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A note on a general integral operator of the bounded boundary rotation[1]

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Abstract

In this note, we consider the classes of bounded radius rotations, bounded radius rotation of order β, bounded boundary rotation. In these classes we study some properties of a general integral operator.

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1 Introduction

Let $\mathcal{P}_k^\lambda(\beta)$ denote the class of analytic functions $p(z)$ in defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ with the following properties:

(i). $p(0) = 1$

(ii). $\displaystyle\int_0^{2\pi} \left| \frac{\Re\{e^{i\lambda} p(z) - \beta \cos\lambda\}}{1-\beta} \right| d\theta \leq k\pi \cos\lambda$

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where, $k \geq 2$, λ real, $|\lambda| < \frac{\pi}{2}$, $0 \leq \beta < 1$ and $z = re^{i\theta}$ for $0 \leq r < 1$.
Let $\mathcal{V}_k^\lambda(\beta)$ [4] denote the class of functions f analytic in $\mathcal{U}$ with the normalized properties $f(0) = f'(0) - 1 = 0$ and

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(\beta), \quad z \in \mathcal{U}$$

where, k, λ and β are as above. For $\beta = 0$ we get the class $\mathcal{V}_k^\lambda$ of functions with bounded boundary rotation studied by Moulis [3].
Any function $f \in \mathcal{V}_k^\lambda(\beta)$ if and only if

$$\Re\left\{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta\cos\lambda, \quad \text{for} \quad |z| < \frac{k - \sqrt{k^2-4}}{2}.$$

A function f defined in $\mathcal{U}$ with the normalization properties $f(0) = 0$ and $f'(0) = 1$ is said to be in the class $\mathcal{U}_k^\lambda(\beta)$ if $\dfrac{zf'}{f} \in \mathcal{P}_k^\lambda(\beta)$.
From the definition of the above classes it follows that $f \in \mathcal{V}_k^\lambda(\beta)$ if and only if $zf' \in \mathcal{U}_k^\lambda(\beta)$.
Now we consider the integral operator $F_n(z)$ [2], defined by

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \ldots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.1}$$

and we study its properties.

Remark 1.1. *We observe that for $n = 1$ and $\alpha_1 = 1$, we obtain the integral operator of Alexander [1], $F(z) = \int_0^z \frac{f(t)}{t}\, dt$.*

2 Main results

Theorem 2.1. *Let α_i be real numbers with the properties $0 \leq \alpha_i < 1$ for $i \in \{1, 2, ..., n\}$ and $\sum_{i=1}^{n} \alpha_i \leq n + 1$. If $f_i \in \mathcal{U}_k^\lambda\left(\frac{1}{\alpha_i}\right)$ then the integral operator defined in (1.1) belongs to $\mathcal{V}_k^\lambda$.*

Proof. Consider,

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} ... \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt.$$

We determine the derivatives of the first and second order for F_n.

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} ... \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \frac{zf_i'(z) - f_i(z)}{z^2} \prod_{j=1,\, j\neq i}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{f_1''(z)}{f_1'(z)} - \frac{1}{z}\right) + ... + \alpha_n \left(\frac{f_n''(z)}{f_n'(z)} - \frac{1}{z}\right)$$

$$\frac{zF_n''(z)}{F_n'(z)} + 1 = \alpha_1 \frac{zf_1''(z)}{f_1'(z)} + ... + \alpha_n \frac{zf_n''(z)}{f_n'(z)} - \alpha_1 - ... - \alpha_n + 1$$

$$\begin{aligned}\Re\left\{e^{i\lambda}\left(\frac{zF_n''(z)}{F_n'(z)} + 1\right)\right\} &= \alpha_1 \Re\left\{e^{i\lambda}\frac{zf_1'(z)}{f_1(z)}\right\} + ... + \alpha_n \Re\left\{e^{i\lambda}\frac{zf_n'(z)}{f_n(z)}\right\} \\ &\quad + \Re\left\{e^{i\lambda}\left(-\alpha_1 - ... - \alpha_n + 1\right)\right\} \\ &= (n+1)\cos\lambda - \sum_{i=1}^{n} \alpha_i \cos\lambda > 0.\end{aligned}$$

Hence $F_n \in \mathcal{V}_k^{\lambda}$.

Corollary 2.2. *For parametric values $k = 2$, $\lambda = 0$, we get the following result [2].*
Let α_i, $i \in \{1, 2, ..., n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, 2, ..., n\}$ and $\sum_{i=1}^{n} \alpha_i \leq n + 1$. We suppose that the functions f_i,

$i \in \{1,2,...,n\}$ are starlike functions of order $\frac{1}{\alpha_i}$, $i \in \{1,2,...,n\}$, that is $f_i \in \mathcal{S}^*\left(\frac{1}{\alpha_i}\right)$ for all $i \in \{1,2,...,n\}$. Then the integral operator defined in (1.1) is convex.

Theorem 2.3. *Let α_i be real numbers with the properties $\alpha_i > 0$ for $i \in \{1,2,...,n\}$ with $\sum_{i=1}^{n} \alpha_i \leq 1$ and $f_i \in \mathcal{U}_k^{\lambda}\left(\frac{1}{\alpha_i}\right)$. Then the integral operator defined in (1.1) belongs to $\mathcal{V}_k^{\lambda}(\alpha)$, where $\alpha = 1 - \sum_{i=1}^{n} \alpha_i$.*

Proof. Consider,

$$\begin{aligned} \frac{zF_n''(z)}{F_n'(z)} &= \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \\ &= \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} - \alpha_1 - ... - \alpha_n. \\ 1 + \frac{zF_n''(z)}{F_n'(z)} &= \alpha_1 \frac{zf_1'(z)}{f_1(z)} + ... + \alpha_n \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - ... - \alpha_n + 1. \\ \Re\left\{ e^{i\lambda}\left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) \right\} &= \alpha_1 \Re\left\{ e^{i\lambda} \frac{zf_1'(z)}{f_1(z)} \right\} + ... + \alpha_n \Re\left\{ e^{i\lambda} \frac{zf_n'(z)}{f_n(z)} \right\} \\ &\quad + \Re\left\{ e^{i\lambda}\left(-\alpha_1 - ... - \alpha_n + 1 \right) \right\}. \end{aligned}$$

But $f_i \in \mathcal{U}_k^{\lambda}$ for all $i \in \{1,2,...,n\}$. Therefore

$$\Re\left\{ e^{i\lambda} \frac{zf_i'(z)}{f_i(z)} \right\} > 0, \quad \forall \ i \in \{1,2,...,n\}.$$

This implies,

$$\Re\left\{ e^{i\lambda}\left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) \right\} > 1 - \sum_{i=1}^{n} \alpha_i = \alpha.$$

Hence $F_n \in \mathcal{U}_k^{\lambda}(\alpha)$.

Corollary 2.4. *For parametric values* $k = 2$, $\lambda = 0$, *we get the following result [2].*
Let α_i, $i \in \{1, 2, ..., n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1, 2, ..., n\}$ *and* $\sum_{i=1}^{n} \alpha_i \leq 1$. *We suppose that the functions* f_i, *with* $i \in \{1, 2, ..., n\}$ *are starlike. Then the integral operator defined in* (1.1) *is convex by order* $1 - \sum_{i=1}^{n} \alpha_i$.

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