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Approximation of the attractor of a countable iterated function system [1]

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Abstract

In this paper we will describe a construction of a sequence of sets which is converging, with respect to the Hausdorff metric, to the attractor of a countable iterated function system on a compact metric space. The importance of that method consists in the fact that the approximation sequence can be constructed of finite sets, hence it is very useful for computer graphic representation.

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1 Introduction

In the famous paper [4], J.E. Hutchinson proves that, given a set of contractions (IFS) $(\omega_n)_{n=1}^k$ in a complete metric space X, there exists a unique nonempty compact set $A \subset X$, named the *attractor* of IFS. This attractor is, generally, a fractal set. These ideas has been extended to infinitely many contractions, a such generalization can be founded in [5] for countable iterated function systems (CIFS) on a compact metric space.

The approximation of the attractor of a IFS has been studied by S. Dubuc, A. Elqortobi, P.M. Centore, E.R. Vrscay, E. de Ámo, I. Chiţescu, C. Díaz, N.A. Secelean (see [1]) and many others.

If we consider a CIFS $(\omega_n)_{n\geq 1}$ whose attractor is A, then A can by approximated (see [5]) by the attractors A_k of the partial IFS $(\omega_n)_{n=1}^k$, $k = 1, 2, \cdots$. However, these attractors are, generally, infinite sets so it cannot be represented by using the computer.

Here, we will construct a sequence of finite sets (which can be subsets of A, hence we use not one point "outward" of A) converging with respect to the Hausdorff metric to A.

As a particular case, we will approximate, by finite subsets, the graph of the countable interpolation function associated of a countable system of data and a corresponding CIFS.

Finally, an example in $\mathbb{R}^2$ which use this method is given.

2 Preliminary Facts

2.1 Iterated Function Systems, Countable Iterated Function Systems

In this subsection we give some well known aspects on Fractal Theory used in the sequel (more complete and rigorous treatments may be found in [4], [3], [5], [7]).

Let (X, d) be a complete metric space and $\mathcal{K}(X)$ be the class of all compact non-empty subsets of X.

The function $\delta : \mathcal{K}(X)\times\mathcal{K}(X) \longrightarrow \mathbb{R}_+$, $\delta(A, B) = \max\{\mathrm{d}(A, B), \mathrm{d}(B, A)\}$, where $\mathrm{d}(A, B) = \sup\limits_{x\in A}\big(\inf\limits_{y\in B} \mathrm{d}(x, y)\big)$, for all $A, B \in \mathcal{K}(X)$, is a metric, namely the *Hausdorff metric.* The set $\mathcal{K}(X)$ is a complete metric space with respect to this metric δ.

THEOREM 1. [5, Th. 1.1] *Let $(A_n)_n$ be an increasing sequence of compact sets in a complete metric space such that the set $A = \bigcup\limits_{n=1}^{\infty} A_n$ is relatively compact. Then $\overline{A} = \lim\limits_{n} A_n$, the limit being taken with respect to the Hausdorff metric, the bar means the closure.*

A set of contractions $(\omega_n)_{n=1}^{k}$, $k \geq 1$, is called an *iterated function system* (IFS). Such a system of maps induces a set function $\mathcal{S}_k : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$, $\mathcal{S}_k(E) = \bigcup\limits_{n=1}^{k} \omega_n(E)$ which is a contraction on $\mathcal{K}(X)$ with contraction ratio $r \leq \max\limits_{1\leq n\leq k} r_n$, r_n being the contraction ratio of ω_n, $n = 1, \ldots, k$. According to the Banach contraction principle, there is a unique set $A_k \in \mathcal{K}(X)$ which is invariant with respect to $\mathcal{S}_k$, that is $A_k = \mathcal{S}_k(A_k) = \bigcup\limits_{n=1}^{k} \omega_n(A_k)$. We say that the set $A_k \in \mathcal{K}(X)$ is the *attractor* of IFS $(\omega_n)_{n=1}^k$.

Now, we suppose further that (X, d) is a compact metric space. The $(\mathcal{K}(X), \delta)$ is also a compact metric space.

A sequence of contractions $(\omega_n)_{n\geq 1}$ on X whose contraction ratios are, respectively, $r_n > 0$, such that $\sup\limits_n r_n < 1$ is called a *countable iterated function system* (CIFS).

If we consider the CIFS $(\omega_n)_{n\geq 1}$, then the set function $\mathcal{S} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$, given by

$$\mathcal{S}(E) = \overline{\bigcup_{n\geq 1} \omega_n(E)} \tag{1}$$

(the bar means the closure of the respective set) is a contraction map on $(\mathcal{K}(X), h)$ with contraction ratio $r \leq \sup\limits_n r_n$. Thus, there exists a unique non-empty compact set $A \subset X$ invariant for the family $(\omega_n)_{n\geq 1}$, that is $A = \mathcal{S}(A) = \overline{\bigcup\limits_{n\geq 1} \omega_n(A)}$. Further, if $B \in \mathcal{K}(X)$, then, by a successive approximation process,

$$\mathcal{S}^p(B) \underset{p}{\longrightarrow} A \tag{2}$$

(with respect to the Hausdorff metric) where $\mathcal{S}^p := \underbrace{\mathcal{S} \circ \cdots \circ \mathcal{S}}_{p\ times}$. The set A invariant under the set function $\mathcal{S}$ is called the *attractor* of CIFS $(\omega_n)_{n\geq 1}$ and it can be obtained as limit, with respect to the Hausdorff metric, of sequence of attractors $(A_k)_{k\geq 1}$ of partial IFS $(\omega_n)_{n=1}^k$, $k = 1, 2, \ldots$ (see [5]).

2.2 Countable fractal interpolation

Now we will describe an extension of the fractal interpolation to the case of the countable system of data (more details can be found in [7]).

Let (Y, d_Y) be a compact metric space. A *countable system of data* is a set of points having the form $\Delta := \{(x_n, F_n) \in \mathbb{R} \times Y : \ n = 0, 1, \dots\}$ where the sequence $(x_n)_{n\geq 0}$ is strictly increasing and bounded and $(F_n)_{n\geq 0}$ is convergent. Denote $a = x_0$, $b = \lim\limits_n x_n$ and $X = [a, b] \times Y$.

An *interpolation function* corresponding to this system of data is a continuous map $f : [a, b] \to Y$ such that $f(x_n) = F_n$ for $n = 0, 1, \dots$. The points $(x_n, F_n) \in \mathbb{R}^2$, $n \geq 0$, are called the *interpolation points*. It can construct a CIFS on X which is associated with Δ.

THEOREM 2. [7, Th.2] *There exists an interpolation function f corresponding to the considered countable system of data such that the graph of f is the attractor A of the associated CIFS. That is*

$$A = \big\{(x, f(x)) : \ x \in [a, b]\big\}.$$

3 Approximation of the attractor of a countable iterated function system

LEMMA 1. *Let us consider two families $(A_i)_{i\in\Im}$, $(B_i)_{i\in\Im}$ of compact subsets of the metric space (X, d) such that the both sets $\bigcup\limits_{i\in\Im} A_i$ and $\bigcup\limits_{i\in\Im} B_i$ are compact. Then*

$$\delta(\bigcup_{i\in\Im} A_i, \bigcup_{i\in\Im} B_i) \leq \sup_i \delta(A_i, B_i).$$

Proof. To establish the inequality from statement, it is enough, because of symmetry, to prove

$$\mathrm{d}(\bigcup_{i\in\Im} A_i, \bigcup_{i\in\Im} B_i) \leq \sup_i \mathrm{d}(A_i, B_i).$$

Let be $x \in \bigcup_{i\in\Im} A_i$. There is $i_x \in \Im$ such that $x \in A_{i_x}$. Therefore

$$\inf_{y\in\bigcup_i A_i} \mathrm{d}(x,y) \le \inf_{y\in B_{i_x}} \mathrm{d}(x,y) \le \mathrm{d}(A_{i_x}, B_{i_x}) \le \sup_i \mathrm{d}(A_i, B_i)$$

and hence

$$\mathrm{d}(\bigcup_{i\in\Im} A_i, \bigcup_{i\in\Im} B_i) = \sup_{x\in\bigcup_i A_i} \big(\inf_{y\in\bigcup_i B_i} \mathrm{d}(x,y)\big) \le \sup_i \mathrm{d}(A_i, B_i).$$

The following lemma describes a standard topological fact:

Lemma 2. *If $(E_i)_{i\in\Im}$ is a family of subsets of a topological space, then* $\overline{\bigcup_{i\in\Im} \overline{E_i}} = \overline{\bigcup_{i\in\Im} E_i}$.

Let $(B_k)_k$ be a sequence of compact nonempty sets on the compact metric space (X, d) converging (with respect to the Hausdorff metric δ) to the compact set $B \subset X$, $B \neq \emptyset$.

We also consider a CIFS $(\omega_n)_n$ on X and denote by A its attractor.

Theorem 3. *Under the above context, A can be approximated by the sequence of compact nonempty sets $(\mathcal{S}_k^p(B_k))_{p,k}$. More precisely, we have*

$$\lim_p \lim_k \mathcal{S}_k^p(B_k) = A,$$

the limiting process being taken with respect to the Hausdorff metric and $\mathcal{S}_k^p := \underbrace{\mathcal{S}_k \circ \cdots \circ \mathcal{S}_k}_{p \text{ times}}$.

Proof. For each $p = 1, 2, \ldots$, we denote $\omega_{i_1 i_2 \ldots i_p} := \omega_{i_1} \circ \cdots \circ \omega_{i_p}$, where $i_1, \ldots, i_p$ are positive integers. Obviously, $\omega_{i_1 i_2 \ldots i_p}$ is a contraction with contraction ratio $r_{i_1} \cdot \ldots \cdot r_{i_p}$, r_n assigning the contraction ratio of ω_n.

Let be $p \geq 1$ and, for each $k \geq 1$, $N_k := \{1, 2, \ldots, k\}$.

First, we prove that

$$\delta\big(\mathcal{S}_k^p(B_k), \mathcal{S}^p(B)\big) \underset{k}{\longrightarrow} 0. \tag{3}$$

One has

(4)

$$\delta\big(\mathcal{S}_k^p(B_k), \mathcal{S}^p(B)\big) \leq \delta\big(\mathcal{S}_k^p(B_k), \mathcal{S}_k^p(B)\big) + \delta\big(\mathcal{S}_k^p(B), \mathcal{S}^p(B)\big), \quad k = 1, 2, \cdots .$$

It is simple to verify that $\mathcal{S}_k^p(E) = \bigcup\limits_{i_1,\ldots,i_p \in N_k} \omega_{i_1\ldots i_p}(E)$ for any arbitrary set $E \subset X$.

Now, in view of Lemma 1,

$$\delta\big(\mathcal{S}_k^p(B_k), \mathcal{S}_k^p(B)\big) = \delta\Big(\bigcup_{i_1,\ldots,i_p \in N_k} \omega_{i_1\ldots i_p}(B_k), \bigcup_{i_1,\ldots,i_p \in N_k} \omega_{i_1\ldots i_p}(B)\Big) \leq \tag{5}$$

$$\leq \sup_{i_1,\ldots,i_p \in N_k} \delta\big(\omega_{i_1\ldots i_p}(B_k), \omega_{i_1\ldots i_p}(B)\big).$$

Since, for all $k = 1, 2, \ldots,$

$$\mathrm{d}\big(\omega_{i_1\ldots i_p}(B_k), \omega_{i_1\ldots i_p}(B)\big) = \sup_{x \in \omega_{i_1\ldots i_p}(B_k)} \inf_{y \in \omega_{i_1\ldots i_p}(B)} \mathrm{d}(x, y) =$$

$$= \sup_{a \in B_k} \inf_{b \in B} \mathrm{d}\big(\omega_{i_1\ldots i_p}(a), \omega_{i_1\ldots i_p}(b)\big) \leq r_n \sup_{a \in B_k} \inf_{b \in B} \mathrm{d}(a, b) = r_{i_1} r_{i_2} \ldots r_{i_p} \mathrm{d}(B_k, B)$$

and, analogously,

$$\mathrm{d}\big(\omega_{i_1\ldots i_p}(B), \omega_{i_1\ldots i_p}(B_k)\big) \leq r_{i_1} r_{i_2} \ldots r_{i_p} \mathrm{d}(B_k, B),$$

one obtain

$$\delta\big(\omega_{i_1\ldots i_p}(B), \omega_{i_1\ldots i_p}(B_k)\big) \leq r_{i_1} r_{i_2} \ldots r_{i_p} \delta(B_k, B) \leq \delta(B_k, B), \ \forall\, i_1, \ldots, i_p \in N_k,$$

so, from (5),

$$\delta\big(\mathcal{S}_k^p(B_k),\mathcal{S}_k^p(B)\big)\le\delta(B_k,B),\ \forall\, k,p\ge 1.$$

Thus, taking into account the hypothesis,

$$\delta\big(\mathcal{S}_k^p(B_k),\mathcal{S}_k^p(B)\big)\underset{k}{\longrightarrow}0,\quad \forall\, p=1,2,\cdots. \tag{6}$$

Afterwards, we observe that

$$\mathcal{S}^p(B)=\overline{\bigcup_{i_1,\dots,i_p\ge 1}\omega_{i_1\dots i_p}(B)}. \tag{7}$$

Indeed, we can proceed by induction. Thus, if we suppose that (7) is true for $p\ge 1$. In view of Lemma 2 and by using the continuity of the functions ω_n, we have

$$\mathcal{S}^{p+1}(B)=\mathcal{S}\Big(\overline{\bigcup_{i_1,\dots,i_p\ge 1}\omega_{i_1\dots i_p}(B)}\Big)=\overline{\bigcup_{i=1}^{\infty}\omega_i\Big(\overline{\bigcup_{i_1,\dots,i_p\ge 1}\omega_{i_1\dots i_p}(B)}\Big)}\subset$$

$$\subset\overline{\bigcup_{i=1}^{\infty}\overline{\omega_i\Big(\bigcup_{i_1,\dots,i_p\ge 1}\omega_{i_1\dots i_p}(B)\Big)}}=\overline{\bigcup_{i=1}^{\infty}\omega_i(\bigcup_{i_1,\dots,i_p\ge 1}\omega_{i_1\dots i_p}(B))}=\mathcal{S}^{p+1}(B).$$

Since the sequence of sets $\Big(\bigcup\limits_{i_1,\dots,i_p\in N_k}\omega_{i_1\dots i_p}(B)\Big)_{k\ge 1}$ is clearly increasing, we can apply Theorem 1 and obtain, by using (7),

$$\lim_k\mathcal{S}_k^p=\lim_k\bigcup_{i_1,\dots,i_p\in N_k}\omega_{i_1\dots i_p}(B)=\overline{\bigcup_{k=1}^{\infty}\Big(\bigcup_{i_1,\dots,i_p\in N_k}\omega_{i_1\dots i_p}(B)\Big)}= \tag{8}$$

$$=\overline{\bigcup_{i_1,\dots,i_p\ge 1}\omega_{i_1\dots i_p}(E)}=\mathcal{S}^p(B).$$

Now, from (6) and (8) becomes (3), so $\lim\limits_k\mathcal{S}_k^p(B_k)=\mathcal{S}^p(B)$.

Finally, from (7) and (2), it follows

$$\lim_p \lim_k \mathcal{S}_k^p(B_k) = \lim_p \mathcal{S}^p(B) = A,$$

completing the proof.

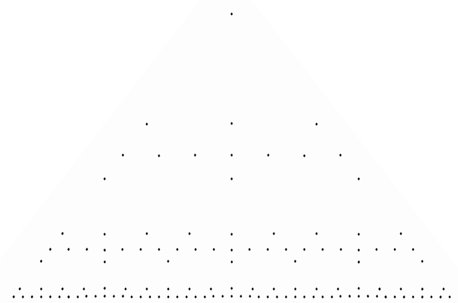
Remark 1. *By taking in the preceding theorem* $B_k := \{e_1, e_2, \dots, e_k\}$, $k = 1, 2, \dots$, e_n *being the fixed point of* ω_n *for every* $n \geq 1$, *one obtain an increasing sequence of subsets of* A *converging, with respect to the Hausdorff metric, to* $B := \overline{\bigcup_{k\geq 1} B_k} = \overline{\{e_1, e_2, \dots\}}$. *Thus, the attractor* A *of CIFS* $(\omega_n)_n$ *can be approximated "from inside", because* $\mathcal{S}_k^p(B_k) \subset A$, *from all* $k, p \geq 1$.

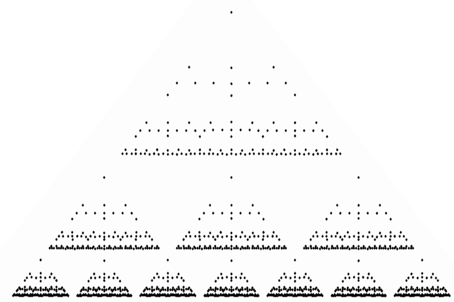
Remark 2. *If* $(B_k)_k$ *are finite sets, then* $\left(\mathcal{S}_k^p(B_k)\right)_{p,k}$ *are finite sets too. As follows, the attractor* A *can be approximated by using a sequence of finite sets. This fact is very useful for the computer graphic representation of the CIFS's attractor in* $\mathbb{R}^2$.

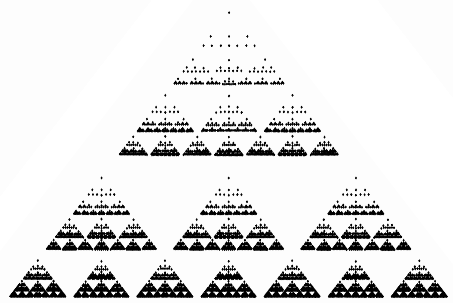
Remark 3. *Let us consider a countable system of data* $\Delta = (x_n, F_n)_{n\geq 1} \subset \mathbb{R} \times Y$ *(see section 2.2) and let be* $B_k := \{(x_n, F_n);\ n = 1, 2, \dots, k\}$. *Then* $B_k \subset A$ *for any* k *and* $\lim_k B_k = \Delta$, *the convergence being considered with respect to the Hausdorff metric. It follows that the graph of the countable interpolation function is approximated "from inside" by a sequence of finite sets.*

Finally, we give an example which shows some progressive steps to approximate an attractor of Sierpinski-infinite type (see [5]), from inside, by some finite sets.

Example *On the space* $X = \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x\}$ *we consider the sequence of contractions* $\omega_{ij}(x,y) = \Big(\frac{1}{2^i}x + (j-1)\frac{1}{2^i}, \frac{1}{2^i}y + (2^i - j - 1)\frac{1}{2^i}\Big)$ *for any* $i = 1, 2, \ldots, \ j = 1, 2, \ldots, \frac{2^i - 1}{2 - 1}$. *Three steps on the attractor's approximation process are represented as follows.*







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