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Some theorems on the explicit evaluations of singular moduli[1]

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Abstract

At scattered places in his notebooks, Ramanujan recorded some theorems for calculating singular moduli and also recorded several values of singular moduli. In this paper, we establish several general theorems for the explicit evaluations of singular moduli. We also obtain some values of class invariants and singular moduli.

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1 Introduction

In Chapter 16, of his second notebook [2], [10] Ramanujan has defined his theta-function as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

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Following Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty,$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,$$

where

$$(a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), \text{ for } n \geq 1, \mid x \mid < 1.$$

The complete elliptic integral of the first kind $K := K(\alpha)$ associated with the modulus α, $0<\alpha<1$, is defined by

$$K(\alpha) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\alpha\sin^2\phi}} = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$$

where the latter representation is achieved by expanding the integrand in a binomial series and integrating termwise. Singular modulus $\alpha := \alpha_n$ is that unique positive number between 0 and 1 satisfying

$$\sqrt{n} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha_n\right)}; \qquad n \in \mathbb{Q}^+. \tag{1.1}$$

At scattered places in his first notebook, Ramanujan recorded several values of singular moduli $\alpha_n := \alpha(e^{-\pi\sqrt{n}})$ in terms of units. On page 82 of his first notebook, he gave some theorems for calculating α_n when n is even. J. M. Borwein and P. B. Borwein [6] had calculated some of Ramanujan's values for α_n. Watson [12] used the formula found in Ramanujan's first notebook [10, vol.1, p. 320], to prove the value of k_{210}, where $\alpha_n = k_n^2$, which Ramanujan wrote in his second letter to Hardy [11, p. xxix].

Let

$$Z(r) := Z(r;x) = {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q_r(x) := exp\left(-\pi \csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}, 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}, 1; x)}\right)$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$.
Let n denote a fixed natural number, and assume that

$$n\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)}, \tag{1.2}$$

where $r = 2, 3, 4$, or 6. Then a modular equation of degree n in theory of elliptic functions of signature r is a relation between moduli α and β induced by (1.2). We often say that β is of degree n over α and $m(r) := \dfrac{Z(r:\alpha)}{Z(r:\beta)}$ is called multiplier. We also use the notations $Z_1 := Z_1(r) = Z(r:\alpha)$ and $Z_n := Z_n(r) = Z(r:\beta)$ to indicate that β has degree n over α. When the context is clear, we omit the argument r in q_r, $Z(r)$ and $m(r)$.
Ramanujan class invariants are defined as

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \quad = \quad \{4\alpha(1-\alpha)\}^{-1/24} \tag{1.3}$$

and

$$g_n := 2^{-1/4}q^{-1/24}\chi(-q) \quad = \quad \{4\alpha(1-\alpha)^{-2}\}^{-1/24}. \tag{1.4}$$

On pages 294–299, Ramanujan recorded table of values for 77 class invariants or monic irreducible polynomials. In [13], [14], Watson proved 24 of Ramanujan's class invariants from Ramanujan's paper [10]. Watson also wrote further four papers [15], [16], [17], [18] on the calculation of class invariants. In [4], B. C. Berndt, H. H. Chan and L. -C. Zhang has used class field theory, Galois theory and Kronecker's limit formula to justify Watson's assumptions and calculated some values of G_n. In [1], N. D. Baruah has established the value of G_{217}. In [9], Mahadeva Naika and K. Sushan Bairy have established several new explicit evaluations of class invariants and singular moduli. In [8], Mahadeva Naika has established several new explicit values for G_n.

In this paper, we establish several general formulas for explicit evaluations of singular moduli using Ramanujan's modular equations. We also obtain some values of class invariant and singular moduli.

2 Preliminary Results

In this section, we collect the identities which are useful in proving our main results.

Lemma 2.1. *If β is of degree 3 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{2.1}$$

For a proof of Lemma 2.1, see Entry 5 of Chapter 19 in [2, pp. 230–231].

Lemma 2.2. *If β is of degree 5 over α, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{2.2}$$

For a proof of Lemma 2.2, see Entry 13 of Chapter 19 in [2, pp. 280–282].

Lemma 2.3. *If β is of degree 7 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{2.3}$$

For a proof of Lemma 2.3 see Entry 19 of Chapter 19 in [2, pp. 314–315].

Lemma 2.4. *If β is of degree 11 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \tag{2.4}$$

For a proof of Lemma 2.4, see Entry 7 of Chapter 20 in [2, p. 363].

Lemma 2.5. *If β is of degree 23 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1. \tag{2.5}$$

For a proof of Lemma 2.5, see Entry 15 of Chapter 20 in [2, p. 411].

Lemma 2.6. *We have*

$$2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}. \tag{2.6}$$

Using (1.4), we obtain (2.6).

Lemma 2.7. *If $a^2 - qb^2 = d^2$, a perfect square, then*

$$\sqrt{a + b\sqrt{q}} = \sqrt{\frac{a+d}{2}} + (sgn\ b)\sqrt{\frac{a-d}{2}}. \tag{2.7}$$

The above lemma is due to Bruce Reznick. For a proof of Lemma 2.7 via Chebyshev polynomials, see [5, p. 150].

Lemma 2.8. *We have*

$$2\sqrt{2}\left[g_n^3 g_{9n}^3 + \frac{1}{g_n^3 g_{9n}^3}\right] = \frac{g_{9n}^6}{g_n^6} - \frac{g_n^6}{g_{9n}^6}, \tag{2.8}$$

$$2\sqrt{2}\left[G_n^3 G_{9n}^3 - \frac{1}{G_n^3 G_{9n}^3}\right] = \frac{G_{9n}^6}{G_n^6} + \frac{G_n^6}{G_{9n}^6}, \tag{2.9}$$

$$2\left[G_n^2 G_{25n}^2 - \frac{1}{G_n^2 G_{25n}^2}\right] = \frac{G_{25n}^3}{G_n^3} + \frac{G_n^3}{G_{25n}^3}. \tag{2.10}$$

Proofs can be found in [2, pp. 231, 282], [7, Th.(4.1), (4.2)].

Lemma 2.9. *We have*

$$G_n^2 G_{9n}^2 = \left(\frac{g_n^2 g_{9n}^2 + \sqrt{g_n^4 g_{9n}^4 + 2g_n^{-2} g_{9n}^{-2}}}{2}\right), \tag{2.11}$$

$$g_n^2 g_{9n}^2 = \left(\frac{G_n^2 G_{9n}^2 + \sqrt{G_n^4 G_{9n}^4 - 2G_n^{-2} G_{9n}^{-2}}}{2}\right), \tag{2.12}$$

$$g_{4n}^2 g_{36n}^2 = g_n^2 g_{9n}^2 \left(g_n^2 g_{9n}^2 + \sqrt{g_n^4 g_{9n}^4 + 2g_n^{-2} g_{9n}^{-2}}\right), \tag{2.13}$$

$$= G_n^2 G_{9n}^2 \left(G_n^2 G_{9n}^2 + \sqrt{G_n^4 G_{9n}^4 - 2G_n^{-2} G_{9n}^{-2}}\right). \tag{2.14}$$

Proofs can be found in [7].

3 Main Theorems

In this section, we obtain several general formulas connecting singular moduli and class invariants.

Theorem 3.1. *We have*

$$\alpha_{9n} = 2G_n^{12}\left[\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}+\frac{1}{2}}+\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}-\frac{1}{2}}\right]^4 \times\left[\sqrt{\frac{1+\sqrt{2}u^3}{4}}-\sqrt{\frac{1-\sqrt{2}u^3}{4}}\right]^8, \tag{3.1}$$

$$= \left(-g_n^{12}+\sqrt{g_n^{24}+1}\right)^{-2}\left[\sqrt{\frac{2+v^6}{2}}-\sqrt{\frac{v^6}{2}}\right]^8, \tag{3.2}$$

where $u=(G_nG_{9n})^{-1}$ *and* $v=g_ng_{9n}$.

Proof of (3.1)**.** Using (1.3) in (2.1), we find that

$$\alpha\beta=\left[\frac{1-\sqrt{1-2u^6}}{2}\right]^4. \tag{3.3}$$

Using (1.3) and (2.7) in the above equation (3.3), we obtain the required result.

Proof of (3.2)**.** Using (1.4) in (2.1), we find that

$$\alpha\beta=\left[\frac{-v^3+\sqrt{v^6+2}}{\sqrt{2}}\right]^8. \tag{3.4}$$

Using (1.4) and (2.7) in the above equation (3.4), we obtain the required result.

Corollary 3.1. *We have*

$$\alpha_{27} = 2^{-14/3}\left(2+\sqrt{3}\right)\left(1-\sqrt{\sqrt[3]{4}-1}\right)^{8}. \tag{3.5}$$

Proof. Using $n = 3$ in (3.1) with

$$G_3 = 2^{1/12}, \tag{3.6}$$

$$G_{27} = 2^{1/12}\left(\sqrt[3]{2}-1\right)^{-1/3}, \tag{3.7}$$

we obtain the required result.

Theorem 3.2. *We have*

$$\begin{aligned}\alpha_{25n} &= 2G_n^{12}\left[\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}+\frac{1}{2}}+\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}-\frac{1}{2}}\right]^4 \\ &\quad \times\left[\sqrt{\frac{1-2u^4+u^6}{4}}-\sqrt{\frac{1-2u^4-u^6}{4}}\right]^4,\end{aligned} \tag{3.8}$$

where $u = (G_nG_{25n})^{-1}$.

Proof. Using (1.3) in (2.2), we deduce that

$$\alpha\beta = \left[\frac{(1-2u^4)-\sqrt{(1-2u^4)^2-u^{12}}}{2}\right]^2. \tag{3.9}$$

Using (1.3) and (2.7) in (3.9), we deduce the required result.

Corollary 3.2. *We have*

$$\alpha_{45} = \frac{1}{2}+\left(255\sqrt{5}-570\right)^{1/2}\left(10\sqrt{3}+17-8\sqrt{5}-\frac{68}{\sqrt{15}}\right). \tag{3.10}$$

Proof. From [3, p.191], we have

$$G_{45} = \left(2+\sqrt{5}\right)^{1/4}\left(\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}}\right)^{1/3}. \tag{3.11}$$

Using (3.11) with $n = \dfrac{9}{5}$ in (2.10), we deduce that

$$G_{\frac{9}{5}} = \left(2+\sqrt{5}\right)^{1/4} \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{1/3}. \tag{3.12}$$

Using (3.11) and (3.12) with $n = \dfrac{9}{5}$ in (3.8), we obtain the required result.
Remark: *A different proof of α_{45} can be found in [3, p. 290].*

Theorem 3.3. *We have*

$$\begin{aligned} \alpha_{49n} &= 2G_n^{12}\left[\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}+\frac{1}{2}}+\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}-\frac{1}{2}}\right]^4 \qquad (3.13)\\ &\quad\times\left[\frac{1-\sqrt{1-2\sqrt{2}u^3}}{2}\right]^8, \\ &= \left(-g_n^{12}+\sqrt{g_n^{24}+1}\right)^{-2}\left[\sqrt{\frac{2\sqrt{2}+v^3}{2\sqrt{2}}}-\sqrt{\frac{v^3}{2\sqrt{2}}}\right]^{16}, \qquad (3.14)\end{aligned}$$

where $u = (G_n G_{49n})^{-1}$ and $v = g_n g_{49n}$.

Proof of (3.13)**.** Using (1.3) in (2.3), we find that

$$\alpha\beta = \left[\frac{1-\sqrt{1-2\sqrt{2}u^3}}{2}\right]^8. \tag{3.15}$$

Using (1.3) and (2.7) in equation (3.15), we obtain the required result (3.13).
Proof of (3.14)**.** Using (1.4) and (2.7) in (2.3), we find that

$$\alpha\beta = \left[\sqrt{\frac{2\sqrt{2}+v^3}{2\sqrt{2}}}-\sqrt{\frac{v^3}{2\sqrt{2}}}\right]^{16}. \tag{3.16}$$

Using (1.4) and (2.7) in the above equation (3.16), we obtain the required result (3.14).

Corollary 3.3. *We have*

$$\alpha_{49} = 2\left(\frac{1-\sqrt{-7-2\sqrt{7}+\sqrt{2}\sqrt[4]{343}+3\sqrt{2}\sqrt[4]{7}}}{2}\right)^8. \tag{3.17}$$

Proof. From [3, p. 191], we have

$$G_{49} = \left(\frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}\right). \tag{3.18}$$

Using $n=1$ with (3.18) in (3.13), we obtain the required result.

Corollary 3.4. *We have*

$$\alpha_{14} = \left(\sqrt{\frac{2\sqrt{2}+1}{2\sqrt{2}}} - \sqrt{\frac{1}{2\sqrt{2}}}\right)^{16} \tag{3.19}$$

$$\times\left[\sqrt{498+352\sqrt{2}-4\sqrt{30926+21868\sqrt{2}}}\right.$$

$$\left.+\sqrt{497+352\sqrt{2}-4\sqrt{30926+21868\sqrt{2}}}\right]^2.$$

Proof. From [3, p. 200], we have

$$g_{14} = \sqrt{\frac{1+\sqrt{2}+\sqrt{2\sqrt{2}-1}}{2}}. \tag{3.20}$$

Using $n = \frac{2}{7}$ in (2.8), we deduce that

$$g_{\frac{2}{7}} = \sqrt{\frac{1+\sqrt{2}-\sqrt{2\sqrt{2}-1}}{2}}. \tag{3.21}$$

Using (3.20) and (3.21) with $n = \frac{2}{7}$ in (3.14), we obtain the required result.

Theorem 3.4. *We have*

$$(3.22)\quad \alpha_{121n} = 2G_n^{12}\left[\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}+\frac{1}{2}}+\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}-\frac{1}{2}}\right]^4 \times\left[\sqrt{\frac{1-2u^2+\sqrt{2}u^3}{4}}-\sqrt{\frac{1-2u^2-\sqrt{2}u^3}{4}}\right]^8,$$

$$(3.23)\quad = \left(-g_n^{12}+\sqrt{g_n^{24}+1}\right)^{-2}\left[\frac{-v(v^2+2)+\sqrt{v^2(v^2+2)^2+2}}{\sqrt{2}}\right]^8,$$

where $u=(G_nG_{121n})^{-1}$ *and* $v=g_ng_{121n}$.

Proof of (3.22)**.** Using (1.3) and (2.7) in (2.4), we deduce that

$$(3.24)\qquad \alpha\beta=\left[\sqrt{\frac{1-2u^2+\sqrt{2}u^3}{4}}-\sqrt{\frac{1-2u^2-\sqrt{2}u^3}{4}}\right]^8.$$

Using (1.3) and (2.7) in (3.24), we deduce the required result (3.22).

Proof of (3.23)**.** Using (1.4) in (2.4), we deduce that

$$(3.25)\qquad \alpha\beta=\left[\frac{-v(v^2+2)+\sqrt{v^2(v^2+2)^2+2}}{\sqrt{2}}\right]^8.$$

Using (1.4) in the above equation (3.25), we find the required result (3.23).

Theorem 3.5. *We have*

$$(3.26)\quad \alpha_{529n} = 2G_n^{12}\left[\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}+\frac{1}{2}}+\sqrt{\sqrt{\frac{G_n^{12}+1}{8}}-\frac{1}{2}}\right]^4 \times\left[\frac{(1-\sqrt{2}u)+\sqrt{(1-\sqrt{2}u)^2-2\sqrt{2}u^3}}{2}\right]^8,$$

$$(3.27)\quad = \left(-g_n^{12}+\sqrt{g_n^{24}+1}\right)^{-2} \times\left[\frac{-\sqrt[4]{2}\sqrt{v}(v+\sqrt{2})+\sqrt{\sqrt{2}v(v+\sqrt{2})^2+4}}{2}\right]^{16},$$

where $u = (G_n G_{529n})^{-1}$ *and* $v = g_n g_{529n}$.

Proof of (3.26)**.** Using (1.3) in (2.5), we deduce that

$$\alpha\beta = \left[\frac{(1-\sqrt{2}u) + \sqrt{(1-\sqrt{2}u)^2 - 2\sqrt{2}u^3}}{2}\right]^8. \tag{3.28}$$

Using (1.3) and (2.7) in (3.28), we obtain the required result (3.26).
Proof of (3.27)**.** Using (1.4) in (2.5), we deduce that

$$\alpha\beta = \left[\frac{-\sqrt[4]{2}\sqrt{v}(v+\sqrt{2}) + \sqrt{\sqrt{2}v(v+\sqrt{2})^2 + 4}}{2}\right]^8. \tag{3.29}$$

Using (1.4) in the above equation (3.29), we find the required result (3.27).

4 Explicit Evaluations of g_n

In this section, we obtain some explicit evaluations of g_n.

Theorem 4.1. *We have*

$$g_{24} = \left(\sqrt{3}+1\right)^{1/4}\left(\sqrt{\frac{45+27\sqrt{3}}{2}} + \sqrt{\frac{43+27\sqrt{3}}{2}}\right)^{1/12}, \tag{4.1}$$

$$g_{\frac{8}{3}} = \left(\sqrt{3}+1\right)^{1/4}\left(\sqrt{\frac{45+27\sqrt{3}}{2}} - \sqrt{\frac{43+27\sqrt{3}}{2}}\right)^{1/12}. \tag{4.2}$$

Proof. In [7, Th.4.5], we have

$$g_{\frac{8}{3}} g_{24} = \sqrt{\sqrt{3}+1}. \tag{4.3}$$

Using (4.3) in (2.8), we deduce that

$$\frac{g_{24}^6}{g_{\frac{8}{3}}^6} = \sqrt{\frac{45+27\sqrt{3}}{2}} + \sqrt{\frac{43+27\sqrt{3}}{2}}. \tag{4.4}$$

From (4.3) and (4.4), we obtain the required results.

Theorem 4.2. *We have*

$$
\begin{aligned}
g_{96} = &\left(1+\sqrt{3}\right)^{1/4}\left(1+\sqrt{3}+\sqrt{3+3\sqrt{3}}\right)^{1/4}\\
&\left(\sqrt{\frac{7110+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}}\right.\\
&\left.+\sqrt{\frac{7106+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}}\right)^{1/12},
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
g_{\frac{32}{3}} = &\left(1+\sqrt{3}\right)^{1/4}\left(1+\sqrt{3}+\sqrt{3+3\sqrt{3}}\right)^{1/4}\\
&\left(\sqrt{\frac{7110+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}}\right.\\
&\left.-\sqrt{\frac{7106+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}}\right)^{1/12},
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
g_{240} =&2^{1/8}\left(\sqrt{3}+1\right)^{1/4}\left(\sqrt{10}+3\right)^{1/4}\left(\sqrt{6}+\sqrt{5}\right)^{1/4}\\
&\left(\sqrt{\frac{3513510\sqrt{10}+6410880\sqrt{3}+11104056+2028510\sqrt{30}}{16}}\right.\\
&\left.+\sqrt{\frac{3513510\sqrt{10}+6410880\sqrt{3}+11104040+2028510\sqrt{30}}{16}}\right)^{1/12},
\end{aligned} \tag{4.7}
$$

$$g_{\frac{80}{3}} = 2^{1/8}\left(\sqrt{3}+1\right)^{1/4}\left(\sqrt{10}+3\right)^{1/4}\left(\sqrt{6}+\sqrt{5}\right)^{1/4}$$

$$\left(\sqrt{\frac{3513510\sqrt{10}+6410880\sqrt{3}+11104056+2028510\sqrt{30}}{16}}\right. \tag{4.8}$$

$$\left.-\sqrt{\frac{3513510\sqrt{10}+6410880\sqrt{3}+11104040+2028510\sqrt{30}}{16}}\right)^{1/12}.$$

Proof of (4.5)**.** Using (4.3) in (2.13), we obtain the required result. Since the proofs of (4.6) – (4.8) are similar to the proof of (4.5), we omit the details.

5 Explicit Evaluations of G_n

In this section, we obtain some explicit evaluations of G_n.

Theorem 5.1. *We have*

$$G_{24} = \left(\frac{1+\sqrt{3}+\sqrt{3+3\sqrt{3}}}{2}\right)^{1/4}\left(\sqrt{\frac{45+27\sqrt{3}}{2}}-\sqrt{\frac{43+27\sqrt{3}}{2}}\right)^{1/12}$$

$$\left(\sqrt{\frac{7110+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}}\right. \tag{5.1}$$

$$\left.+\sqrt{\frac{7106+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}}\right)^{1/12},$$

$$G_{\frac{8}{3}} = \left(\frac{1+\sqrt{3}+\sqrt{3+3\sqrt{3}}}{2}\right)^{1/4} \left(\sqrt{\frac{45+27\sqrt{3}}{2}}+\sqrt{\frac{43+27\sqrt{3}}{2}}\right)^{1/12}$$

$$\left(\sqrt{\frac{7110+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}} - \sqrt{\frac{7106+4104\sqrt{3}+27\sqrt{141706+81814\sqrt{3}}}{4}}\right)^{1/12}. \tag{5.2}$$

Theorem (5.1) is obtained by using Theorem (4.1) and the formula $g_{4n} = 2^{\frac{1}{4}} G_n g_n$.

6 Explicit Evaluations of α_n

In this section, we obtain some explicit evaluations of α_n.

Theorem 6.1. *We have*

(6.1)

$$\alpha_{24} = \left(69+40\sqrt{3}-28\sqrt{6}-48\sqrt{2}+6\sqrt{256+153\sqrt{3}}-2\sqrt{2333+1347\sqrt{3}}\right)^2 \times \left(\sqrt{6+3\sqrt{3}}-\sqrt{5+3\sqrt{3}}\right)^8,$$

(6.2)

$$\alpha_{\frac{8}{3}} = \left(69+40\sqrt{3}-28\sqrt{6}-48\sqrt{2}+6\sqrt{256+153\sqrt{3}}-2\sqrt{2333+1347\sqrt{3}}\right)^{-2},$$

(6.3)

$$\alpha_{39} = 2^4 \left(\frac{\sqrt{13}+3}{2}\right)^2 \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right)^{12} \left(\frac{1}{2} - \frac{1}{4}\sqrt{\frac{-3+3\sqrt{13}}{2}}\right)^4$$
$$\times \left(\sqrt{1295056 + 359184\sqrt{13} - 48\sqrt{1455864558 + 403784178\sqrt{13}}}\right.$$
$$\left. + \sqrt{1295055 + 359184\sqrt{13} - 48\sqrt{1455864558 + 403784178\sqrt{13}}}\right),$$

(6.4)

$$\alpha_{\frac{13}{3}} = 2^{-4} \left(\frac{\sqrt{13}-3}{2}\right)^2 \left(\sqrt{\frac{5+\sqrt{13}}{8}} + \sqrt{\frac{\sqrt{13}-3}{8}}\right)^{12}$$
$$\times \left(\sqrt{1295056 + 359184\sqrt{13} - 48\sqrt{1455864558 + 403784178\sqrt{13}}}\right.$$
$$\left. - \sqrt{1295055 + 359184\sqrt{13} - 48\sqrt{1455864558 + 403784178\sqrt{13}}}\right).$$

Proofs are similar to the proof of corollary 3.1, so we omit the details.

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