}$, $(1-x)^{1/2}$, $(1-x^2)^{1/2}$ and $\arcsin x$.

– It is related to the so called integrals of Wallis, $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x$.

– It is related to the formula of Wallis

(2) $$\lim_{n\to\infty} W_n = \frac{\pi}{2},$$

where

(3) $$W_n = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots \cdot 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1)(2n+1)},$$

because of the relation

(4) $$W_n = \frac{1}{\Omega_n^2} \frac{1}{2n+1}.$$

Just because of these, the expression Ω_n was intensively studied.

Firstly, from the inequality

(5) $$0 < \Omega_n < \frac{1}{\sqrt{2n+1}}$$

it results $\lim_{n\to\infty} \Omega_n = 0$.

From (2) and (4) we obtain $\lim_{n\to\infty} \Omega_n\sqrt{n} = 1/\sqrt{\pi}$, i.e.

$$\Omega_n = O\left(\frac{1}{\sqrt{\pi n}}\right). \tag{6}$$

A two sided estimation of Ω_n (more accurate than (5)), namely

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi n}} \tag{7}$$

is called in a famous book of *D.S.Mitrinović* and *P. M. Vasić* [5] "the inequality of *Wallis*".

It was refined by *D. N. Kazarinoff* in 1956 (see [1])

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \tag{8}$$

respectively *L. Panaitopol* in 1985 (see [6])

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{4n}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} \tag{9}$$

and later we have given its asymptotic expansion

$$\Omega_n = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}+\ldots}} \tag{10}$$

(see [7], [8]).

3 The expressions considered by us

Let $r \in \mathbb{N}^*$, $r > 1$ be. Also let α and β be the real numbers, such that $0 < \alpha < \beta \leq r$. Consider the sequence $\left(\Omega_{n,r}^{[\frac{\alpha}{\beta}]}\right)_{n\geq 1}$ with general term defined by the equality

$$\Omega_{n,r}^{[\frac{\alpha}{\beta}]} \stackrel{\text{def}}{=} \frac{\alpha(\alpha+r)(\alpha+2r)\cdot\ldots\cdot(\alpha+(n-1)r)}{\beta(\beta+r)(\beta+2r)\cdot\ldots\cdot(\beta+(n-1)r)}.$$

This generalizes Ω_n, which can be obtained for $r = 2$, $\alpha = 1$ and $\beta = 2$.

We are interested to obtain the order of magnitude of $\Omega_{n,r}^{[\frac{\alpha}{\beta}]}$.

4 The order of magnitude

We have

$$\Omega_{n,r}^{[\frac{\alpha}{\beta}]} = \frac{r^n \frac{\alpha}{r}\left(\frac{\alpha}{r}+1\right)\left(\frac{\alpha}{r}+2\right)\cdot\ldots\cdot\left(\frac{\alpha}{r}+(n-1)\right)}{r^n \frac{\beta}{r}\left(\frac{\beta}{r}+1\right)\left(\frac{\beta}{r}+2\right)\cdot\ldots\cdot\left(\frac{\beta}{r}+(n-1)\right)}. \tag{11}$$

From the formula $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we obtain by iteration

$$\Gamma(x+p+1) = x(x+1)(x+2)\cdot\ldots\cdot(x+p)\Gamma(x)$$

($p \in \mathbb{N}$, $x, x+p \notin \{-1,-2,-3,\ldots\}$) i.e.

$$x(x+1)(x+2)\cdot\ldots\cdot(x+p) = \frac{\Gamma(x+p+1)}{\Gamma(x)}. \tag{12}$$

Applying (12) two times in (11) we obtain

$$\Omega_{n,r}^{[\frac{\alpha}{\beta}]} = \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)} \cdot \frac{\Gamma\left(\frac{\alpha}{r}+n\right)}{\Gamma\left(\frac{\beta}{r}+n\right)}, \tag{13}$$

which is a first expression of $\Omega_{n,r}^{[\alpha \atop \beta]}$.

To obtain the order of magnitude of $\Omega_{n,r}^{[\alpha \atop \beta]}$, we will use the *Stirling* approximation of Γ, namely

$$\Gamma(x+1) \sim x^x \mathrm{e}^{-x}\sqrt{2\pi x} \qquad (x>0).$$

So we obtain

$$\Omega_{n,r}^{[\alpha \atop \beta]} \sim \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)} \frac{\left(n+\frac{\alpha}{r}-1\right)^{n+\frac{\alpha}{r}-1} \cdot \mathrm{e}^{-\left(n+\frac{\alpha}{r}-1\right)}\sqrt{2\pi\left(n+\frac{\alpha}{r}-1\right)}}{\left(n+\frac{\beta}{r}-1\right)^{n+\frac{\beta}{r}-1} \cdot \mathrm{e}^{-\left(n+\frac{\beta}{r}-1\right)}\sqrt{2\pi\left(n+\frac{\beta}{r}-1\right)}},$$

i.e.

$$\Omega_{n,r}^{[\alpha \atop \beta]} \sim \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)} \left(\frac{n+\frac{\alpha}{r}-1}{n+\frac{\beta}{r}-1}\right)^{n+\frac{\alpha}{r}-1} \cdot \frac{1}{\left(n+\frac{\beta}{r}-1\right)^{\frac{\beta-\alpha}{2}}} \cdot$$

$$\cdot \sqrt{\frac{n+\frac{\alpha}{r}-1}{n+\frac{\beta}{r}-1}} \cdot \frac{1}{\mathrm{e}^{-\frac{\beta-\alpha}{r}}}.$$

Because of the equality

$$\lim_{n\to\infty} \left(\frac{n+\frac{\alpha}{r}-1}{n+\frac{\beta}{r}-1}\right)^{n+\frac{\alpha}{r}-1} = \mathrm{e}^{\frac{\alpha-\beta}{r}},$$

we obtain

$$\lim_{n\to\infty} \frac{\Omega_{n,r}^{[\alpha \atop \beta]}}{\frac{1}{n^{\frac{\beta-\alpha}{r}}}} = \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)}. \tag{14}$$

This conducts us to find the magnitude of $\Omega_{n,r}^{[\alpha\atop\beta]}$, namely we have obtained the

Theorem 1 *We have*

$$\boxed{\Omega_{n,r}^{[\alpha\atop\beta]} \sim \frac{\Gamma\left(\dfrac{\beta}{r}\right)}{\Gamma\left(\dfrac{\alpha}{r}\right)} \cdot \frac{1}{n^{\frac{\beta-\alpha}{r}}}} \tag{15}$$

The proof has given before.

In the case of Ω_n ($r = 2$, $\alpha = 1$ and $\beta = 2$) the formula (13) gives

$$\Omega_n = \frac{\Gamma\left(n+\dfrac{1}{2}\right)}{\Gamma\left(\dfrac{1}{2}\right)\Gamma(n+1)}. \tag{16}$$

The first author remembers a view in a first time of the formula (16) starting directly from the definition of Ω_n, in a conversation with the regretted Professor *Alexandru Lupaş*. This conducted us to two joint papers [3], [4] and stimulated us to consider and study the general case of $\Omega_{n,r}^{[\alpha\atop\beta]}$.

The formula (15) becomes

$$\Omega_n \sim \frac{1}{\sqrt{\pi n}},$$

finding again (6).

The presence of $\sqrt{n}$ has now a natural explanation by our overview. The apparition of $\sqrt{\pi}$ is related only to the well-known relation between $\Gamma\,(1/2)$ and $\sqrt{\pi}$.

References

[1] D.Kazarinoff, *On Wallis's formula*, Edinb. Math.Not. **40**,19-21.

[2] I. B. Lazarević, A. Lupaş, *Functional Equations for Wallis and Gamma Functions*, Publ. Electr. Fac. Univ. u Beogradu, Série: Electr., Téléc., Aut., No. 461-497 (1974), 245-251.

[3] A. Lupaş, A. Vernescu, *Asupra unei inegalităţi*, G. M. Seria A, **17** (**96**) (1999), 201-210.

[4] A. Lupaş, A. Vernescu, *Asupra unei conjecturi*, G. M. Seria A, **19** (**97**) (2001), 212-217.

[5] D. S. Mitrinović, P. M. Vasić, *Analytic Inequalities*, Springer Verlag, Berlin-Heidelberg-New York, 1970.

[6] L. Panaitopol, *O rafinare a formulei lui Stirling*, G. M. **90** (1985), No. 9, 329-332.

[7] A. Vernescu, *Sur l'approximation et le developpement asymptotique de la suite de terme général* $\frac{(2n-1)!!}{(2n)!!}$, Proc.of the Ann. Meet. of the Rom. Soc. of Math. Sc., Tome 1, Bucharest, 1998.

[8] A. Vernescu, *The Natural Proof of the Inequalities of Wallis Type*, Libertas Mathematica, **24** (2004), 183-190.

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