

Iterations of asymptotically pseudocontractive mappings in Banach spaces¹

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Abstract

In this paper, we establish the strong convergence for a modified two-step iterative scheme with errors due to Chang [1] for asymptotically pseudocontractive mappings in real Banach spaces.

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1 Introduction

Let E be a real normed space and K be a nonempty convex subset of E . Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 \text{ and } \|f^*\| = \|x\|\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j .

Let $T : D(T) \subset E \rightarrow E$ be a mapping with domain $D(T)$ in E .

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Definition 1. The mapping T is said to be uniformly L -Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$

$$\|T^n x - T^n y\| \leq L \|x - y\|.$$

Definition 2. T is said to be nonexpansive if for all $x, y \in D(T)$, the following inequality holds:

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in D(T).$$

Definition 3. T is said to be asymptotically nonexpansive [4], if there exists a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in D(T), n \geq 1.$$

Definition 4. T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ and there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 \text{ for all } x, y \in D(T), n \geq 1.$$

Remark 1. 1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L -Lipschitzian.

2. If T is asymptotically nonexpansive mapping then for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\begin{aligned} \langle T^n x - T^n y, j(x - y) \rangle &\leq \|T^n x - T^n y\| \|x - y\| \\ &\leq k_n \|x - y\|^2, n \geq 1. \end{aligned}$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

3. Rhoades in [10] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

4. The asymptotically pseudocontractive mappings were introduced by Schu [11].

In recent years, Mann and Ishikawa iterative schemes [7,9] have been studied extensively by many authors.

Let $T : K \rightarrow K$ be a mapping.

(a) The Mann iteration process is defined by the sequence $\{x_n\}_{n \geq 0}$

$$(1.1) \quad \begin{cases} x_1 \in K, \\ x_{n+1} = (1 - b_n)x_n + b_nTx_n, \quad n \geq 0, \end{cases}$$

where $\{b_n\}_{n \geq 0}$ is a sequence in $[0, 1]$.

(b) The sequence $\{x_n\}_{n \geq 0}$ defined by

$$(1.2) \quad \begin{cases} x_1 \in K, \\ x_{n+1} = (1 - b_n)x_n + b_nTy_n, \\ y_n = (1 - b'_n)x_n + b'_nTx_n, \quad n \geq 0, \end{cases}$$

where $\{b_n\}_{n \geq 0}, \{b'_n\}_{n \geq 0}$ are sequences in $[0, 1]$ is known as the Ishikawa iteration process.

In 1995, Liu [8] introduced iterative schemes with errors as follows:

(c) The sequence $\{x_n\}_{n \geq 0}$ in K iteratively defined by:

$$(1.3) \quad \begin{cases} x_1 \in K, \\ x_{n+1} = (1 - b_n)x_n + b_nTy_n + u_n, \\ y_n = (1 - b'_n)x_n + b'_nTx_n + v_n, \quad n \geq 1 \end{cases}$$

where $\{b_n\}_{n \geq 0}, \{b'_n\}_{n \geq 0}$ are sequences in $[0, 1]$ and $\{u_n\}_{n \geq 0}, \{v_n\}_{n \geq 0}$ are sequences in K satisfying $\sum_{n=0}^{\infty} \|u_n\| < \infty, \sum_{n=0}^{\infty} \|v_n\| < \infty$, is known as Ishikawa iterative scheme with errors.

(d) The sequence $\{x_n\}_{n \geq 0}$ iteratively defined by:

$$(1.4) \quad \begin{cases} x_1 \in K, \\ x_{n+1} = (1 - b_n)x_n + b_nTx_n + u_n, \quad n \geq 1 \end{cases}$$

where $\{b_n\}_{n \geq 0}$ is a sequence in $[0, 1]$ and $\{u_n\}_{n \geq 0}$ a sequence in K satisfying $\sum_{n=0}^{\infty} \|u_n\| < \infty$, is known as Mann iterative scheme with errors.

While it is clear that consideration of error terms in iterative schemes is an important part of the theory, it is also clear that the iterative schemes with errors introduced by Liu [8], as in (c) and (d) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (c) and (d) which say that they tend to zero as n tends to infinity are, therefore, unreasonable.

In 1998, Xu [12] introduced a more satisfactory error term in the following iterative schemes:

(e) The sequence $\{x_n\}_{n \geq 0}$ iteratively defined by:

$$(1.5) \quad \begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 0, \end{cases}$$

with $\{u_n\}_{n \geq 0}$ is a bounded sequence in K and $a_n + b_n + c_n = 1$, is known as Mann iterative scheme with errors.

(f) The sequence $\{x_n\}_{n \geq 0}$ iteratively defined by:

$$(1.6) \quad \begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad n \geq 0, \end{cases}$$

with $\{u_n\}_{n \geq 0}, \{v_n\}_{n \geq 0}$ bounded sequences in K and $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$, is known as Ishikawa iterative scheme with errors.

Note that the error terms are now improved, Mann and Ishikawa iterative schemes follow as special cases of the above schemes respectively.

In [11], Schu proved the following theorem:

Theorem 1. *Let K be a nonempty bounded closed convex subset of a Hilbert space H , $T : K \rightarrow K$ a completely continuous, uniformly L -Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1, \infty)$; $q_n = 2k_n - 1$,*

$\forall n \in N; \sum(q_n^2 - 1) < \infty; \{\alpha_n\}, \{\beta_n\} \subset [0, 1]; \epsilon < \alpha_n < \beta_n \leq b, \forall n \in N,$
 $\epsilon > 0$ and $b \in (0, L^{-2}[(1 + L^2)^{\frac{1}{2}} - 1]); x_1 \in K$ for all $n \in N$, define

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ converges to some fixed point of T .

The recursion formula of Theorem 1 is a modification of the well-known Mann iteration process (see [9]).

Recently, Chang [1] extended Theorem 1 to real uniformly smooth Banach space using the following Ishikawa type iteration process:

Algorithm 1. Let $T : K \rightarrow K$ be a given mapping. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n \geq 0}$ by the iterative scheme

$$(1.7) \quad \begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n, \\ y_n &= a'_n x_n + b'_n T^n x_n + c'_n v_n, \quad n \geq 0, \end{aligned}$$

which is called the modified two-step iterative process with errors, where $\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0}, \{a'_n\}_{n \geq 0}, \{b'_n\}_{n \geq 0}, \{c'_n\}_{n \geq 0}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}_{n \geq 0}, \{v_n\}_{n \geq 0}$ are bounded sequences in K .

It may be noted that the iteration schemes (1.1-1.6) may be viewed as the special case of (1.7).

Also he proved the following theorem:

Theorem 2. Let K be a nonempty bounded closed convex subset of a real uniformly smooth Banach space E , $T : K \rightarrow K$ an asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n \geq 0} \subset [1, \infty), \lim_{n \rightarrow \infty} k_n = 1$, and let $F(T) \neq \phi$. Let $\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:

i) $b_n + c_n \leq 1, b'_n + c'_n \leq 1;$

$$\text{ii) } \lim_{n \rightarrow \infty} b_n = 0 = \lim_{n \rightarrow \infty} b'_n = \lim_{n \rightarrow \infty} c'_n;$$

$$\text{iii) } \sum_{n \geq 0} b_n = \infty, \sum_{n \geq 0} c_n \leq \infty.$$

For arbitrary $x_0 \in K$ let $\{x_n\}$ be iteratively defined by (1.7).

1. If $\{x_n\}$ converges strongly to a fixed point x^* of T in K , then there exists a strictly increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$, $\psi(0) = 0$ such that

$$(C) \quad \langle T^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \psi(\|x - x^*\|), \quad \forall n \in N,$$

2. Conversely, if there exists a strictly increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$, $\psi(0) = 0$ satisfying (C), then $x_n \rightarrow x^* \in F(T)$.

In this paper, we establish the strong convergence for a modified two-step iterative scheme with errors due to Chang [1] for asymptotically pseudocontractive mappings in real Banach spaces. We also significantly extend Theorem 2 from uniformly smooth Banach space to arbitrary real Banach space. The boundedness assumption imposed on K in the theorem is also dispensed with.

2 Main Results

The following lemmas are now well known.

Lemma 1. *Let $J : E \rightarrow 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$.

Lemma 2. Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, $\{\lambda_n\}$ be a real sequence satisfying

$$0 \leq \lambda_n \leq 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a positive integer n_0 such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \psi(\theta_{n+1}) + \sigma_n,$$

for all $n \geq n_0$, with $\sigma_n \geq 0$, $\forall n \in \mathbb{N}$, and $\sigma_n = o(\lambda_n)$, then $\lim_{n \rightarrow \infty} \theta_n = 0$.

Theorem 3. Let K be a nonempty closed convex subset of a real Banach space E , $T : K \rightarrow K$ a uniformly continuous asymptotically pseudocontractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0}, \{a'_n\}_{n \geq 0}, \{b'_n\}_{n \geq 0}, \{c'_n\}_{n \geq 0}$ be real sequences in $[0, 1]$ satisfying the following conditions:

i) $\lim_{n \rightarrow \infty} b_n = 0 = \lim_{n \rightarrow \infty} b'_n = \lim_{n \rightarrow \infty} c'_n$;

ii) $\sum_{n \geq 0} b_n = \infty$;

iii) $c_n = o(b_n)$.

For arbitrary $x_0 \in K$ let $\{x_n\}_{n \geq 0}$ be iteratively defined by (1.7). Suppose there exists a strictly increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 - \psi(\|x - p\|), \quad \forall x \in K.$$

Then $\{x_n\}_{n \geq 0}$ converges strongly to $p \in F(T)$.

Proof. Since p is a fixed point of T , then the set of fixed points $F(T)$ of T is nonempty. Also the condition $c_n = o(b_n)$, implies $c_n = t_n b_n$; $t_n \rightarrow 0$ as $n \rightarrow \infty$. Since T has bounded range, we set

$$(2.1) \quad M_1 = \|x_0 - p\| + \sup_{n \geq 0} \|T^n y_n - p\| + \sup_{n \geq 0} \|u_n - p\| + \sup_{n \geq 0} \|v_n - p\|.$$

Obviously $M_1 < \infty$.

It is clear that $\|x_0 - p\| \leq M_1$. Let $\|x_n - p\| \leq M_1$. Next we will prove that $\|x_{n+1} - p\| \leq M_1$.

Consider

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|a_n x_n + b_n T^n y_n + c_n u_n - p\| \\
 &\leq (1 - b_n) \|x_n - p\| + b_n \|T^n y_n - p\| + c_n \|u_n - p\| \\
 &\leq \|x_0 - p\| + (1 - b_n) [\sup_{n \geq 0} \|T^n y_n - p\| + \sup_{n \geq 0} \|u_n - p\|] \\
 &\quad + b_n \|T^n y_n - p\| + b_n \|u_n - p\| \\
 &\leq M_1.
 \end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n \geq 0}$ is bounded. Let $M_2 = \sup_{n \geq 0} \|x_n - p\|$.

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

A real valued function f defined on an interval (or on any convex subset C of some vector space) is called **generalized convex** if for any three points x, y and z in its domain C and any a, b, c in $[0, 1]$; $a + b + c = 1$, we have

$$(2.2) \quad f(ax + by + cz) \leq af(x) + bf(y) + cf(z).$$

The real function $f : [0, \infty) \rightarrow [0, \infty)$, $f(t) = t^2$ is increasing and generalized convex. For all $a, b, c \in [0, 1]$ with $a + b + c = 1$ and $t_1, t_2, t_3 > 0$, we have

$$(2.3) \quad (at_1 + bt_2 + ct_3)^2 \leq at_1^2 + bt_2^2 + ct_3^2.$$

Consider

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|a_n x_n + b_n T^n y_n + c_n u_n - p\|^2 \\
 &= \|a_n(x_n - p) + b_n(T^n y_n - p) + c_n(u_n - p)\|^2 \\
 &\leq [a_n \|x_n - p\| + b_n \|T^n y_n - p\| + c_n \|u_n - p\|]^2 \\
 (2.4) \quad &\leq \|x_n - p\|^2 + M^2 b_n + M^2 c_n.
 \end{aligned}$$

Now from Lemma 1 for all $n \geq 0$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|a_n x_n + b_n T^n y_n + c_n u_n - p\|^2 \\
&= \|a_n(x_n - p) + b_n(T^n y_n - p) + c_n(u_n - p)\|^2 \\
&\leq (1 - b_n)^2 \|x_n - p\|^2 + 2b_n \langle T^n y_n - p, j(x_{n+1} - p) \rangle \\
&\quad + 2c_n \langle u_n - p, j(x_{n+1} - p) \rangle \\
&\leq (1 - b_n)^2 \|x_n - p\|^2 + 2b_n \langle T^n x_{n+1} - p, j(x_{n+1} - p) \rangle \\
&\quad + 2b_n \langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - p) \rangle + 2M^2 c_n \\
&\leq (1 - b_n)^2 \|x_n - p\|^2 + 2b_n k_n \|x_{n+1} - p\|^2 - 2b_n \psi(\|x_{n+1} - p\|) \\
&\quad + 2b_n \|T^n y_n - T^n x_{n+1}\| \|x_{n+1} - p\| + 2M^2 c_n \\
&\leq (1 - b_n)^2 \|x_n - p\|^2 + 2b_n k_n \|x_{n+1} - p\|^2 \\
(2.5) \quad &\quad - 2b_n \psi(\|x_{n+1} - p\|) + 2b_n d_n + 2M^2 c_n,
\end{aligned}$$

where

$$(2.6) \quad d_n = M \|T^n y_n - T^n x_{n+1}\|.$$

From (1.7) we have

$$\begin{aligned}
\|y_n - x_{n+1}\| &= \|b'_n(T^n x_n - x_n) + c'_n(v_n - x_n) + b_n(x_n - T^n y_n) - c_n(u_n - x_n)\| \\
&\leq b'_n \|T^n x_n - x_n\| + c'_n \|v_n - x_n\| + b_n \|x_n - T^n y_n\| + c_n \|u_n - x_n\| \\
(2.7) \quad &\leq 2M(b_n + c_n + b'_n + c'_n).
\end{aligned}$$

From the conditions $\lim_{n \rightarrow \infty} b_n = 0 = \lim_{n \rightarrow \infty} b'_n = \lim_{n \rightarrow \infty} c'_n$, $c_n = o(b_n)$ and (2.7), we obtain

$$\lim_{n \rightarrow \infty} \|y_n - x_{n+1}\| = 0,$$

and the uniform continuity of T leads to

$$\lim_{n \rightarrow \infty} \|T^n y_n - T^n x_{n+1}\| = 0.$$

Thus, we have

$$(2.8) \quad \lim_{n \rightarrow \infty} d_n = 0.$$

Substituting (2.4) in (2.5) and with the help of condition $c_n = o(b_n)$, we obtain

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq [(1 - b_n)^2 + 2b_n k_n] \|x_n - p\|^2 - 2b_n \psi(\|x_{n+1} - p\|) \\
 &\quad + 2b_n [M^2 (k_n (b_n + c_n) + t_n) + d_n] \\
 &= [1 + b_n^2 + 2b_n (k_n - 1)] \|x_n - p\|^2 - 2b_n \psi(\|x_{n+1} - p\|) \\
 &\quad + 2b_n [M^2 (k_n (b_n + c_n) + t_n) + d_n] \\
 &\leq \|x_n - p\|^2 - 2b_n \psi(\|x_{n+1} - p\|) + b_n [M^2 (2(k_n - 1) + b_n) \\
 (2.9) \quad &\quad + 2(M^2 (k_n (b_n + c_n) + t_n) + d_n)].
 \end{aligned}$$

Denote

$$\begin{aligned}
 \theta_n &= \|x_n - p\|, \\
 \lambda_n &= 2b_n, \\
 \sigma_n &= b_n [M^2 (2(k_n - 1) + b_n) + 2(M^2 (k_n (b_n + c_n) + t_n) + d_n)].
 \end{aligned}$$

Condition $\lim_{n \rightarrow \infty} b_n = 0$ ensures the existence of a rank $n_0 \in \mathbb{N}$ such that $\lambda_n = 2b_n \leq 1$, for all $n \geq n_0$. Now with the help of the conditions $\lim_{n \rightarrow \infty} b_n = 0$, $\sum_{n \geq 0} b_n = \infty$, $c_n = o(b_n)$, (2.8) and Lemma 2, we obtain from (2.9) that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = 0,$$

completing the proof.

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