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On Fuzzy $\varphi\psi$-Continuous Function Between L-fuzzy Uniform Spaces

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Abstract

In this paper, by means of operations (is called in [1, 2, 3] φ, ψ) we shall define $\varphi\psi$-continuity between two L-fuzzy quasi-uniform spaces. We shall prove that $\varphi\psi$-continuity between two L-fuzzy quasi-uniformity induces $\varphi\psi$-continuity between L-fuzzy topology generated by them. We shall investigate some Theorems on L-fuzzy uniform spaces.

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1 Introduction

Fuzzy versions of uniformity theory were established by B. Hutton , R. Lowen , U. Höhle , A. K. Katsaras , etc. Fuzzy uniformity in Hutton's sense has been accepted by many authors and has attracted wide attention in the literature.

In an analogous way to that in general topology, A. K. Katsaras introduced proximity structures in fuzzy spaces and investigated the F-topology generated by these proximities. However, as was pointed in [6], that those F-topologies are always crisp. Thus a new and more reasonable definition was given in [6] independently. An bijective correspondence between the fuzzy uniformities in the sense of Hutton and proximities on a set X was obtained in [6], and it was also proved that an L-fts is completely regular if and only if it can be generated by a proximity.

In the present paper, by means of operations (is called in [1, 2, 3] φ, ψ) we shall define L-fuzzy quasi-uniformly $\varphi\psi$-continuous mapping between two L-fuzzy quasi-uniform spaces. We shall prove that $\varphi\psi$-continuous mapping defined between two L-fuzzy quasi-uniformity induces $\varphi\psi$-continuity between L-fuzzy topology generated by them. Finally we shall investigate some Theorems on L-fuzzy uniform spaces and L-fuzzy proximity spaces.

2 Preliminaries

bf Definition 2.1 (Liu et al. [6]). Let $L_\circ$, L_1 be complete lattices. Denote by $T(L_\circ, L_1)$ the family of all the arbitrary join preserving mapping from $L_\circ$ to L_1. Equip $T(L_\circ, L_1)$ with the partial order $\leq$ as follows :

$$\forall f, g \in T(L_\circ, L_1), f \leq g \text{ implies } \forall a \in L, f(a) \leq g(a).$$

A self mapping $f : L^X \to L^X$ on L^X, L^X always means an L-fuzzy space such that L is a F-lattice, is called value increasing, if $f(a) \geq a$ for every $a \in L^X$. Denote by $\varepsilon(L^X)$ the family of all the value increasing and arbitrary join preserving self mapping on L^X. Equip $\varepsilon(L^X)$ with the partial order $\leq$ defined in $T(L^X, L^X) \supset \varepsilon(L^X)$, i.e.

$$\forall f, g \in \varepsilon(L^X),\ f \leq g\ \ implies\ \ \forall a \in L^X, f(a) \leq g(a).$$

Definition 2.2 (Liu et al. [6]). Let $\mathcal{D} \subset \varepsilon(L^X).\mathcal{D}$ is called an L-fuzzy quasi-uniformity on X, if $\mathcal{D}$ fulfils the following conditions (UF1)- (UF3) :
(UF1) $f \in \varepsilon(L^X), g \in \mathcal{D},\ f \geq g \to f \in \mathcal{D}$.
(UF2) $f, g \in \mathcal{D}\ implies\ f \wedge g \in \mathcal{D}$.
(UF3) $f \in \mathcal{D}\ implies\ \exists g \in \mathcal{D}, \quad g \circ g \leq f$.

$\mathcal{D}$ is called an L-fuzzy uniformity on X, if $\mathcal{D}$ fulfills the above conditions $(UF1) - (UF3)$ and the following condition :

$$\text{(UF4) } f \in\ \mathcal{D}\ \ implies\ f^{\triangleleft} \in\ \mathcal{D}.$$

Call $(L^X, \mathcal{D})$ an L-fuzzy quasi-uniform space (or L-fuzzy uniform space, respectively), if $\mathcal{D}$ is an L-fuzzy quasi-uniformity (or an L-fuzzy uniformity, respectively) on X.

Definition 2.3 Let L be a complete lattice with order-reversing involution $\prime : L \to L$.

$\lambda \in L^{\mathbf{R}}$ is called monotonically decreasing, if $s, t \in \mathbf{R}, s \leq t\ implies\ \lambda(s) \leq \lambda(t)$. Denote the family of all the monotonically decreasing mappings $\lambda \in L^{\mathbf{R}}$ fulfilling the following conditions by $md_{\mathbf{R}}(L)$:
$\vee_{t\in\mathbf{R}}\, \lambda(t) = 1, \qquad \wedge_{t\in\mathbf{R}}\lambda(t) = 0.$

Denote the family of all the elements in $md_{\mathbf{R}}(L)$ fulfilling the following conditions by $md_I(L)$:

$t < 0$ *implies* $\lambda(t) = 1$, $\qquad t > 1$ *implies* $\lambda(t) = 0$.

For every $\lambda \in md_{\mathbf{R}}(L)$ and every $t \in \mathbf{R}$, define

$\lambda(t-) = \wedge_{s<t}\lambda(s), \qquad \lambda(t+) = \vee_{s>t}\lambda(s).$

Define the equivalent relation $\sim$ on $md_{\mathbf{R}}(L)$ as follows:

$\forall \lambda, \mu \in md_{\mathbf{R}}(L), \quad \lambda \sim \mu \ \ iff \ \ \forall t \in \mathbf{R}, \lambda(t-) = \mu(t-), \lambda(t+) = \mu(t+).$

$[\lambda] = \{\mu \in md_{\mathbf{R}}(L) : \mu \sim \lambda\}.$

Denote the family of all equivalent classes in $md_{\mathbf{R}}(L)$ with respect to $\sim$ by $\mathbf{R}[L]$. For every $t \in \mathbf{R}$, define $L_t, R_t \in L^{\mathbf{R}[L]}$ as follows:

$\forall \lambda \in \mathbf{R}[L], \qquad L_t(\lambda) = \lambda(t-)', \qquad R_t(\lambda) = \lambda(t+).$

Also denote the restrictions of $L_t, R_t : \mathbf{R}[L] \to L$ on $I[L]$ by L_t, R_t for every $t \in \mathbf{R}$ respectively.

Denote:

$S_L^I = \{L_t, R_t \in L^{I[L]} : \quad t \in \mathbf{R}\}.$

$B_L^I = \{\bigwedge \mathcal{F} : \quad \mathcal{F} \in [S_L^I]^{<\omega} \backslash \emptyset\},$

$\mathcal{T}_L^I = \{\bigvee \mathcal{A} : \quad \mathcal{A} \subset \mathcal{B}_L^I\}.$

Remark 2.1 L-fuzzy unit interval $I(L)$ is uniformizable.

For convenience, is defined

$L_{-\infty} = \underline{0}_{I[L]}, \quad L_{+\infty} = \underline{1}_{I[L]},$

$R_{-\infty} = \underline{1}_{I[L]}, \quad R_{+\infty} = \underline{0}_{I[L]},$

$S = S_L^I \bigcup \{L_{-\infty}, L_{+\infty}, R_{-\infty}, R_{+\infty}\},$

and denoted $X = I[L],\ \delta = \mathcal{T}_L^I$, then S is a subbase of δ. For every $A \in L^X$, let

$$S(A) = \{s \in \mathbf{R} : A \leq L_s'\}, \quad u(A) = \bigvee S(A), \qquad (2.2)$$

$$T(A) = \{t \in \mathbf{R} : A \leq R'_t\}, \quad l(A) = \bigwedge T(A), \qquad (2.3)$$

then we always have $(-\infty, 0) \subset S(A)$, $(1, +\infty) \subset T(A)$. So both $S(A)$ and $T(A)$ are always nonempty. For every $\epsilon > 0$ and every $A \in L^X$, let

$$f_\epsilon(A) = R_{u(A)-\epsilon}, \qquad (2.4)$$

then the family $\{f_\epsilon \ : \ \epsilon > 0\}$ possesses the following properties,

(i) $f_\epsilon \in \varepsilon(L^X)$, and $f_\epsilon \geq f_\rho$ whenever $\epsilon \geq \rho > 0$.

(ii) For every $\epsilon > 0$ and every $A \in L^X$,

$$f_\epsilon^{\triangleleft}(A) = L_{l(A)+\epsilon}. \qquad (2.5)$$

(iii) For every $\epsilon > 0$ and every $\rho > 0$,

$$f_\epsilon \circ f_\rho = f_{\epsilon+\rho}. \qquad (2.6)$$

(iv) $\varepsilon = \{f \in \varepsilon(L^X) : \exists \epsilon > 0, \ f \leq f_\epsilon \wedge f_\epsilon^{\triangleleft}\}$ is an L-fuzzy uniformity on X.

(v) The L-fuzzy topology on X generated by ε is just $\mathcal{T}_L^I$.

Definition 2.4 (Daraby et al. [1, 2]). Let $(L^X, \mathcal{D})$ be an L-fuzzy quasi-uniform space. A mapping $\alpha : L^X \longrightarrow L^X$ is called an operation on L^X if for each $A \in L^X \setminus \{\emptyset\}$, $int(A) \leq A^\alpha$ and $\emptyset^\alpha = \emptyset$ where A^α denotes the value of α in A.

3 L-fuzzy quasi-uniformly $\varphi\psi$-continuous mapping

Definition 3.1 Let $(L^X, D_\circ)$, $(L^Y, \mathcal{D}_1)$ be L-fuzzy quasi-uniform spaces (uniform spaces, respectively), $F^\rightarrow : L^X \longrightarrow L^Y$ an L-fuzzy mapping.

Let φ, ψ be two operations on L^X, L^Y respectively. $F^{\rightarrow}$ is called quasi-uniformly continuous (uniformly $\varphi\psi$-continuous, respectively), if for every $g \in \mathcal{D}_1$, there exists $f \in \mathcal{D}_\circ$ such that $F^{\rightarrow} \circ f^{\varphi} \leq g^{\psi} \circ F^{\rightarrow}$. Particularly, an L-fuzzy uniformly $\varphi\psi$-continuous mapping $F^{\rightarrow} : (L^X, \mathcal{D}) \longrightarrow I(L)$ defined on an L-fuzzy uniform space $(L^X, \mathcal{D})$ is called an L-fuzzy uniformly $\varphi\psi$-continuous function.

Theorem 3.1 Let φ, ψ be two operations on L^X, L^Y respectively. If $F^{\rightarrow}$: $(L^X, \mathcal{D}_\circ) \longrightarrow (L^Y, \mathcal{D}_1)$ an L-fuzzy quasi-uniformly $\varphi\psi$-continuous mapping, then $F^{\rightarrow} : (L^X, \delta(\mathcal{D}_\circ)) \longrightarrow (L^Y, \delta(\mathcal{D}_1))$ is L-fuzzy $\varphi\psi$-continuous mapping.

Proof Suppose that $B \in \delta(\mathcal{D}_1)$ including $F^{\rightarrow}(A), A \in \delta(\mathcal{D}_\circ)$ such that $i_\circ(B) = B$. We know that

$i_1(B) = \bigvee\{C \in L^Y : \quad \exists g \in \mathcal{D}_1; \quad g(C) \leq B\} = B$, so we have

$$B^{\psi} = \bigvee\{C \in L^Y : \ \exists g \in \mathcal{D}_1; \, g^{\psi}(C) \leq B^{\psi}\}. \quad (I)$$

From L-fuzzy quasi-uniformly $\varphi\psi$-continuity $F^{\rightarrow} : (L^X, \mathcal{D}_\circ) \longrightarrow (L^Y, \mathcal{D}_1)$ for each $g \in \mathcal{D}_1$, there exists $f \in \mathcal{D}_\circ$ such that $F^{\rightarrow} \circ f^{\varphi} \leq g^{\psi} \circ F^{\rightarrow}$, then

$$f(F^{\leftarrow}(B)) \leq F^{\leftarrow}F^{\rightarrow}fF^{\leftarrow}(B) \leq F^{\leftarrow}gF^{\rightarrow}F^{\leftarrow}(B) \leq F^{\leftarrow}(g(B)).$$

It follows that $f^{\varphi}(F^{\leftarrow}(B)) \leq F^{\leftarrow}(g^{\psi}(B))$.

Hence $\bigvee f^{\varphi}(F^{\leftarrow}(B)) \leq \bigvee F^{\leftarrow}(g^{\psi}(B))$.

We know that f is value increasing, so $F^{\leftarrow}(B) \leq f(F^{\leftarrow}(B))$. Hence $(F^{\leftarrow}(B))^{\varphi} \leq (f(F^{\leftarrow}(B))^{\varphi} = f^{\varphi}(F^{\leftarrow}(B))$. It follows that $(F^{\leftarrow}(B))^{\varphi} \leq \bigvee F^{\leftarrow}(g^{\psi}(B))$. Since $F^{\leftarrow}$ is arbitrary join preserving and order preserving, it follows that

$$(F^{\leftarrow}(B))^{\varphi} \leq F^{\leftarrow} \bigvee(g^{\psi}(B)). \quad \text{(II)}$$

From relations (I) and (II), we have

$$F^{\leftarrow}(B))^{\varphi} \leq F^{\leftarrow}(B^{\psi}).$$

So $F^{\rightarrow}(F^{\leftarrow}(B))^{\varphi} \leq F^{\rightarrow}(F^{\leftarrow}(B^{\psi}))$. It follows that $F^{\rightarrow}(F^{\leftarrow}(B))^{\varphi} \leq B^{\psi}$. Thus $F^{\rightarrow} : (L^X, \delta(\mathcal{D}_\circ)) \longrightarrow (L^Y, \delta(\mathcal{D}_1))$ is L-fuzzy $\varphi\psi$-continuous mapping. □

Theorem 3.2 Let $(L^X, \mathcal{D})$ be an L-fuzzy uniform space, $f \in \mathcal{D}$ and $A, B \in L^X$ such that $f(A) \leq B$. Then there exists an L-fuzzy uniformity $\varphi\psi$-continuous mapping $F^{\rightarrow} : (L^X, \mathcal{D}) \longrightarrow I(L)$ such that

$A \leq F^{\leftarrow}(L_1') \leq F^{\leftarrow}(R_\circ) \leq B.$ (3.1)

Proof For every $r \in \mathbf{R}$, take $A_r \in L^X$ as follows : as the first step, let $A_r = \underline{1}$ if $r < 0$, $A_r = B$ if $r = 0$, $A_r = A$ if $r = 1$ and $A_r = \underline{0}$ if $r > 1$.

Denote

$\mathbf{N}_\circ = \{0\} \cup \mathbf{N}$,

$\mathbf{B}(n) = \{\dfrac{2i-1}{2^k} : k \leq n, 0 < i \leq 2^{k-1}\}, \forall n \in \mathbf{N}_\circ$,

$\mathbf{B}_\circ = \{\dfrac{2i-1}{2^k} : k \in \mathbf{N}_\circ, 0 < i \leq 2^{k-1}\}$.

Then

$$\forall n \in N_\circ, \quad \mathbf{B}(n) \subset \mathbf{B}(n+1) \subset \mathbf{B}_\circ, \qquad \bigcup_{n \in N_\circ} \mathbf{B}(n) = \mathbf{B}_\circ. \quad (3.2)$$

We shall construct $\{h_{\frac{1}{2^n}} : n \in \mathbf{N}_\circ\} \subset \mathcal{D}$ and $\mathcal{A} = \{A_r : r \in \mathbf{B}_\circ\} \subset L^X$ such that for every $n \in \mathbf{N}_\circ$ and $1 \leq i \leq 2^{n-1}$,

(I) $k \in \mathbf{N}$ *implies* $h_{\frac{1}{2^k}} \circ h_{\frac{1}{2^k}} \leq h_{\frac{1}{2^{k-1}}}$,

(II) $m \leq n$, $\frac{1}{2^m} \leq \frac{2i-1}{2^n}$ *implies* $h^{\varphi}_{\frac{1}{2^m}}(A_{\frac{2i-1}{2^n}}) \leq A^{\psi}_{\frac{2i-1}{2^n} - \frac{1}{2^m}}$,

(III) $r, s \in \mathbf{B}(n)$, $r \leq s$ *implies* $A_r \leq A_s$.

Let $h_1 = f \in \mathcal{D}$. Suppose for some $n \in \mathbf{N}$ and every $k \in n$ we have constructed $h_{\frac{1}{2^k}}$ fulfilling condition (I), then by condition (UF3) of uniformity, there exists $h_{\frac{1}{2^{n+1}}} \in \mathcal{D}$ such that $h_{\frac{1}{2^{n+1}}} \circ h_{\frac{1}{2^{n=1}}} \leq h_{\frac{1}{2^n}}$. By inductive method, we have constructed $\{h_{\frac{1}{2^n}} : n \in \mathbf{N}_\circ\} \subset \mathcal{D}$ satisfying condition (I). Suppose $n \in \mathbf{N}_\circ,\ i \leq 2^n$, denote

$\xi(n,i) = \{h_{\frac{1}{2^{m_\circ}}} \circ \cdots \circ h_{\frac{1}{2^{m_k}}} : k \in \mathbf{N}_\circ,\ m_\circ, \cdots, m_k \leq n, \sum_{j=0}^{k} \frac{1}{2^{m_j}} \leq 1 - \frac{1}{2^n}\}$, (3.3)

then

$m, n \in \mathbf{N}_\circ,\ i \leq 2^m,\ j \leq 2^n,\ \frac{i}{2^m} \leq \frac{j}{2^n}$ *implies* $\xi(m,i) \supset \xi(n,j)$. (3.4)

We have had $A_\circ \geq A_1$, which satisfy condition (II) and (III). Suppose for some $n \in \mathbf{N}_\circ$ we have constructed $\{A_r : r \in \mathbf{B}(n)\} \subset L^X$ satisfying conditions (II) and (III). Define $\{A^{\psi}_{\frac{2i-1}{2^{n+1}}} : 1 \leq i \leq 2^n\}$ as follows:

$$A^{\psi}_{\frac{2i-1}{2^{n+1}}} = (\bigvee\{g(A_1) : g \in \xi(n+1, 2i-1)\})^{\varphi},$$

then by (3.4), we have constructed the family $\{A^{\psi}_r : r \in \mathbf{B}(n+1)\} \subset L^X$. Let $1 \leq i \leq 2^n, m \leq n+1$, we have

$$\begin{aligned}
&h^{\varphi}_{\frac{1}{2^m}}(A_{\frac{2i-1}{2^{n+1}}}) = h^{\varphi}_{\frac{1}{2^m}}(\bigvee\{g(A_1) : g \in \xi(n+1, 2i-1)\}) \\
&= \varphi(\bigvee\{h_{\frac{1}{2^m}} \circ h_{\frac{1}{2^{m_\circ}}} \circ \cdots \circ h_{\frac{1}{2^{m_k}}}(A_1) : k \in \mathbf{N}_\circ, m_\circ, \cdots, m_k \leq n+1, \\
&\quad \sum_{j=0}^{k} \frac{1}{2^{m_j}} \leq 1 - \frac{2i-1}{2^{n+1}}\} \\
&= \bigvee\{h_{\frac{1}{2^m}} \circ h_{\frac{1}{2^{m_\circ}}} \circ \cdots \circ h_{\frac{1}{2^{m_k}}}(A_1) : k \in \mathbf{N}_\circ, m_\circ, \cdots, m_k \leq n+1, \\
&\quad \frac{1}{2^m} + \sum_{j=0}^{k} \frac{1}{2^{m_j}} \leq 1 - (\frac{2i-1}{2^{n+1}} - \frac{1}{2^m})\}^{\varphi} \\
&\leq \bigvee\{h_{\frac{1}{2^{m_\circ}}} \circ \cdots \circ h_{\frac{1}{2^{m_k}}}(A_1) : k \in \mathbf{N}_\circ,\ m_\circ, \cdots,\ m_k \leq n+1, \\
&\quad \sum_{j=0}^{k} \frac{1}{2^{m_j}} \leq 1 - (\frac{2i-1}{2^{n+1}} - \frac{1}{2^m})\}^{\varphi} \\
&= \bigvee\{g(A_1) : g \in \xi(n+1, (2i-1) - 2^{n+1-m})\}^{\varphi} \\
&= A^{\psi}_{\frac{(2i-1)-2^{n+1-m}}{2^{n+1}}} \\
&= A^{\psi}_{\frac{2i-1}{2^{n+1}} - \frac{1}{2^m}}.
\end{aligned}$$

condition (II) holds for $n+1$. To prove (III), note that every number in the form of $\frac{2i}{2^n}$ can be represented as a member in $\mathbf{B}_\circ$, we need only prove the following inequalities for $i \in \{1, \cdots, 2^n\}$:

$$A_{\frac{2i-2}{2^{n+1}}} \geq A_{\frac{2i-1}{2^{n+1}}} \geq A_{\frac{2i}{2^{n+1}}}. \quad (3.5)$$

Suppose the irreducible form of $\frac{2i-2}{2^{n+1}}$ is $\frac{2j-1}{2^k}$, i.e. $\frac{2i-2}{2^{n+1}} = \frac{2j-1}{2^k}$, then $\frac{2j-1}{2^k} < \frac{2i-1}{2^{n+1}}$, by (3.4) the first inequality of (3.5) is true. Similarly, the second inequality of (3.5) holds also. Therefore, for $n+1$, we have proved condition (I) - (III). By inductive method and (3.2), we have constructed $\{h_{\frac{1}{2^n}} \ : \ n \in \mathbf{N}_\circ\} \subset \mathcal{D}$ and $\mathcal{A} = \{A_r \ : \ r \in \mathbf{B}_\circ\} \subset L^X$ such that the conditions (I) - (III) hold.

Since every $h_{\frac{1}{2^m}}$ is arbitrary join preserving, $h_{\frac{1}{2^m}}(\underline{0}) = \underline{0}$, and hence by the previous definition of A_r, for every $n < \omega$ and $i > 2^n$, we have

$$h_{\frac{1}{2^m}}(A_{\frac{i}{2^n}}) = h_{\frac{1}{2^m}}(\underline{0}) = \underline{0} \leq A_{\frac{i}{2^n} - \frac{1}{2^m}};$$

since $h_{\frac{1}{2^m}}$ is value increasing, for $i \leq 0$,

$$h_{\frac{1}{2^m}}(A_{\frac{i}{2^n}}) = h_{\frac{1}{2^n}}(\underline{1}) \geq \underline{1},$$
$$h_{\frac{1}{2^m}}(A_{\frac{i}{2^n}}) = \underline{1} = A_{\frac{i}{2^n}} - \frac{1}{2^m}.$$

Therefore, condition (II) holds for every $i \in \mathbf{Z}$.

Denote

$$\mathbf{B} = \{\tfrac{2i-1}{2^n} \quad : \quad n < \omega, \quad i \in \mathbf{Z}\}. \qquad (3.6)$$

For every $t \in \mathbf{R}$ and every $m \in \mathbf{N}_\circ$, let

$$\mathbf{B}_m(< t) = \{\tfrac{2i-1}{2^n} \in \mathbf{B} : n \geq m, \tfrac{2i-1}{2^n} < t\},$$
$$\mathbf{B}_m(> t) = \{\tfrac{2i-1}{2^n} \in \mathbf{B} : n \geq m, \tfrac{2i-1}{2^n} > t\},$$
$$A_t = \bigvee\{A_s : s \in \mathbf{B}, \ s > t\}, \quad \forall \in [0,1],$$

then $\mathbf{B}$ is dense in $\mathbf{R}$, for every $m \in \mathbf{N}_\circ, \mathbf{B}_m(< t)$ is dense in $(-\infty, t), \mathbf{B}_m(> t)$ is dense in $(t, = \infty)$. By the previous definition of A_r, we obtain a family $\mathcal{A}^* = \{A_t : \ t \in \mathbf{R}\} \subset L^X$. For every $x \in X$ and every $t \in \mathbf{R}$, if let $\lambda(t) = A_t(x)$, then by condition (III), $\lambda : \mathbf{R} \to L$ is monotonically decreasing and $\lambda(t) = 1$ for every $t < 0, \lambda(t) = 0$ for every $t > 1$. So $\lambda \in md_I(L)$ (see [9]), $[\lambda] \in I[L]$. Therefore, following the stipulation about the meaning of $F(x)(t)$ in the statement of this theorem, we can reasonably define an ordinary mapping $F : X \to I[L]$ as follows:

$$\forall x \in X, \forall t \in \mathbf{R}, \quad F(x)(t) = A_t(x).$$

Then

$$A(x) = A_1(x) \leq \bigwedge\nolimits_{\epsilon>0} A_{1-\epsilon}(x) = F(x)(1-) = F^{\leftarrow}(L'_1)(x),$$
$$B(x) = A_\circ(x) \geq \bigvee\nolimits_{\epsilon>0} A_\epsilon(x) = F(x)(0+) = F^{\leftarrow}(R_\circ)(x),$$

(3.1) holds.

Now we turn to the proof of the uniform $\varphi\psi$-continuity of $F^{\rightarrow}$. First of all, we prove the following two equalize : $\forall r \in \mathbf{R}, m \in \mathbf{N}_\circ, r \in \mathbf{R}$ *implies* $F^{\leftarrow}(R_r) = \bigvee_{s \in \mathbf{B}_m(>r)} A_s, \ F^{\leftarrow}(L_r) = \bigvee_{s \in \mathbf{B}_m(<r)} A'_s$. (3.7)
To prove (3.1), suppose $x \in X, F(x) = [\lambda]$, then for every $t \in \mathbf{R}, A_t(x) = \lambda(t)$. Since $\mathbf{B}_m(> r)$ is dense in $(r, = \infty)$,
$F^{\leftarrow}(R_r)(x) = R_r(F(x)) = R_r([\lambda]) = \lambda(r+)$
$= \bigvee_{s \in \mathbf{B}_m(>r)} \lambda(s) = \bigvee_{s \in \mathbf{B}_m(>r)} A_s(x)$
$= (\bigvee_{s \in \mathbf{B}_m(>r)} A_s(x)$.

So $F^{\leftarrow}(R_r) = \bigvee_{s \in \mathbf{B}_m(>r)} A_s$. Similarly prove the second equality in (3.7).
To consider the members of the canonical uniformity of $I(L)$, suppose f_ϵ is

defined by (2.4) for $\epsilon > 0$, then $f_\epsilon^{\triangleleft}$ is determined by (2.5), and there exist $g_1, g_2 \in \mathcal{D}$ such that

$$F^{\rightarrow} \circ g_1^{\varphi} \leq f_\epsilon^{\psi} \circ F^{\rightarrow}, \qquad F^{\rightarrow} \circ g_2^{\varphi} \leq f_\epsilon^{\triangleleft\psi} \circ F^{\rightarrow}. \tag{3.8}$$

In fact, fix a $\frac{1}{2^m} \in (0, \epsilon)$, then for every $r \in \mathbf{R}$, by condition (II) (It has been proved above, (II) holds for every $i \in \mathbf{Z}$) and (3.7),

$F^{\rightarrow} h^{\varphi}_{\frac{1}{2^m}} F^{\leftarrow}(R_r)$
$= F^{\rightarrow} h^{\varphi}_{\frac{1}{2^m}} (\bigvee_{s \in \mathbf{B}_m(>r)} A_s)$
$= F^{\rightarrow} (\bigvee_{s \in \mathbf{B}_m(>r)} h^{\varphi}_{\frac{1}{2^m}} (A_s))$
$\leq F^{\rightarrow} (\bigvee_{s \in \mathbf{B}_m(>r)} A^{\psi}_{s-\frac{1}{2^m}})$
$= F^{\rightarrow} F^{\leftarrow} (R^{\psi}_{r-\frac{1}{2^m}})$
$\leq R^{\psi}_{r-\frac{1}{2^m}}$
$\leq R^{\psi}_{r-\epsilon}$.

On the other hand, since $R_r \leq L'_s$ if and only if $R_r \leq R_s$. By (2.4),

$$f_\epsilon(R_r) = R_{u(R_r)-\epsilon} = R_{\bigvee\{s: R_r \leq L'_s\}-\epsilon} = R_{\bigvee\{s: R_r \leq R_s\}-\epsilon} = R_{r-\epsilon}.$$

So

$$F^{\rightarrow} h^{\phi}_{\frac{1}{2^m}} F^{\leftarrow}(R_r) \leq f_\epsilon^{\psi}(R_r).$$

Suppose $C \in L^X \setminus \{\underline{0}\}$. If $u(C) > 0$ for the function u defined in (2.2, 2.3), take

$\sigma \in (0, \epsilon) \cap (0, u(C))$, then by (2.6),

$$f_\epsilon(C) = f_{\epsilon-\sigma}(f_\sigma(C)) = f_{\epsilon-\sigma}(R_{u(C)-\sigma}).$$

Take $m \in \mathbf{N}$ such that $\frac{1}{2^m} < \epsilon - \sigma$, then by $C \leq f_\sigma(C) = R_{u(C)-\sigma}$,

$F^{\rightarrow} h^{\varphi}_{\frac{1}{2^m}} F^{\leftarrow}(C) \leq F^{\rightarrow} h^{\varphi}_{\frac{1}{2^m}} F^{\leftarrow}(R_{u(C)-\sigma})$

$\leq f^{\psi}_{\epsilon-\sigma}(R_{u(C)-\sigma})$
$= f^{\psi}_{\epsilon-\sigma}(f_{\sigma}(C))$
$= f^{\psi}_{\epsilon}(C)$.

So the following implication holds for arbitrary $C \in L^X \setminus \{\underline{0}\}$ such that $u(C) > 0$:

$$\epsilon > 0, m \in \mathbf{N}, \frac{1}{2^m} < \epsilon \textit{ implies } F^{\rightarrow} h^{\varphi}_{\frac{1}{2^m}} F^{\leftarrow}(C) \leq f^{\psi}_{\epsilon}(C). \quad (3.9)$$

If $u(C)$ not > 0, then by $u(C) \geq 0$ we have $u(C) = \bigvee\{s \in \mathbf{R} : C \leq L'_s\} = 0$. Hence
$f_{\epsilon}(C) = R_{u(C)-\epsilon} = R_{-\epsilon} = \underline{1}$,
the implication (3.9) still holds. Since (3.9) is true for $C = \underline{0}$, so (3.9) holds for every $C \in L^X$. Therefore, $F^{\rightarrow} h^{\varphi}_{\frac{1}{2^m}} F^{\leftarrow} \leq f^{\psi}_{\epsilon}$. Take $g_1 = h_{\frac{1}{2^m}}$ we obtain the first inequality in (3.8). Similarly prove the second inequality in (3.8).

Suppose h is a member of the canonical uniformity ε of I(L), then there exists $\epsilon > 0$ such that $h \geq f_{\epsilon} \wedge f^{\triangleleft}_{\epsilon}$, since ψ is monotonous operation, so $h^{\psi} \geq f^{\psi}_{\epsilon} \wedge (f^{\triangleleft}_{\epsilon})^{\psi}$. Take $g_1, g_2 \in \mathcal{D}$ such that (3.8) holds, let $g = g_1 \wedge g_2$,then
$F^{\rightarrow} g^{\varphi} F^{\leftarrow} \leq ((F^{\rightarrow} g^{\varphi}_1 F^{\leftarrow}) \wedge (F^{\rightarrow} g^{\varphi}_2 F^{\leftarrow}) \leq f^{\psi}_{\epsilon} \wedge f^{\triangleleft\psi}_{\epsilon} \leq h^{\psi}$,
hence $F^{\rightarrow} \circ g^{\varphi} \leq h^{\psi} \circ F^{\rightarrow}$. This completes the proof of the uniform $\varphi\psi$-continuity of $F^{\rightarrow}$.

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