General Mathematics Vol. 17, No. 3 (2009), 203–220

On quadrature formulas of Gauss-Turán and Gauss-Turán-Stancu type [1]

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Abstract

In this paper we study the quadrature formulas of Gauss-Turán and Gauss-Turán-Stancu type, the determination of the nodes and the coefficients using the s-orthogonal and σ-orthogonal polynomials.

2000 Mathematical Subject Classification: 65D30, 65D32

Key words and phrases: Gauss-Turán, Gauss-Turán-Stancu quadrature formulas, s-orthogonality, $\sigma-$orthogonality

1 Introduction

Let $\mathbb{P}_m$ be the set of all algebraic polynomials of degree at most m. In 1950 P.$Turán$ [17] was studied numerical quadratures of the form :

$$(1)\qquad \int_{-1}^{1} f(x)dx = \sum_{k=1}^{n}\sum_{\nu=0}^{s-1} A_{k,\nu} f^{(\nu)}(x_k) + R_{n,s}(f),$$

[1] *Received 25 March, 2009*

Accepted for publication (in revised form) 7 April, 2009

where the nodes $-1 \leq x_1 < \cdots < x_n \leq 1$ are arbitrary, $A_{k,\nu} = \int_{-1}^{1} l_{k,\nu}(x)dx$, $(k = \overline{1,n}\ ;\ \nu = \overline{0,s-1})$ and $l_{k,\nu}(x)$ are the fundamental polynomials of Hermite interpolation. The formula (1) is exact for any $f \in \mathbb{P}_{sn-1}$.

One raise the problem to determine, if it is possible the nodes $\{x_i,\ i = \overline{1,n}\}$ so that the quadrature formula is exact for all $f \in \mathbb{P}_{(s+1)n-1}$. *Turán* showed that the nodes must have odd multiplicities to obtain an increase of degree of exactness and these nodes must be the zeros of the *monic* polynomial $\pi_n^*(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, which minimizes the value of the integral $\int_{-1}^{1} [\pi_n(x)]^{s+1} dx$.

If one consider the odd orders of multiplicity of the nodes to be $2s+1$ then one obtain the *Gauss-Turán* type quadrature formula :

$$\int_a^b f(x)d\lambda(x) = \sum_{k=1}^{n} \sum_{\nu=0}^{2s} A_{k,\nu} f^{(\nu)}(x_k) + R_{n,2s}(f), \tag{2}$$

where $d\lambda(x)$ is a nonnegative measure on the interval (a,b) which can be the real axis $\mathbb{R}$, with compact or infinite support for which all moments: $\mu_k = \int_a^b x^k d\lambda(x),\ k = 0, 1, \ldots,$ exists, are finite, and $\mu_0 > 0$.

If the nodes $\{x_k,\ k = \overline{1,n}\}$ in (2) are chosen the zeros of the *monic* polynomial $\pi_{n,s} = \pi_{n,s}(x)$ which minimizes the integral.

$$F(a_0, a_1, \ldots, a_{n-1}) = \int_a^b [\pi_n(x)]^{2s+2} d\lambda(x), \tag{3}$$

then the formula (2) is exact for all polynomials of degree at most $2(s+1)n - 1$, that is, $R_{n,2s}(f) = 0,\ \forall f \in \mathbb{P}_{2(s+1)n-1}$. The condition (3) is equivalent with the following conditions:

$$\int_a^b [\pi_n(x)]^{2s+1} x^k d\lambda(x) = 0,\ (k = \overline{0, n-1}). \tag{4}$$

Let denote, $\pi_{n,s}(x)$ by $P_{n,s}(x)$. The case $d\lambda(x) = w(x)dx$ on $[a, b]$ has been studied by *Osscini* and *Ghizzetti.*

2 The construction of GAUSS-TURÁN Quadrature Formulas by using s-Orthogonal and σ-Orthogonal Polynomials

In order to numerically construct the s-orthogonal polynomials with respect to the measure $d\lambda(x)$, one can use the orthogonality conditions (4). Let n and s be given, and the measure : $d\mu(x) = d\mu_{n,s}(x) = (\pi_n(x))^{2s}d\lambda(x)$. Then the orthogonality conditions can be written as: $\int_a^b \pi_k^{n,s}(x)t^\nu d\mu(x) = 0$, $(\nu = \overline{0, k-1})$, where $\{\pi_k^{n,s}\}_{k\in\mathbb{N}}$ is a sequence of *monic* orthogonal polynomials with respect to the new measure $d\mu(x)$.

So, the polynomials $\pi_k^{n,s}$, which we will denote by $\pi_k = \pi_k(x)$ satisfies a three-term recurrence relation of the form :

(5) $$\pi_{k+1}(x) = (x - \alpha_k)\pi_k(x) - \beta_k\pi_{k-1}(x),$$

where $\pi_{-1}(x) = 0$, $\pi_0(x) = 1$, and we have from the orthogonality property: $\beta_0 = \int_a^b d\mu(x)$,

(6)
$$\alpha_k = \frac{< x\pi_k, \pi_k >}{< \pi_k, \pi_k >} = \frac{\int_a^b x\pi_k^2(x)d\mu(x)}{\int_a^b \pi_k^2(x)d\mu(x)},\ \beta_k = \frac{< \pi_k, \pi_k >}{< \pi_{k-1}, \pi_{k-1} >} = \frac{\int_a^b \pi_k^2(x)d\mu(x)}{\int_a^b \pi_{k-1}^2(x)d\mu(x)}.$$

One can calculate the coefficients α_k, β_k, $(k = \overline{0, n-1})$, and are obtained the first $n + 1$ orthogonal polynomials $\pi_0, \pi_1, \ldots, \pi_n$, and let denote them by $P_{n,s} = \pi_n^s$.

Let define the function on the Euclidian space $\mathbf{R}^n$

$$\Phi(x_1, \dots, x_n) = \int_a^b (x - x_1)^{2s+2} \dots (x - x_n)^{2s+2} d\lambda(x). \tag{7}$$

If $d\lambda(x)$ is a positive measure, it was proven that this function is continuous and positive. Then the function $\Phi(x_1, \dots, x_n)$ has an lower bound μ_0 and this value is attained for $a < x_1 < \dots < x_n < b$ (see [8] T.Popoviciu).

Let consider the polynomial $P_{n,s}^{2s+2}(x) = \prod_{k=1}^n (x - x_k)^{2s+2}$ with the zeros $a < x_1 < \dots < x_n < b$.

Then the function $\Phi(x_1, \dots, x_n)$ have a relative minimum point and we have: $-\frac{1}{2s+2}\frac{\partial \Phi}{\partial x_k} = I(P_k) = 0$, where $P_k(x) = \frac{P_{n,s}^{2s+2}(x)}{x - x_k}$. Then one must have:

$$\int_a^b P_{n,s}^{2s+1} l_k(x) d\lambda(x) = 0,\ k = \overline{1,n}\ ,\ \text{where } l_k(x),\ k = \overline{1,n}$$

are the Lagrange's fundamental interpolation polynomials corresponding to the nodes : $x_1, \dots, x_n$, which are linearly independent. Thus, one obtain that the polynomial $P_{n,s}^{2s+1}$ satisfies the orthogonality conditions :

$$\int_a^b \left[P_{n,s}(x)\right]^{2s+1} x^k d\lambda(x) = 0,\ k = \overline{0, n-1}.$$

From the condition to have a relative minimum we obtain:

$$\frac{\partial \Phi}{\partial x_k} = 0, \quad \frac{\partial^2 \Phi}{\partial x_k \partial x_j} = 0, \quad \frac{\partial^2 \Phi}{\partial x_k^2} > 0, \quad k, j = \overline{1,n}\ ,\ k \neq j.$$

It was showed that the remainder in (2) can be expressed as

$$R(f) = \frac{f^{(N)}(\xi)}{N!} \int_a^b P_{n,s}^{2s+2} d\lambda(x),\ N = 2(s+1)n. \tag{8}$$

Now, we consider the following expression of the remainder in the quadrature formula (2) $R(f;d\lambda)=\int_a^b U(x)D(f;x)d\lambda(x)$, where

$$u(x)=\prod_{k=1}^{n}(x-x_k)^{2s+1},\ U(x)=u(x)(x-x_1)\dots(x-x_n)=\prod_{k=1}^{n}(x-x_k)^{2s+2}$$

and $D(f;x)=\begin{bmatrix} x, & x_1, & x_2, & \dots & x_n; & f \\ 1 & 2s+1 & 2s+1 & \dots & 2s+1 & \end{bmatrix}$.

If $f\in C^N(a,b)$, by using the *Peano's Theorem*, then the remainder can be expressed as $R[f]=\int_a^b K_N(t)f^{(N)}(t)d\lambda(t)$, with $N=2(s+1)n$, where the *Peano's Kernel* have the expression :

$K_N(t)=R_x[\frac{(x-t)_+^{N-1}}{(N-1)!}]$, which is a spline function of degree $N-1$ with the interpolation points in the nodes of the quadrature formula and the compact support $[a,b]$. Then we have:

$$K_N(t)=\int_a^b\frac{(x-t)_+^{N-1}}{(N-1)!}d\lambda(t)-\sum_{k=1}^{n}\sum_{\nu=0}^{2s}(N-1)^{[\nu]}\frac{(x_k-t)_+^{N-\nu-1}}{(N-1)!}. \tag{9}$$

Let $n\in\mathbf{N}$, $\sigma=(s_1,\dots,s_n)$ be a sequence of nonnegative integers, and the nodes x_k ordered, say $a\le x_1<x_2<\dots<x_n\le b$, with odd multiplicities $2s_1+1,\dots,2s_n+1$, respectively.

A generalization of the quadrature formula of *Gauss-Turán* type was given independently by *Chakalov* [2] and *T.Popoviciu,* [8], for the nodes x_k with different multiplicities $2s_k+1$, $k=\overline{1,n}$ of the following form

$$\int_a^b f(x)d\lambda(x)=\sum_{k=1}^{n}\sum_{\nu=0}^{2s_k}A_{k,\nu}f^{(\nu)}(x_k)+R(f), \tag{10}$$

which have $d_{max}=2\sum_{k=1}^{n}s_k+2n-1$, if and only if

$$\int_a^b\prod_{\nu=1}^{n}(x-x_\nu)^{2s_\nu+1}x^k d\lambda(x)=0,\ k=\overline{0,n-1}. \tag{11}$$

The conditions (11) defines a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in\mathbf{N}_0}$, $\pi_{n,\sigma}(x) = \prod_{k=1}^{n}(x-x_k)$, such that $\int_a^b \pi_{k,\sigma}(x)\prod_{\nu=1}^{n}(x-x_\nu)^{2s_\nu+1}d\lambda(x)=0,\ k=\overline{0,n-1}$.

These polynomials are called σ-orthogonal polynomials and they corresponds to the sequence $\sigma=(s_1,s_2,\ldots,s_n)$ of nonnegative integers.

Definition 1 *The polynomials* $P_{n,\sigma}(x)=\prod_{\nu=1}^{n}(x-x_\nu^{n,\sigma})$ *are called* $\sigma-$*orthogonal, if they satisfies the orthogonality conditions* $\int_a^b P_{n,\sigma}(x)\ x^j\ w_{n,\sigma}(x)dx=0,\ j=\overline{0,n-1}$, *with respect to the weight* $w_{n,\sigma}(x)=w(x)\prod_{\nu=1}^{n}(x-x_\nu^{n,\sigma})^{2s_\nu}$.

It can be proved that the $\sigma-$*orthogonal* polynomial $P_{n,\sigma}$ can be obtained by the minimization of the integral $\int_a^b w(x)\prod_{\nu=1}^{n}(x-x_\nu)^{2s_\nu+2}dx$.

If we consider the vector of multiplicity orders $\sigma=(2s+1,2s+1,\ldots,2s+1)$, then the above polynomials reduces to the s-orthogonal polynomials.

Let consider the *Lagrange-Hermite* interpolation polynomial

$$(L_Hf)(x)=L\begin{pmatrix} x_k, & \gamma_j, & x; & f\\ 2s_k+1 & 1 & 1 & \end{pmatrix} \tag{12}$$

on the nodes x_k with the multiplicities $2s_k+1,\ k=\overline{1,n}$ and we apply *the parameters method of D.D. Stancu.*

Then L_Hf can be expressed in the following form

(13)

$$(L_Hf)(x)=v(x)L_H\begin{pmatrix} x_k, & x; & f_1\\ 2s_k+1 & 1 & \end{pmatrix}+u(x)L_H\begin{pmatrix} \gamma_j, & x; & f_2\\ 1 & 1 & \end{pmatrix}, \text{where}$$

$u(x)=(x-x_1)^{2s_1+1}(x-x_2)^{2s_2+1}\ldots(x-x_n)^{2s_n+1},\ v(x)=(x-\gamma_1)(x-\gamma_2)\ldots(x-\gamma_n)$,

$$f_1(x) = f(x)/v(x), \quad f_2(x) = f(x)/u(x).$$

Note that $v(x)$ is the polynomial of undetermined nodes. Then we have the following interpolation formula

(14) $$f(x) = (L_H f)(x) + (rf)(x), \text{ where}$$

(15)
$$(rf)(x) = u(x)v(x) \begin{bmatrix} x_1, & \dots, & x_n, & \gamma_1, & \dots, & \gamma_n & x; & f \\ 2s_1+1, & \dots, & 2s_n+1 & 1, & \dots, & 1 & 1 & \end{bmatrix}.$$

By multiplying the *Lagrange-Hermite* formula (13) with the weight function $w = w(x)$ and by integrating on (a, b) with respect to the measure $d\lambda(x) = w(x)dx$, we obtain the quadrature formula

(16) $$I(w; f) = Q(f) + G(f) + R(f),$$

where $R(f) = I(w, rf)$, and

(17) $$G(f) = \sum_{j=1}^{n} B_j f(\gamma_j).$$

One can observe that in (15) , the divided difference which appears have the order $N + 1 = 2\sum_{k=1}^{n} s_k + 2n = 2S + 2n$, where $S = \sum_{k=1}^{n} s_k$.

Thus, the degree of exactness of (16) is $N = 2S + 2n - 1$.

Remark 1 *One must determine the nodes* x_k, $k = \overline{1, n}$ *with the multiplicities* $2s_k + 1$, $(k = \overline{1, n})$, *so that* $B_1 = \cdots = B_n = 0$, *for any values of the* *parameters* γ_j, $j=\overline{1,n}$, *and it is necessary and sufficient that*

(18)
$$\int_a^b \prod_{\nu=1}^{n}(x-x_\nu)^{2s_\nu+1}x^k d\lambda(x)=0,\ k=\overline{0,n-1}\ ,\ \textit{where } d\lambda(x)=w(x)dx.$$

One can prove that the system (18) with the unknowns $x_1, x_2, \ldots, x_n$ has at least a solution with distinct values. If $f\in C^{N+1}(a,b)$, then the expression for the remainder will be $R(f)=f^{(2S+2n)}(\xi)K_{2S+2n}$, where

$$K_{2S+2n}=\frac{1}{(2S+2n)!}I(w;U_{2S+2n}),\ U_{2S+2n}=\prod_{k=1}^{n}(x-x_k)^{2s_k+2}.$$

a) The determination of the Gaussian nodes

Let denote $\tau_k := x_k$ the nodes of the quadrature (10), and $\{p_j\}_{j\in\mathbb{N}_0,}$ let be a sequence of orthonormal polynomials with respect to the measure, $d\lambda(t)$ on $\mathbb{R}$. Then, these polynomials satisfy the three-term recurrence relation

$$\sqrt{\beta_{j+1}}\ p_{j+1}(t)+\alpha_j\ p_j(t)+\sqrt{\beta_j}\ p_{j-1}(t)=tp_j(t),\ j=0,1,\ldots, \tag{19}$$

where $p_{-1}(t)=0,\ p_0(t)=1/\sqrt{\beta_0},\ \beta_0=\mu_0=\int_a^b d\lambda(t)$.

For a given sequence $\sigma=(s_1,s_2,\ldots,s_n)$, the orthogonality conditions (18) can be written as

$$F_j(t)=\int_{\mathbb{R}}p_{j-1}(t)\left[\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\right]d\lambda(t)=0,\ j=\overline{1,n}, \tag{20}$$

where $\mathbf{t}=(\tau_1,\ldots,\tau_n)^T$, $\mathbf{F}(t)=[F_1(t),\ F_2(t),\ \ldots,\ F_n(t)]^T$, which is a non linear system of equations.

To solve the system (20) can be used the *Newton-Kantorovic* method (see [7]). One can construct the iterative formula

$$\mathbf{t}^{(k+1)}=\mathbf{t}^{(k)}-W^{-1}(\mathbf{t}^{(k)})\mathbf{F}(\mathbf{t}^{(k)}),\ k=0,1,2,\ldots$$

where $\mathbf{t}^{(k)} = (\tau_1^{(k)}, \tau_2^{(k)}, \ldots, \tau_n^{(k)})^T$, and $W = W(\mathbf{t}) = [w_{j,k}]_{n\times n} = [\frac{\partial F_j}{\partial \tau_k}]_{n\times n}$, is the *Jacobian* of $\mathbf{F(t)}$, whose elements can be calculated by

$$w_{j,k} = \frac{\partial F_j}{\partial \tau_k} = -(2s_k+1)\int_{\mathbb{R}} \frac{p_{j-1}(t)}{t-\tau_k}\left[\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\right]d\lambda(t),\ \ j,k=\overline{1,n}.$$

But, $w_{0,k} = 0$ and

$$w_{1,k} = -\frac{2s_k+1}{\sqrt{\beta_0}}\int_{\mathbb{R}}(t-\tau_k)^{2s_k}\Big[\prod_{\nu=1,\nu\neq k}^{n}(t-\tau_\nu)^{2s_\nu+1}\Big]d\lambda(t), \tag{21}$$

then, by integrating (19) one obtain

(22)

$$\sqrt{\beta_{j+1}}\ w_{j+2,k} = (\tau_k-\alpha_j)\ w_{j+1,k} - \sqrt{\beta_j}\ w_{j,k} - (2s_k+1)F_{j+1},\ \ j=\overline{0,n-2}.$$

Thus, knowing only F_j and $w_{1,j}$, $(j=\overline{1,n})$, one can calculate the elements of the Jacobian matrix by the nonhomogenous recurrence relation (22).

The integrals (20), (21), can be calculated by using a *Gauss-Christoffel* quadrature formula, (w.r.t. the measure $d\lambda(t)$) of the following form

$$\int_a^b g(t)d\lambda(t) = \sum_{k=1}^{L} A_k^{(L)} g(\tau_k^{(L)}) + R_L(g),$$

with $L = \sum_{k=1}^n s_k + n$, which is exact for $\forall f \in \mathbb{P}_{2L-1}$, where $2L-1 = 2\sum_{k=1}^n s_k + 2n - 1$.

For a sufficiently good approximation $t^{(0)}$, the convergence of the method for the calculation of $t^{(k+1)}$ is quadratic (see [7]).

If one consider $\sigma = (s, s, \ldots, s)$, and the quadrature formula (2) then, in order to determine the coefficients α_ν, β_ν from the recurrence relation (5), can be used the discretized Stieltjes procedure for infinite intervals of orthogonality. From (5) one obtain the following nonlinear system

$$f_0 \equiv \beta_0 - \int_{\mathbb{R}} \pi_n^{2s}(t) d\lambda(t) = 0, f_{2\nu+1} \equiv \int_{\mathbb{R}} (\alpha_\nu - t)\pi_\nu^2(t)\pi_n^{2s}(t) d\lambda(t) = 0, (\nu = \overline{0, n-1}),$$

$$f_{2\nu} \equiv \int_{\mathbb{R}} \left[\beta_\nu \pi_{\nu-1}^2(t) - \pi_\nu^2(t)\right] \pi_n^{2s}(t) d\lambda(t) = 0, (\nu = \overline{0, n-1}).$$

The polynomials $\pi_0, \pi_1, \ldots, \pi_n$ can be expressed in terms of $\alpha_\nu, \beta_\nu,\ \nu = \overline{0, n},$ by the recurrence relation (5).

By using the *Newton-Kantorovic's* method, one obtain the following relations for the determination of the coefficients in (5), namely $x^{(k+1)} = x^{(k)} - W^{-1}(x^{(k)}) f(x^{(k)}),\ k = 0, 1, \ldots,$ where the zeros $\tau = \tau(s, n),\ (\nu = \overline{1, n})$ of $\pi_n^{s,n}$ are the nodes of *Gauss-Turan's* type quadrature formula.

Note that these zeros can be obtained by using the QR algotithm, which determines the eigenvalues of a symmetric tridiagonal Jacobi matrix J_n

$$J_n = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & 0 & \ldots & 0 & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \sqrt{\beta_{n-2}} & \alpha_{n-2} & \sqrt{\beta_{n-1}} \\ 0 & 0 & 0 & \ldots & 0 & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{pmatrix}.$$

This algorithm can be used to determine the $s-$ or $\sigma-$orthogonal polynomials by constructing MATLAB routines for some *Gauss-Christoffel* quadrature formulas and routines to solve some systems of equations.

b) The determination of the coefficients

Let denote $U(t) = \prod_{k=1}^{n} (t - \tau_k)^{2s_k+1}$, and let consider the Hermite interpolation formula

$$(23)\qquad f(t)=(Hf)(t)+(Rf)(t)=\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu}h_{\nu,i}(t)f^{(i)}(\tau_\nu)+(Rf)(t),\text{where}$$

$$h_{\nu,i}(t)=\frac{(t-\tau_\nu)^i}{i!}\Big[\sum_{k=0}^{2s_\nu-i}\frac{(t-\tau_\nu)^k}{k!}\Big(\frac{1}{U_\nu(t)}\Big)^{(k)}_{t=\tau_\nu}\Big]U_\nu(t),U_\nu(t)=\prod_{k=1}^{n}(t-\tau_k)^{2s_k+1}/(t-\tau_\nu)^{2s_\nu+1}.$$

By integrating (23), one obtain

$$A_{\nu,i}=\int_a^b h_{\nu,i}(t)d\lambda(t)=\frac{1}{i!}\sum_{k=0}^{2s_\nu-i}\frac{1}{k!}\left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{U(t)}\right]^{(k)}_{t=\tau_\nu}\int_a^b(t-\tau_\nu)^{i+k}\frac{U(t)d\lambda(t)}{(t-\tau_\nu)^{2s_\nu+1}}=$$

$$=\frac{1}{i!}\sum_{k=0}^{2s_\nu-i}\frac{1}{k!}\left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{U(t)}\right]^{(k)}_{t=\tau_\nu}\int_a^b\frac{U(t)}{(t-\tau_\nu)^{2s_\nu-i-k+1}}d\lambda(t).$$

Let denote $U_{\nu;i+k}(t)=\frac{U(t)}{(t-\tau_\nu)^{2s_\nu-i-k+1}}=$

$$=(t-\tau_\nu)^{i+k}\times(t-\tau_1)^{2s_1+1}\dots(t-\tau_{\nu-1})^{2s_{\nu-1}+1}(t-\tau_{\nu+1})^{2s_{\nu+1}+1}\dots(t-\tau_n)^{2s_n+1},\text{where}$$

$$deg(U_{\nu;i+k})\le 2s_\nu+(2s_1+1)+\dots+(2s_{\nu-1}+1)+(2s_{\nu+1}+1)+\dots+(2s_n+1)=$$

$$=2\sum_{\nu=1}^{n}s_\nu+n-1\le 2(\sum_{\nu=1}^{n}s_\nu+n)-1=2N-1=d_{max},\ N=2(S+n)$$

So, one obtain

$$(24)\qquad A_{\nu,i}=\frac{1}{i!}\sum_{k=0}^{2s_\nu-i}\frac{1}{k!}\left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{U(t)}\right]^{(k)}_{t=\tau_\nu}\int_a^b U_{\nu;i+k}(t)d\lambda(t),$$

for $\nu=\overline{1,n};\ i=0,1,\dots,2s_\nu$ and $deg(U_{\nu;i+k})\le 2N-1$.

The integrals $\int_a^b U_{\nu,i+k}(t)d\lambda(t),\ \nu=\overline{1,n};\ i=\overline{0,2s_\nu},\ k=\overline{0,2s_\nu-i}$, can be calculated by applying the quadrature formula

$$\int_a^b g(t)d\lambda(t)=\sum_{k=1}^{N}A_k^{(N)}g(\tau_k^{(N)})+R_N(g),$$

with $N=\sum_{\nu=1}^{n}s_\nu+n$ nodes.

3 A generalization given by D.D.Stancu to the Gauss-Turán type quadrature formula

A generalization of the *Turán* quadrature formula (2) to quadratures having nodes with arbitrary multiplicities was derived independently by *Chakalov* [2] and *T. Popoviciu* [8].

D.D. Stancu in [14], [16], was bring very important contributions in this domain, by investigating and constructing so-called Gauss-Stancu quadrature formulas having multiple fixed nodes and simple or multiple free (Gaussian) nodes.

Let $a_i,\ i=\overline{1,n}$ fixed (or prescribed) nodes, with the given multiplicities $m_i,\ i=\overline{1,n}$, and $x_1 < x_2 < \cdots < x_m$ be the free nodes with given multiplicities $n_1,\ldots,n_m$. Then, we have the general quadrature of *Gauss-Stancu* type for the integral
$I[f]=\int_a^b f(x)d\lambda(x),\ (d\lambda(x)=w(x)dx)$ of the form

(25) $$Q[f]=\sum_{i=1}^{n}\sum_{\nu=0}^{m_i-1}B_{i,\nu}f^{(\nu)}(a_i)+\sum_{k=1}^{m}\sum_{\nu=0}^{n_k-1}A_{k,\nu}f^{(\nu)}(x_k).$$

We denote

(26)
$$\omega(x)=\alpha\prod_{i=1}^{n}(x-a_i)^{m_i},\ u(x)=\prod_{k=1}^{m}(x-x_k)^{n_k},\ M=\sum_{i=1}^{n}m_i, N=\sum_{k=1}^{m}n_k.$$

The quadrature formula (25) have *interpolatory type* with the algebraic degree of exactness at least $d^*=M+N-1$, if $I(f)=Q(f),\ \forall f\in\mathbb{P}_{M+N-1}$.

The free nodes $x_k,\ k=\overline{1,m}$ can be chosen to increase the degree of exactness, and so one can obtain $I[f] = Q[f], \ \forall \ f \in \mathbb{P}_{M+N+n-1}$.

D.D. Stancu gave the following characterizations

Theorem 1 *The nodes $x_1, \dots, x_m$ are the Gaussian nodes if and only if*

$$\int_a^b x^k \omega(x) u(x) d\lambda(x) = 0, \ \forall k = \overline{0, m-1}. \tag{27}$$

Theorem 2 *If the multiplicities of the Gaussian nodes are all odd , $n_k = 2s_k + 1$, $(k = \overline{1, m})$ and if the multiplicities of the fixed nodes are even, $m_i = 2r_i$, $i = \overline{1, n}$, then there exist the real distinct nodes: $x_k, \ k = \overline{1, m}$, which are the Gaussian nodes for the quadrature formula of* Gauss-Turán-Stancu *type (25).*

In this case, the orthogonality conditions (27) can be written as

$$\int_a^b x^k \pi_m(x) d\mu(x), \ k = \overline{0, m-1}, \text{ where } \pi_m(x) = \prod_{k=1}^{m} (x - x_k),$$

$$d\mu(x) = (\prod_{k=1}^{m} (x - x_k)^{2s_k})(\prod_{i=1}^{n} (x - a_i)^{2r_i}) d\lambda(x).$$

This fact means that the polynomial $\pi_m(x)$ is orthogonal with respect to the new nonnegative measure $d\mu(x)$, and therefore , all zeros $x_1, \dots, x_m$ are simple, real and belongs to $supp(d\mu) = supp(d\lambda)$.

One can observe that the measure $d\mu(x)$, contains the nodes $x_1, \dots, x_m$, i.e. the unknown polynomial $\pi_m(t)$ is implicitly defined.

Let now consider the sets of fixed and Gaussian nodes $F_n = \{a_1, \dots, a_n\}$, $G_m = \{x_1, \dots, x_m\}$ and let $F_n \bigcap G_m = \emptyset$, and denote $X_p = \{\xi_1, \dots, \xi_p\} := F_n \bigcup G_m, \ (p = n + m)$ with the multiplicity of the node ξ_k be $r_k, \ k = \overline{1, p}$.

Then can be determined the coefficients $C_{i,\nu}$ (i.e. $A_{i,\nu}$ and $B_{i,\nu}$) by using an interpolatory formula of the form

$$\int_a^b f(t)d\lambda(t) = \sum_{i=1}^{p}\sum_{\nu=0}^{r_\nu-1} C_{i,\nu}f^{(\nu)}(\xi_i) + R_p(f). \tag{28}$$

Note that the multiplicity of the Gaussian nodes are odd numbers.

Example 3.1

If $(a,b) = (-1,1)$, $w(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$, and $a_0 = -1, a_1 = 1$ are simple fixed nodes, x_0 is a simple free node, then the highest degree of exactness will be $D = (1+1)+1 = 3$ which will be obtained for $x_0 = \frac{\beta-\alpha}{\alpha+\beta+4}$. The corresponding quadrature formula of *Gauss-Christoffel-Stancu* type will be

$$\int_{-1}^1 (1-x)^\alpha(1+x)^\beta f(x)dx = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\alpha+2)(\beta+2)\Gamma(\alpha+\beta+4)}\big[(\alpha+1)(\alpha+2)^2 f(-1)+$$
$$+(\alpha+1)(\beta+1)(\alpha+\beta+4)^2 f(\frac{\beta-\alpha}{\alpha+\beta+4}+(\beta+1)(\beta+2)^2 f(1)\big]-$$
$$-2^{\alpha+\beta+2}\frac{\Gamma(\alpha+3)\Gamma(\beta+3)}{3(\alpha+\beta+4)\Gamma(\alpha+\beta+6)}f^{IV}(\xi).$$

Example 3.2 Let $u(x) = \prod_{i=0}^{m+1}(x-x_i)^{r_i}$, be the polynomial of nodes with the following multiplicities $x_0 = a,\ r_0 = p+1, x_{m+1} = b,\ r_{m+1} = q+1$, the fixed nodes and the Gaussian nodes $x_i,\ r_i = 2s+1,\ (i = \overline{1,m})$.

Then we can construct the quadrature formula of Gauss-Stancu type with fixes nodes $x_0 = a,\ x_{m+1} = b$, with the above given multiplicity orders.

$$\int_a^b f(x)w(x)dx = \sum_{i=0}^{p} A_{0,i}f^{(i)}(a) + \sum_{j=0}^{q} A_{m+1,j}f^{(j)}(b) + \sum_{k=1}^{m}\sum_{\nu=0}^{2s} A_{k,\nu}f^{(\nu)}(x_k) + R(f), \tag{29}$$

with the polynomial of fixed nodes $\omega(x) = (x-a)^{p+1}(b-x)^{q+1}$. For a given $s \in \mathbb{N}$, the polynomial $P_{m,s}$ is orthogonal on $[a,b]$ with respect to the weight function $w(x)$, if this polynomial is chosen as the solution of the extremal problem $\int_a^b P_{m,s}^{2s+2} w(x)dx = \min$, which is equivalent with the condition that $\int_a^b P_{m,s}^{2s+1}(x)x^k w(x)dx = 0, k = \overline{0, m-1}$.

Then the last one condition can be interpreted as a orthogonality condition with respect to the weight function $p(x) = \omega(x)P_{m,s}^{2s}(x)$.

We use a method given by D.D Stancu in [12]. Let consider the auxiliary function

$$\varphi_i(x) = \frac{1}{u_i(x)} \int_a^b \frac{u(x)-u(t)}{x-t} w(t)dt, \text{where } u_i(x) = u(x)/(x-x_i)^{r_i}. \tag{30}$$

We have

$$\sum_{k=1}^{m}\sum_{\nu=0}^{2s} A_{k,\nu} f^{(\nu)}(x_k) = \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\left[\int_a^b h_{i,k}(x)w(x)dx\right] f^{(k)}(x_i), \text{where}$$

$$h_{i,k}(x) = \frac{(x-x_i)^k}{k!}\sum_{j=0}^{r_i-1-k}\left[\frac{(x-x_i)^j}{j!}\left(\frac{1}{u_i(x)}\right)_{x_i}^{(j)}\right] u_i(x).$$

Let $n_i = r_i - k - 1$, and calculate the expression using the Leibniz's formula

$$\varphi_i^{(n_i)}(x) = \sum_{j=0}^{n_i}\binom{n_i}{j}\left(\frac{1}{u_i(x)}\right)^{(j)}\left[\int_a^b \frac{u(x)-u(t)}{x-t}w(t)dt\right]^{(n_i-j)}, \text{where}$$

$$\left[\int_a^b \frac{u(x)-u(t)}{x-t}w(t)dt\right]^{(k)} = \sum_{\nu=0}^{k}\binom{k}{\nu}\int_a^b \left(\frac{1}{x-t}\right)^{(\nu)}[u(x)-u(t)]^{(k-\nu)}w(t)dt.$$

If $x = \alpha$ is a zero of order $r,\ r > k$ for the polynomial $u(x)$, then one obtain

$$\left[\int_a^b \frac{u(x)-u(t)}{x-t}w(t)dt\right]_{x=\alpha}^{(k)} = -\int_a^b \left(\frac{1}{x-t}\right)_{x=\alpha}^{(k)} u(t)w(t)dt = \dots$$

$$= k!\int_a^b \frac{u(t)}{(x-\alpha)^{k+1}}w(t)dt.$$

Then one obtain the expression

$$\varphi_i^{(r_i-k-1)}(x_i) = (r_i-k-1)!\sum_{j=0}^{r_i-k-1}\frac{1}{j!}\Big(\frac{1}{u_i(x)}\Big)_{x_i}^{(j)}\int_a^b \frac{u(x)}{(x-x_i)^{r_i-k-j}}w(x)dx =$$

$$= (r_i-k-1)!\int_a^b \frac{u(x)}{(x-x_i)^{r_i-k}}\Big[\sum_{j=0}^{r_i-k-1}\frac{(x-x_i)^j}{j!}\Big(\frac{1}{u_i(x)}\Big)_{x_i}^{(j)}\Big]w(x)dx.$$

By integrating the Lagrange-Hermite interpolation formula and using the expression of $h_{i,k}(x)$, finally one obtain the following expression for the coefficients of the quadrature formula

$$A_{i,k} = \frac{1}{k!(r_i-k-1)!}\varphi_i^{(r_i-k-1)}(x_i).$$

Note that the quadrature formula (29) is called the Turan-Ionescu formula.

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