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Certain Convolution Properties of Multivalent Analytic Functions Associated with a Linear Operator [1]

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Abstract

Very recently N.E.Cho, O.S.Kwon and H.M.Srivastava (J.Math. Anal. Appl. 292(2004), 470-483) have introduced and investigated a special linear operator $\mathscr{I}_p^\lambda(a,c)$ defined by the Haramard product (or convolution). In this paper we consider some inclusion properties of a class $\mathscr{B}_p^\lambda(a,c,\alpha;h)$ of multivalent analytic functions associated with the operator $\mathscr{I}_p^\lambda(a,c)$. We have made use of differential subordinations and properties of convolution in geometric function theory.

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1 Introduction and Preliminaries

Let $\mathscr{A}(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_nz^{n+p}\quad (p\in\mathbb{N}:=\{1,2,3,\cdots\}), \tag{1.1}$$

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which are analytic in the open unit disk $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let the Hadamard product (or convolution) of two functions

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{n+p} \quad (j = 1, 2),$$

be given by

$$(f_1 * f_2)(z) := z^p + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n+p} =: (f_2 * f_1)(z).$$

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, we say that the function $g(z)$ is subordinate to $f(z)$ and write $g(z) \prec f(z) \quad (z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathbb{U})$ such that $g(z) = f(w(z)) \quad (z \in \mathbb{U})$. In particular, if $f(z)$ is univalent in $\mathbb{U}$, we have the following equivalence

$$g(z) \prec f(z) \quad (z \in \mathbb{U}) \Leftrightarrow g(0) = f(0) \text{ and } g(\mathbb{U}) \subset f(\mathbb{U}).$$

A function $f(z) \in \mathscr{A}(1)$ is said to be in the class $\mathscr{S}^*(\rho)$ if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (z \in \mathbb{U})$$

for some $\rho (\rho < 1)$. When $0 \leq \rho < 1$, $\mathscr{S}^*(\rho)$ is the class of starlike functions of order ρ in $\mathbb{U}$. A function $f(z) \in \mathscr{A}(1)$ is said to be prestarlike of order ρ in $\mathbb{U}$ if

$$\frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in \mathscr{S}^*(\rho) \quad (\rho < 1).$$

We note this class by $\mathscr{R}(\rho)$ (see [6]). Clearly a function $f(z) \in \mathscr{A}(1)$ is in the class $\mathscr{R}(0)$ if and only if $f(z)$ is convex univalent in $\mathbb{U}$ and

$$\mathscr{R}\left(\frac{1}{2}\right) = \mathscr{S}^*\left(\frac{1}{2}\right).$$

In [7] Saitoh introduced a linear operator

$$\mathscr{L}_p(a, c) : \mathscr{A}(p) \to \mathscr{A}(p)$$

defined by

(1.2) $$\mathscr{L}_p(a,c)f(z) := \phi_p(a,c;z) * f(z) \quad (z \in \mathbb{U}; f \in \mathscr{A}(p))$$

where

$$\phi_p(a,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}$$

(1.3) $$(a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- := \{0,-1,-2,\cdots\}; z \in \mathbb{U}).$$

and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1 & \text{for } n = 0, \\ x(x+1)\cdots(x+n-1) & \text{for } n \in \mathbb{N}. \end{cases}$$

The operator $\mathscr{L}_p(a,c)$ is an extension of the Carlson-Shaffer operator [1]. Very recently, Cho, Kwon and Srivastava [2] introduced the following linear operator $\mathscr{I}_p^\lambda(a,c)$ analogous to $\mathscr{L}_p(a,c)$:

$$\mathscr{I}_p^\lambda(a,c) : \mathscr{A}(p) \to \mathscr{A}(p)$$

$$\mathscr{I}_p^\lambda(a,c)f(z) := \phi_p^\dagger(a,c;z) * f(z)$$

(1.4) $$(a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p \,; z \in \mathbb{U}; f \in \mathscr{A}(p)),$$

where $\phi_p^\dagger(a,c;z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition

(1.5) $$\phi_p(a,c;z) * \phi_p^\dagger(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}},$$

where $\phi_p(a,c;z)$ is given by (1.3). It is well known that for $\lambda > -p$

(1.6) $$\frac{z^p}{(1-z)^{\lambda+p}} = \sum_{n=0}^{\infty} \frac{(\lambda+p)_n}{n!} z^{n+p} \quad (z \in \mathbb{U}).$$

Therefore the function $\phi_p^{\dagger}(a,c;z)$ has the following form

$$\phi_p^{\dagger}(a,c;z)=\sum_{n=0}^{\infty}\frac{(\lambda+p)_n(c)_n}{n!(a)_n}z^{n+p}\quad(z\in\mathbb{U}). \tag{1.7}$$

Cho, Kwon and Srivastava [2] have obtained the following properties of the operator $\mathscr{I}_p^{\lambda}(a,c)$:

$$\mathscr{I}_p^{1}(p+1,1)f(z)=f(z),\quad \mathscr{I}_p^{1}(p,1)f(z)=\frac{zf'(z)}{p}, \tag{1.8}$$

$$z(\mathscr{I}_p^{\lambda}(a+1,c)f(z))'=a\mathscr{I}_p^{\lambda}(a,c)f(z)-(a-p)\mathscr{I}_p^{\lambda}(a+1,c)f(z), \tag{1.9}$$

and

$$z(\mathscr{I}_p^{\lambda}(a,c)f(z))'=(\lambda+p)\mathscr{I}_p^{\lambda+1}(a,c)f(z)-\lambda\mathscr{I}_p^{\lambda}(a,c)f(z). \tag{1.10}$$

Many interesting results of multivalent analytic functions associated with the linear operator $\mathscr{I}_p^{\lambda}(a,c)$ have been given in [2]. Also, the authors [2] presented a long list of papers connected with the operators (1.2) and (1.4) and classes of functions defined by means of those operators.

Let $\mathscr{P}$ be the class of functions $h(z)$ with $h(0)=1$, which are analytic and convex univalent in $\mathbb{U}$.

In this paper, we shall introduce and investigate the following subclass of $\mathscr{A}(p)$ associated with the operator $\mathscr{I}_p^{\lambda}(a,c)$.

Definition 1. *A function $f(z)\in\mathscr{A}(p)$ is said to be in the class $\mathscr{B}_p^{\lambda}(a,c,\alpha;h)$ if it satisfies the subordination condition*

$$(1-\alpha)z^{-p}\mathscr{I}_p^{\lambda}(a,c)f(z)+\frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^{\lambda}(a,c)f(z))'\prec h(z), \tag{1.11}$$

where α is a complex number, $a,c\in\mathbb{R}\setminus\mathbb{Z}_0^-$ and $h(z)\in\mathscr{P}$.

The following lemmas will be used in our investigation.

Lemma 1. *(see [4,5]) Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{1.12}$$

where $Re\mu \geq 0$ and $\mu \neq 0$, then $g(z) \prec h(z)$.

Lemma 2. *(see [6]) Let $\rho < 1$, $f(z) \in \mathscr{S}^*(\rho)$ and $g(z) \in \mathscr{R}(\rho)$. Then, for any analytic function $F(z)$ in $\mathbb{U}$,*

$$\frac{g * (fF)}{g * f}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U})),$$

where $\overline{co}(F(\mathbb{U}))$ denotes the closed convex hull of $F(\mathbb{U})$.

2 Inclusion Properties Involving the Operator $\mathscr{I}_p^{\lambda}(a,c)$

Theorem 1. *Let $0 \leq \alpha_1 < \alpha_2$. Then*

$$\mathscr{B}_p^{\lambda}(a, c, \alpha_2; h) \subset \mathscr{B}_p^{\lambda}(a, c, \alpha_1; h).$$

Proof. Let $0 \leq \alpha_1 < \alpha_2$ and suppose that

$$g(z) = z^{-p} \mathscr{I}_p^{\lambda}(a, c) f(z) \tag{2.1}$$

for $f(z) \in \mathscr{B}_p^{\lambda}(a, c, \alpha_2; h)$. Then the function $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$. Differentiating both sides of (2.1) with respect to z and using (1.11), we have

$$\begin{aligned} &(1-\alpha_2) z^{-p} \mathscr{I}_p^{\lambda}(a, c) f(z) + \frac{\alpha_2}{p} z^{-p+1} (\mathscr{I}_p^{\lambda}(a, c) f(z))' \\ &= g(z) + \frac{\alpha_2}{p} z g'(z) \prec h(z). \end{aligned} \tag{2.2}$$

Hence an application of Lemma 1 yields

$$g(z) \prec h(z). \tag{2.3}$$

Noting that $0 \leq \frac{\alpha_1}{\alpha_2} < 1$ and that $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}
&(1-\alpha_1)z^{-p}\mathscr{I}_p^{\lambda}(a,c)f(z) + \frac{\alpha_1}{p}z^{-p+1}(\mathscr{I}_p^{\lambda}(a,c)f(z))' \\
&= \frac{\alpha_1}{\alpha_2}\left((1-\alpha_2)z^{-p}\mathscr{I}_p^{\lambda}(a,c)f(z) + \frac{\alpha_2}{p}z^{-p+1}(\mathscr{I}_p^{\lambda}(a,c)f(z))'\right) + \left(1-\frac{\alpha_1}{\alpha_2}\right)g(z) \\
&\prec h(z).
\end{aligned}$$

Thus $f(z) \in \mathscr{B}_p^{\lambda}(a,c,\alpha_1;h)$ and the proof of Theorem 1 is completed.

Theorem 2. *Let*

$$Re\{z^{-p}\phi_p(a_1,a_2;z)\} > \frac{1}{2} \quad (z \in \mathbb{U}), \tag{2.4}$$

where $\phi_p(a_1,a_2;z)$ *is defined as in (1.3). Then*

$$\mathscr{B}_p^{\lambda}(a_1,c,\alpha;h) \subset \mathscr{B}_p^{\lambda}(a_2,c,\alpha;h).$$

Proof. For $f(z) \in \mathscr{A}(p)$ it is easy to verify that

$$z^{-p}\mathscr{I}_p^{\lambda}(a_2,c)f(z) = (z^{-p}\phi_p(a_1,a_2;z)) * (z^{-p}\mathscr{I}_p^{\lambda}(a_1,c)f(z)) \tag{2.5}$$

and

$$z^{-p+1}(\mathscr{I}_p^{\lambda}(a_2,c)f(z))' = (z^{-p}\phi_p(a_1,a_2;z)) * (z^{-p+1}(\mathscr{I}_p^{\lambda}(a_1,c)f(z))'). \tag{2.6}$$

Let $f(z) \in \mathscr{B}_p^{\lambda}(a_1,c,\alpha;h)$. Then from (2.5) and (2.6) we deduce that

$$\begin{aligned}
&(1-\alpha)z^{-p}\mathscr{I}_p^{\lambda}(a_2,c)f(z) + \frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^{\lambda}(a_2,c)f(z))' \\
&= (z^{-p}\phi_p(a_1,a_2;z)) * \psi(z)
\end{aligned} \tag{2.7}$$

and

$$\psi(z) = (1-\alpha)z^{-p}\mathscr{I}_p^{\lambda}(a_1,c)f(z) + \frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^{\lambda}(a_1,c)f(z))' \prec h(z). \tag{2.8}$$

In view of (2.4), the function $z^{-p}\phi_p(a_1,a_2;z)$ has the Herglotz representation

$$z^{-p}\phi_p(a_1,a_2;z) = \int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \tag{2.9}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x|=1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.7), (2.8) and (2.9) that

$$\begin{aligned}&(1-\alpha)z^{-p}\mathscr{I}_p^{\lambda}(a_2,c)f(z) + \frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^{\lambda}(a_2,c)f(z))'\\ &= \int_{|x|=1}\psi(xz)d\mu(x) \prec h(z).\end{aligned}$$

This shows that $f(z) \in \mathscr{B}_p^{\lambda}(a_2,c,\alpha;h)$.

Theorem 3. *Let*

$$Re\{z^{-p}\phi_p(c_1,c_2;z)\} > \frac{1}{2} \quad (z \in \mathbb{U}), \tag{2.10}$$

where $\phi_p(c_1,c_2;z)$ is defined as in (1.3). Then

$$\mathscr{B}_p^{\lambda}(a,c_2,\alpha;h) \subset \mathscr{B}_p^{\lambda}(a,c_1,\alpha;h).$$

Proof. For $f(z) \in \mathscr{A}(p)$ it is easy to verify that

$$z^{-p}\mathscr{I}_p^{\lambda}(a,c_1)f(z) = (z^{-p}\phi_p(c_1,c_2;z)) * (z^{-p}\mathscr{I}_p^{\lambda}(a,c_2)f(z))$$

and

$$z^{-p+1}(\mathscr{I}_p^{\lambda}(a,c_1)f(z))' = (z^{-p}\phi_p(c_1,c_2;z)) * (z^{-p+1}(\mathscr{I}_p^{\lambda}(a,c_2)f(z))').$$

The remaining part of the proof of Theorem 3 is similar to that of Theorem 2 and hence we omit it.

Theorem 4. *Let* $0 < a_1 < a_2$. *Then*

$$\mathscr{B}_p^\lambda(a_1, c, \alpha; h) \subset \mathscr{B}_p^\lambda(a_2, c, \alpha; h).$$

Proof. Define

$$g(z) = z + \sum_{n=1}^{\infty} \frac{(a_1)_n}{(a_2)_n} z^{n+1} \quad (z \in \mathbb{U}; 0 < a_1 < a_2).$$

Then

$$z^{-p+1}\phi_p(a_1, a_2; z) = g(z) \in \mathscr{A}(1), \tag{2.11}$$

where $\phi_p(a_1, a_2; z)$ is defined as in (1.3), and

$$\frac{z}{(1-z)^{a_2}} * g(z) = \frac{z}{(1-z)^{a_1}}\ . \tag{2.12}$$

By (2.12) we see that

$$\frac{z}{(1-z)^{a_2}} * g(z) \in \mathscr{S}^*\left(1 - \frac{a_1}{2}\right) \subset \mathscr{S}^*\left(1 - \frac{a_2}{2}\right)$$

for $0 < a_1 < a_2$ which shows that

$$g(z) \in \mathscr{R}\left(1 - \frac{a_2}{2}\right). \tag{2.13}$$

Let $f(z) \in \mathscr{B}_p^\lambda(a_1, c, \alpha; h)$. Then we deduce from (2.7) and (2.8) (used in the proof of Theorem 2) and (2.11) that

$$\begin{aligned} &(1-\alpha)z^{-p}\mathscr{I}_p^\lambda(a_2, c)f(z) + \frac{\alpha}{p} z^{-p+1}(\mathscr{I}_p^\lambda(a_2, c)f(z))' \\ &= \frac{g(z)}{z} * \psi(z) = \frac{g(z) * (z\psi(z))}{g(z) * z}, \end{aligned} \tag{2.14}$$

where

$$\psi(z) = (1-\alpha)z^{-p}\mathscr{I}_p^\lambda(a_1, c)f(z) + \frac{\alpha}{p} z^{-p+1}(\mathscr{I}_p^\lambda(a_1, c)f(z))'. \tag{2.15}$$

Since the function z belongs to $\mathscr{S}^*\left(1 - \frac{a_2}{2}\right)$ and $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.13), (2.14), (2.15) and Lemma 2 that

$$(1-\alpha)z^{-p}\mathscr{I}_p^\lambda(a_2, c)f(z) + \frac{\alpha}{p} z^{-p+1}(\mathscr{I}_p^\lambda(a_2, c)f(z))' \prec h(z).$$

Thus $f(z) \in \mathscr{B}_p^\lambda(a_2, c, \alpha; h)$ and the proof is completed.

Theorem 5. *Let* $0 < c_1 < c_2$. *Then*

$$\mathscr{B}_p^{\lambda}(a, c_2, \alpha; h) \subset \mathscr{B}_p^{\lambda}(a, c_1, \alpha; h).$$

Proof. Define

$$g(z) = z + \sum_{n=1}^{\infty} \frac{(c_1)_n}{(c_2)_n} z^{n+1} \quad (z \in \mathbb{U}; 0 < c_1 < c_2).$$

Then

$$z^{-p+1}\phi_p(c_1, c_2; z) = g(z) \in \mathscr{A}(1),$$

where $\phi_p(c_1, c_2; z)$ is defined as in (1.3), and

$$\frac{z}{(1-z)^{c_2}} * g(z) = \frac{z}{(1-z)^{c_1}} \ . \tag{2.16}$$

From (2.16) we see that

$$\frac{z}{(1-z)^{c_2}} * g(z) \in \mathscr{S}^*\left(1 - \frac{c_1}{2}\right) \subset \mathscr{S}^*\left(1 - \frac{c_2}{2}\right)$$

for $0 < c_1 < c_2$ which shows that

$$g(z) \in \mathscr{R}\left(1 - \frac{c_2}{2}\right).$$

The remaining part of the proof is similar to that of Theorem 4 and we omit it.

Theorem 6. *Let* $f(z) \in \mathscr{B}_p^{\lambda}(a, c, \alpha; h)$,

$$g(z) \in \mathscr{A}(p) \ \textit{and} \ Re\{z^{-p}g(z)\} > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{2.17}$$

Then

$$(f * g)(z) \in \mathscr{B}_p^{\lambda}(a, c, \alpha; h).$$

Proof. For $f(z) \in \mathscr{B}_p^\lambda(a,c,\alpha;h)$ and $g(z) \in \mathscr{A}(p)$, we have

$$(1-\alpha)z^{-p}\mathscr{I}_p^\lambda(a,c)(f*g)(z) + \frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^\lambda(a,c)(f*g)(z))'$$

$$= (1-\alpha)(z^{-p}g(z)) * (z^{-p}\mathscr{I}_p^\lambda(a,c)f(z)) + \frac{\alpha}{p}(z^{-p}g(z)) * (z^{-p+1}(\mathscr{I}_p^\lambda(a,c)f(z))')$$

$$= (z^{-p}g(z)) * \psi(z), \tag{2.18}$$

where

$$\psi(z) = (1-\alpha)z^{-p}\mathscr{I}_p^\lambda(a,c)f(z) + \frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^\lambda(a,c)f(z))' \prec h(z). \tag{2.19}$$

In view of (2.17), the function $z^{-p}g(z)$ has the Herglotz representation

$$z^{-p}g(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \tag{2.20}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.18) to (2.20) that

$$(1-\alpha)z^{-p}\mathscr{I}_p^\lambda(a,c)(f*g)(z) + \frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^\lambda(a,c)(f*g)(z))'$$
$$= \int_{|x|=1} \psi(xz)d\mu(x) \prec h(z).$$

This shows that $(f*g)(z) \in \mathscr{B}_p^\lambda(a,c,\alpha;h)$ and the theorem is proved.

Theorem 7. *Let* $f(z) \in \mathscr{B}_p^\lambda(a,c,\alpha;h)$,

$$g(z) \in \mathscr{A}(p) \text{ and } z^{-p+1}g(z) \in \mathscr{R}(\rho) \quad (\rho < 1).$$

Then

$$(f*g)(z) \in \mathscr{B}_p^\lambda(a,c,\alpha;h).$$

Proof. For $f(z) \in \mathscr{B}_p^\lambda(a, c, \alpha; h)$ and $g(z) \in \mathscr{A}(p)$, from (2.18) we can write

$$(2.21) \qquad (1-\alpha)z^{-p}\mathscr{I}_p^\lambda(a,c)(f*g)(z) + \frac{\alpha}{p}z^{-p+1}(\mathscr{I}_p^\lambda(a,c)(f*g)(z))' = \frac{(z^{-p+1}g(z))*(z\psi(z))}{(z^{-p+1}g(z))*z} \quad (z \in \mathbb{U}),$$

where $\psi(z)$ is defined as in (2.19).

Since $h(z)$ is convex univalent in $\mathbb{U}$,

$$\psi(z) \prec h(z), z^{-p+1}g(z) \in \mathscr{R}(\rho) \text{ and } z \in \mathscr{S}^*(\rho) \quad (\rho < 1),$$

it follows from (2.21) and Lemma 2 the desired result.

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