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REPRESENTATION OF SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS OF ELASTICITY BY A SUM OF THE SOLUTIONS OF OTHER BOUNDARY VALUE PROBLEMS

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Abstract. Basic static boundary value problems of elasticity are considered for a semi-infinite curvilinear prism $\Omega = \{\rho_0 < \rho < \rho_1,\ \alpha_0 < \alpha < \alpha_1,\ 0 < z < \infty\}$ in generalized cylindrical coordinates ρ, α, z with Lamé coefficients $h_\rho = h_\alpha = h(\rho, \alpha)$, $h_z = 1$. It is proved that the solution of some boundary value problems of elasticity can be reduced to the sum of solutions of other boundary value problems of elasticity. Besides its cognitive significance, this fact also enables one to solve some non-classical elasticity problems.

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Introduction. When some boundary value problems of elasticity are solved in generalized cylindrical coordinates ρ, α, z for a semi-infinite curvilinear prism $\Omega = \{\rho_0 < \rho < \rho_1,\ \alpha_0 < \alpha < \alpha_1,\ 0 < z < \infty\}$ it turns out that the solution of some boundary value problems for a semi-infinite prism can be represented as a sum of solutions of some other boundary value problems for the same prism. This fact is stated in the paper in the form of two theorems, which underlie some non-classical elasticity problems for semi-infinite curvilinear prisms in Remarks 1, 2, 3 and 4. A solution method of some usual and mixed boundary value problems is also given.

Generalized cylindrical coordinates ρ, α, z are coordinates where ρ, α form a curvilinear orthogonal system of coordinates on the plane and z is a linear coordinate ($-\infty < z < \infty$), with Lamé coefficients $h_\rho = h_\alpha = h(\rho, \alpha)$, $h_z = 1$ of the system. The main coordinates of this type are listed below [1].

1. Cartesian coordinates x, y, z ($-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$).

2. Circular cylindrical coordinates r, α, z ($0 \le r < \infty$, $0 \le \alpha < 2\pi$, $-\infty < z < \infty$); $x = r\cos\alpha$, $y = r\sin\alpha$, $h = r$. If ρ, α, z imply circular cylindrical coordinates, then $\frac{\partial}{\partial\rho}$ is substituted by $r\frac{\partial}{\partial r}$.

3. Cylindrical elliptical coordinates ρ, α, z ($0 \le \rho < \infty$, $0 \le \alpha < 2\pi$, $-\infty < z < \infty$); $x = c\cosh\rho\cos\alpha$, $y = c\sinh\rho\sin\alpha$, $h = c\sqrt{\cosh^2\rho - \cos^2\alpha}$, where c is a scale factor.

4. Cylindrical parabolic coordinates ρ, α, z ($-\infty < \rho < \infty$, $0 \le \alpha < \infty$, $-\infty < z < \infty$); $x = c\,\frac{\rho^2-\alpha^2}{2}$, $y = c\rho\alpha$, $h = c\sqrt{\rho^2+\alpha^2}$.

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5. Cylindrical bipolar coordinates ρ, α, z $(-\infty < \rho < \infty,\ 0 \le \alpha < 2\pi,\ -\infty < z < \infty)$; $x = \dfrac{c \sinh \rho}{\cosh \rho + \cos \alpha}$, $y = \dfrac{c \sin \alpha}{\cosh \rho + \cos \alpha}$, $h = \dfrac{c}{\cosh \rho + \cos \alpha}$.

Some particular cases of semi-infinite curvilinear prisms are the following: a semi-infinite tetrahedral prism $\Omega = \{0 < x < x_1,\ 0 < y < y_1,\ 0 < z < \infty\}$, semi-infinite circular cylinders $\Omega = \{0 \le r < r_1,\ 0 \le \alpha < 2\pi,\ 0 < z < \infty\}$ and $\Omega = \{r_0 < r < r_1,\ 0 \le \alpha < 2\pi,\ 0 < z < \infty\}$, semi-infinite elliptical cylinders $\Omega = \{0 < \rho < \rho_1,\ 0 \le \alpha < 2\pi,\ 0 < z < \infty\}$ and $\Omega = \{\rho_0 < \rho < \rho_1,\ 0 \le \alpha < 2\pi,\ 0 < z < \infty\}$, etc.

1. In the generalized cylindrical coordinates consider the elastic equilibrium of a homogeneous and isotropic semi-infinite curvilinear prism $\Omega = \{\rho_0 < \rho < \rho_1,\ \alpha_0 < \alpha < \alpha_1,\ 0 < z < \infty\}$ with the following boundary conditions

$$\begin{aligned} &\text{with } \rho = \rho_j \text{ we have: } a)\ u = 0,\ \ \omega_z = 0,\ \ Z_\rho = 0 \ \text{ or} \\ &\qquad\qquad\qquad\qquad\quad b)\ e = 0,\ v = 0,\ w = 0, \end{aligned} \tag{1}$$

$$\begin{aligned} &\text{with } \alpha = \alpha_j \text{ we have: } a)\ v = 0,\ \ Z_\alpha = 0,\ \ \omega_z = 0 \ \text{ or} \\ &\qquad\qquad\qquad\qquad\quad b)\ e = 0,\ \ w = 0,\ \ u = 0. \end{aligned} \tag{2}$$

In (1) and (2) we have

$$\omega_z = \operatorname{rot}_z \vec{U} = \frac{1}{h^2}\left[\frac{\partial(hv)}{\partial \rho} - \frac{\partial(hu)}{\partial \alpha}\right],$$
$$e = \operatorname{div} \vec{U} = \frac{1}{h^2}\left[\frac{\partial(hu)}{\partial \rho} + \frac{\partial(hv)}{\partial \alpha}\right] + \frac{\partial w}{\partial z}.$$

With $z = 0$ we have:

$$\begin{aligned} &a)\ Z_z = F_1(\rho,\alpha),\ \ hZ_\rho = F_2(\rho,\alpha),\ \ hZ_\alpha = F_3(\rho,\alpha) \ \text{ or} \\ &b)\ Z_z = F_1(\rho,\alpha),\ \ hu = f_2(\rho,\alpha),\ \ hv = f_3(\rho,\alpha), \ \text{ or} \\ &c)\ w = f_1(\rho,\alpha),\ \ hZ_\rho = F_2(\rho,\alpha),\ \ hZ_\alpha = F_3(\rho,\alpha), \ \text{ or} \\ &d)\ w = f_1(\rho,\alpha),\ \ hu = f_2(\rho,\alpha),\ \ hv = f_3(\rho,\alpha). \end{aligned} \tag{3}$$

In (1), (2) and (3): $j = 0, 1$; u, v, w are components of the displacement vector $\vec{U}$ along the tangents to the coordinate lines ρ, α, z; R_ρ, A_α, Z_z are normal stresses and $Z_\rho = R_z$, $R_\alpha = A_\rho$, $A_z = Z_\alpha$ are tangential stresses. The functions F_1, F_2, F_3 and f_1, f_2, f_3 are chosen so that the compatibility conditions hold on the edges of the semi- infinite prism and their differential properties ensure regularity of the solutions of boundary value problems of elastic equilibrium for a semi-infinite prism (regularity of the solutions implies the same as in [2] but, in addition, the displacements and stresses vanish when $z \to \infty$).

Hooke's law in generalized cylindrical coordinates can be written as

$$\frac{1+\nu}{E} R_\rho = \frac{\nu}{(1-2\nu)}\, e + \left(\frac{1}{h}\frac{\partial u}{\partial \rho} + \frac{1}{h^2}\frac{\partial h}{\partial \alpha} v\right),$$
$$\frac{1+\nu}{E} A_\alpha = \frac{\nu}{(1-2\nu)}\, e + \left(\frac{1}{h}\frac{\partial v}{\partial \alpha} + \frac{1}{h^2}\frac{\partial h}{\partial \rho} u\right),$$

$$\frac{1+\nu}{E} Z_z = \frac{\nu}{(1-2\nu)}\, e + \frac{\partial w}{\partial z},$$
$$\frac{1+\nu}{E} Z_\rho = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{1}{h}\frac{\partial w}{\partial \rho}\right), \quad \frac{1+\nu}{E} Z_\alpha = \frac{1}{2}\left(\frac{1}{h}\frac{\partial w}{\partial \alpha} + \frac{\partial v}{\partial z}\right),$$
$$\frac{1+\nu}{E} A_\rho = \frac{1}{2}\left[\frac{\partial}{\partial \rho}\left(\frac{v}{h}\right) + \frac{\partial}{\partial \alpha}\left(\frac{u}{h}\right)\right],$$

where E is the elasticity modulus and ν is the Poisson coefficient.

We shall give a short technical interpretation of the boundary conditions: (1a), (2a), (3c) with $f_1 = 0$, $F_2 = F_3 = 0$ are symmetry-type conditions (SM) and (1b), (2b), (3b) with $F_1 = 0$, $f_2 = f_3 = 0$ are antisymmetry-type conditions (ASM).

In the case of SM conditions it can be assumed that the cylindrical or plane boundary S of the semi-finite curvilinear prism is connected to the absolutely smooth cylindrical or plane boundary surface S, respectively, of an absolutely rigid body (S does not denote the whole boundary surface of the prism, but only its part, in particular, $\rho = \rho_j$, $\alpha = \alpha_j$ or $z = 0$). By virtue of the absolute rigidity of the body, the component of the displacement vector normal to S vanishes and because of the absolute smoothness of S we have $\omega_z|_{\rho=\rho_j} = 0$, $Z_\rho|_{\rho=\rho_j} = 0$ or $Z_\alpha|_{\alpha=\alpha_j} = 0$, $\omega_z|_{\alpha=\alpha_j} = 0$, or $Z_\rho|_{z=0} = 0$, $Z_\alpha|_{z=0} = 0$.

In the case of ASM conditions we can assume that an absolutely flexible, but absolutely nontensile and noncompressible thin plate is glued onto the cylindrical or plane boundary surface S. Since the plate is absolutely nontensile and noncompressible we have $v|_{\rho=\rho_j} = 0$, $w|_{\rho=\rho_j} = 0$ or $w|_{\alpha=\alpha_j} = 0$, $u|_{\alpha=\alpha_j} = 0$, or $u|_{z=0} = 0$, $v|_{z=0} = 0$, and since it is absolutely flexible we have $e = 0$ (with $z = 0$ the conditions $e = 0$, $u = 0$, $v = 0$ are equivalent to the conditions $Z_z = 0$, $u = 0$, $v = 0$).

If with $\rho = \rho_j$ and $\alpha = \alpha_j$, in addition to conditions (1) and (2), we also consider the conditions

$$\begin{aligned} &a)\quad u|_{\rho=\rho_j} = 0,\ A_\rho|_{\rho=\rho_j} = 0,\ Z_\rho|_{\rho=\rho_j} = 0 \text{ or}\\ &b)\quad R_\rho|_{\rho=\rho_j} = 0,\ v|_{\rho=\rho_j} = 0,\ w|_{\rho=\rho_j} = 0;\\ &c)\quad v|_{\alpha=\alpha_j} = 0,\ Z_\alpha|_{\alpha=\alpha_j} = 0,\ R_\alpha|_{\alpha=\alpha_j} = 0 \text{ or}\\ &d)\quad A_\alpha|_{\alpha=\alpha_j} = 0,\ w|_{\alpha=\alpha_j} = 0,\ u|_{\alpha=\alpha_j} = 0, \end{aligned} \tag{4}$$

then it can be easily shown that conditions (1a) and (1b) are equivalent to conditions (4a) and (4b), respectively, in the case when $\rho = \rho_j$ is a plane, while conditions (2a) and (2b) are equivalent to conditions (4c) and (4d), respectively, when $\alpha = \alpha_j$ is a plane. If $\rho = \rho_j$ and $\alpha = \alpha_j$ are planes, then (4a) and (4c) are simmetry conditions while (4b) and (4c) are antisymmetry conditions.

Write the elastic equilibrium equation as [3]

$$\Delta \vec{U} + \frac{1}{1-2\nu}\ \text{grad div}\,\vec{U} = 0, \tag{5}$$

where $\Delta = \frac{1}{h^2}\left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \alpha^2}\right) + \frac{\partial^2}{\partial z^2}$.

Denote boundary value problem (5), (1), (2), (3a) by $A(F_1, F_2, F_3)$, problem (5), (1), (2), (3b) by $B(F_1, f_2, f_3)$, problem (5), (1), (2), (3c) by $C(f_1, F_2, F_3)$, problem (5), (1), (2), (3d) by $D(f_1, f_2, f_3)$. In particular, if in problem (5), (1), (2), (3a) we have $F_2 = 0$ and $F_3 = 0$, then the notation will have the form $A(F_1, 0, 0)$, while if $F_1 = 0$, then $A(0, F_2, F_3)$, etc.

It should be noted that in the boundary value problems under consideration the principal vector and the principal moment are assumed to be equal to zero (to ensure correctness of the problems).

It follows from [2] that for each of the boundary problems $A(F_1, F_2, F_3)$, $B(F_1, f_2, f_3)$, $C(f_1, F_2, F_3)$ and $D(f_1, f_2, f_3)$, a unique solution can be effectively constructed (the solution of problems $B(F_1, 0, 0)$ and $C(0, F_2, F_3)$ is given in Section 2).

The validity of the following representation

$$a)\ A(F_1, 0, 0) = B(F_1, 0, 0) - C(0, F_{2B}, F_{3B}) + A(F_{1C}, 0, 0), \tag{6}$$

can be easily verified, it implies that the components of the displacement vector and the stress tensor in the problem $A(F_1, 0, 0)$ can be obtained by summing up the corresponding displacements and the corresponding stresses of the problem $B(F_1, 0, 0)$ and $A(F_{1C}, 0, 0)$ and by subtraction of the corresponding displacements and the corresponding stresses of the problem $C(0, F_{2B}, F_{3B})$. The functions F_{2B} and F_{3B} which are tangential stresses in the problem $C(0, F_{2B}, F_{3B})$ with $z=0$, are at the same time the tangential stresses in the problem $B(F_1, 0, 0)$ with $z = 0$ and are taken from the latter after its solution. In quite a similar way the function F_{1C} in the problem $A(F_{1C}, 0, 0)$ is taken from the problem $C(0, F_{2B}, F_{3B})$ after its solution.

It may seem that Equality (6a) makes no sense since the problem $A(F_1, 0, 0)$ can be represented as a difference of the problems $B(F_1, 0, 0)$ and $C(0, F_{2B}, F_{3B})$ and the problem $A(F_{1C}, 0, 0)$, which is similar to the problem $A(F_1, 0, 0)$, but further on proportionality of the stresses in the problems $A(F_1, 0, 0)$ and $A(F_{1C}, 0, 0)$, is established, which gives meaning to Equality (6a).

Similarly to Equality (6a), we can write

$$\begin{aligned} &b)\ B(F_1, 0, 0) = A(F_1, 0, 0) - D(0, f_{2A}, f_{3A}) + B(F_{1D}, 0, 0), \\ &c)\ C(f_1, 0, 0) = D(f_1, 0, 0) - A(0, F_{2D}, F_{3D}) + C(f_{1A}, 0, 0), \\ &d)\ D(f_1, 0, 0) = C(f_1, 0, 0) - B(0, f_{2C}, f_{3C}) + D(f_{1B}, 0, 0). \end{aligned} \tag{6}$$

Note that Equalities (6) can be also written in the following form:

$$\begin{aligned} A(F_1, 0, 0) &= B(F_1, 0, 0) + C(0, -F_{2B}, -F_{3B}) + A(-F_{1C}, 0, 0), \\ B(F_1, 0, 0) &= A(F_1, 0, 0) + D(0, -f_{2A}, -f_{3A}) + B(-F_{1D}, 0, 0), \\ C(f_1, 0, 0) &= D(f_1, 0, 0) + A(0, -F_{2D}, -F_{3D}) + C(-f_{1A}, 0, 0), \\ D(f_1, 0, 0) &= C(f_1, 0, 0) + B(0, -f_{2C}, -f_{3C}) + D(-f_{1B}, 0, 0). \end{aligned}$$

2. For the considered class of boundary value problems of elasticity the general solution is given in [2], which has the following form

$$\begin{aligned}\varkappa h u &= z\frac{\partial^2\Phi_3}{\partial z\partial\rho}+\varkappa\frac{\partial\Phi_3}{\partial\rho}+2\frac{\partial\Phi_2}{\partial\rho}+\frac{4(1-\nu^2)}{E}\frac{\partial\Phi_1}{\partial\alpha},\\ \varkappa h v &= z\frac{\partial^2\Phi_3}{\partial z\partial\alpha}+\varkappa\frac{\partial\Phi_3}{\partial\alpha}+2\frac{\partial\Phi_2}{\partial\alpha}-\frac{4(1-\nu^2)}{E}\frac{\partial\Phi_1}{\partial\rho},\\ \varkappa w &= z\frac{\partial^2\Phi_3}{\partial z^2}-(\varkappa-1)\frac{\partial\Phi_3}{\partial z}+2\frac{\partial\Phi_2}{\partial z},\end{aligned}\tag{7}$$

where $\varkappa = 2(1-\nu)$, (since $0<\nu<\frac{1}{2}$, we have $1<\varkappa<2$), $\Delta\Phi_i = 0$ $(i = 1,2,3)$. Using (7) in Hooke's law we can write

$$\begin{aligned}Z_z &= \frac{E}{1-\nu^2}\left(\frac{z}{2}\frac{\partial^3\Phi_3}{\partial z^3}+\frac{\partial^2\Phi_2}{\partial z^2}\right),\\ hZ_\rho &= \frac{E}{1-\nu^2}\frac{\partial^2}{\partial\rho\partial z}\left(\frac{z}{2}\frac{\partial\Phi_3}{\partial z}+\Phi_2\right)+\frac{\partial^2\Phi_1}{\partial\alpha\partial z},\\ hZ_\alpha &= \frac{E}{1-\nu^2}\frac{\partial^2}{\partial\alpha\partial z}\left(\frac{z}{2}\frac{\partial\Phi_3}{\partial z}+\Phi_2\right)-\frac{\partial^2\Phi_1}{\partial\rho\partial z}.\end{aligned}\tag{8}$$

If we apply the method of separation of variables, the harmonic functions Φ_1, Φ_2, and Φ_3, will take the following form

$$\begin{aligned}\Phi_1 &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{1mn}e^{-p_1 z}\tilde{\psi}_{mn},\\ \Phi_2 &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{2mn}e^{-pz}\psi_{mn},\\ \Phi_3 &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{3mn}e^{-pz}\psi_{mn}\end{aligned}\tag{9}$$

for the considered type of problems. In (9) ψ_{mn} are eigenfunctions of the following regular Sturm–Liouville problem [4]:

$$\begin{aligned}&\frac{1}{h^2}\left(\frac{\partial^2\psi_{mn}}{\partial\rho^2}+\frac{\partial^2\psi_{mn}}{\partial\alpha^2}\right)+p^2(m,n)\cdot\psi_{mn}=0;\\ &\text{with } \rho=\rho_j \text{ we have}: \quad \psi_{mn}=0 \text{ or } \frac{\partial\psi_{mn}}{\partial\rho}=0,\\ &\text{with } \alpha=\alpha_j \text{ we have } \ \psi_{mn}=0 \text{ or } \frac{\partial\psi_{mn}}{\partial\alpha}=0.\end{aligned}\tag{10}$$

If in (10) the constant $p(m,n)$ is substituted by the constant $p_1(m,n)$ and the function ψ_{mn} is replaced by the function $\tilde{\psi}_{mn}$, we shall obtain eigenfunctions $\tilde{\psi}_{mn}$. The boundary conditions in (10) follow from conditions (1) and (2).

For the Cartesian coordinate system ψ_{mn} is the product of trigonometric functions; in the case of cylindrical coordinates ψ_{mn} is the product of trigonometric and Bessel functions; for a cylindrical elliptic coordinate system ψ_{mn} is the product of Mattieu functions; for a cylindrical parabolic coordinate system ψ_{mn} is the product of Weber functions. As for the cylindrical bipolar coordinate system, even in the Laplace equation the variables cannot be entirely separated for this system.

Note that although the coordinate surfaces of various coordinate systems enable us to consider elastic equilibrium of different shapes of bodies, the mathematical aggregate of the solution remains unchanged. The geometric shape of an elastic body and the form of the functions ψ_{mn} and $\tilde{\psi}_{mn}$ are defined by the parameter $h(\rho,\alpha)$.

Now we shall consider Equation (6a) in more detail and establish the connection between the boundary value problems $A(F_1,0,0)$ and $A(F_{1C},0,0)$. First of all, we shall need the solutions of the boundary value problems $B(F_1,0,0)$ and $C(0,F_{2B},F_{3B})$, appearing in (6a), so we shall write them out.

Following [2] , the boundary conditions for the problem $B(F_1,0,0)$ with $z=0$

$$Z_z = F_1(\rho,\alpha),\ \ u=0,\ \ v=0$$

taking into account (7) and (8), can be written in an equivalent form, i.e.,

$$\begin{aligned}
&Z_z\big|_{z=0} = \frac{E}{1-\nu^2}\left(\frac{\partial^2\Phi_2}{\partial z^2}\right)_{z=0} = F_1(\rho,\alpha) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{1mn}\psi_{mn},\\
&\frac{1}{h^2}\left[\frac{\partial(hu)}{\partial\rho}+\frac{\partial(hv)}{\partial\alpha}\right]_{z=0} = -\left[\left(\frac{\partial^2\Phi_3}{\partial z^2}\right)_{z=0}+\frac{1}{1-\nu}\left(\frac{\partial^2\Phi_2}{\partial z^2}\right)_{z=0}\right]=0,\\
&\frac{1}{h^2}\left[\frac{\partial(hv)}{\partial\rho}-\frac{\partial(hu)}{\partial\alpha}\right]_{z=0} = \frac{2(1+\nu)}{E}\left(\frac{\partial^2\Phi_1}{\partial z^2}\right)_{z=0}=0,
\end{aligned}\tag{11}$$

where F_{1mn} is a Fourier coefficient of the function $F_1(\rho,\alpha)$, expanded into a Fourier series with respect to the eigenfunctions ψ_{mn}. The proof of the equivalence of the boundary conditions $Z_z|_{z=0} = F_1(\rho,\alpha)$, $u|_{z=0}=0$, $v|_{z=0}=0$ and (11) is also given in [2].

Taking into account (9), it follows from (11) that

$$A_{1mn}=0,\quad A_{2mn}=\frac{1-\nu^2}{E}\,\frac{F_{1mn}}{p^2},\quad A_{3mn}=-\frac{1+\nu}{E}\,\frac{F_{1mn}}{p^2}\,.$$

Therefore

$$\Phi_1=0,\quad \Phi_2=\frac{1-\nu^2}{E}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{F_{1mn}}{p^2}e^{-pz}\psi_{mn},$$

$$\Phi_3=-\frac{1+\nu}{E}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{F_{1mn}}{p^2}e^{-pz}\psi_{mn}.$$

Hence the problem $B(F_1, 0, 0)$ has been solved. Having the solution of the problem $B(F_1, 0, 0)$, we can define

$$\begin{aligned}\frac{1}{h^2}\left[\frac{\partial(hZ_\rho)}{\partial\rho}+\frac{\partial(hZ_\alpha)}{\partial\alpha}\right]_{z=0} &= -\frac{E}{1-\nu^2}\left(\frac{\partial^3\Phi_2}{\partial z^3}+\frac{1}{2}\frac{\partial^3\Phi_3}{\partial z^3}\right)_{z=0}\\ &= \frac{1-2\nu}{2(1-\nu)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} pF_{1mn}\psi_{mn}, \qquad (12)\\ \frac{1}{h^2}\left[\frac{\partial(hZ_\alpha)}{\partial\rho}-\frac{\partial(hZ_\rho)}{\partial\alpha}\right]_{z=0} &= \left(\frac{\partial^3\Phi_1}{\partial z^3}\right)_{z=0}=0.\end{aligned}$$

As in the previous problem, the boundary conditions

$$w=0,\ hZ_\rho=F_{2B}(\rho,\alpha),\ hZ_\alpha=F_{3B}(\rho,\alpha)$$

with $z=0$ are written out for the problem $C(0, F_{2B}, F_{3B})$, taking into account (7), (8) and (12), in their equivalent form, i.e., as

$$\begin{aligned}w|_{z=0} &= -\frac{1-2\nu}{2(1-\nu)}\left(\frac{\partial\Phi_3}{\partial z}\right)_{z=0}+\frac{1}{(1-\nu)}\left(\frac{\partial\Phi_2}{\partial z}\right)_{z=0}=0,\\ \frac{1}{h^2}\left[\frac{\partial(hZ_\rho)}{\partial\rho}+\frac{\partial(hZ_\alpha)}{\partial\alpha}\right]_{z=0} &= -\frac{E}{1-\nu^2}\left(\frac{\partial^3\Phi_2}{\partial z^3}+\frac{1}{2}\frac{\partial^3\Phi_3}{\partial z^3}\right)_{z=0}\\ &= \frac{1-2\nu}{2(1-\nu)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} pF_{1mn}\psi_{mn}, \qquad (13)\\ \frac{1}{h^2}\left[\frac{\partial(hZ_\alpha)}{\partial\rho}-\frac{\partial(hZ_\rho)}{\partial\alpha}\right]_{z=0} &= \left(\frac{\partial^3\Phi_1}{\partial z^3}\right)_{z=0}=0.\end{aligned}$$

Taking (9) into account, (13) implies

$$A_{1mn}=0,\quad A_{2mn}=\frac{(1+\nu)(1-2\nu)^2}{4(1-\nu)E}\,\frac{F_{1mn}}{p^2},\quad A_{3mn}=\frac{(1+\nu)(1-2\nu)}{2(1-\nu)E}\,\frac{F_{1mn}}{p^2},$$

therefore

$$\Phi_1=0,\quad \Phi_2=\frac{(1+\nu)(1-2\nu)^2}{4(1-\nu)E}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{F_{1mn}}{p^2}e^{-pz}\psi_{mn},$$

$$\Phi_3=\frac{(1+\nu)(1-2\nu)}{2(1-\nu)E}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{F_{1mn}}{p^2}e^{-pz}\psi_{mn}.$$

Hence the problem $C(0, F_{2B}, F_{3B})$ has been also solved. Having the solution of the problem $C(0, F_{2B}, F_{3B})$, we can define

$$\begin{aligned}Z_z|_{z=0} &= \frac{E}{1-\nu^2}\left(\frac{\partial^2\Phi_2}{\partial z^2}\right)_{z=0}=\left[\frac{1-2\nu}{2(1-\nu)}\right]^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_{1mn}\psi_{mn}\\ &= \left(\frac{\varkappa-1}{\varkappa}\right)^2 F_1(\rho,\alpha)\end{aligned}$$

Then for the function $F_{1C}(\rho,\alpha)$, appearing in the problem $A(F_{1C},0,0)$, we shall have

$$F_{1C}(\rho,\alpha)=\left(\frac{\varkappa-1}{\varkappa}\right)^2 F_1(\rho,\alpha).$$

Hence in Equality (6a) the state of stress in the problem $A(F_1,0,0)$ is proportional to the state of stress in the problem $A(F_{1C},0,0)$. Proportionality of the stresses in the problems $B(F_1,0,0)$ and $B(F_{1D},0,0)$, $C(f_1,0,0)$ and $(f_{1A},0,0)$, $D(f_1,0,0)$ and $D(f_{1B},0,0)$, is proved in a similar way, therefore the following lemma is valid.

3. Lemma. *The states of stress of an elastic semi-infinite curvilinear prism in the problems $A(F_1,0,0)$ and $A(F_{1C},0,0)$, $B(F_1,0,0)$ and $B(F_{1D},0,0)$, $C(f_1,0,0)$ and $(f_{1A},0,0)$, $D(f_1,0,0)$ and $D(f_{1B},0,0)$, appearing in equalities* (6) *are proportional, in particular, we have*

$$\begin{aligned}
A(F_{1C},0,0)&=k_1A(F_1,0,0)=A(k_1F_1,0,0),\\
B(F_{1D},0,0)&=k_2B(F_1,0,0)=B(k_2F_1,0,0),\\
C(f_{1A},0,0)&=k_2C(f_1,0,0)=C(k_2f_1,0,0),\\
D(f_{1B},0,0)&=k_1D(f_1,0,0)=D(k_1f_1,0,0),
\end{aligned}$$

where $k_1=\left(\frac{\varkappa-1}{\varkappa}\right)^2$, $k_2=-\frac{(\varkappa-1)^2}{2\varkappa-1}$.

Lemma with Equalities (6) in mind leads to Theorem 1.

Theorem 1.

$$\begin{aligned}
a)\ \ A(F_1,0,0)&=g[B(F_1,0,0)-C(0,F_{2B},F_{3B})]\\
&=B(gF_1,0,0)-C(0,gF_{2B},gF_{3B}),\\
b)\ \ B(F_1,0,0)&=\frac{1}{g}\,[A(F_1,0,0)-D(0,f_{2A},f_{3A})]\\
&=A\left(\frac{1}{g}\,F_1,0,0\right)-D\left(0,\frac{1}{g}\,f_{2A},\frac{1}{g}\,f_{3A}\right),\\
c)\ \ C(f_1,0,0)&=\frac{1}{g}\,[D(f_1,0,0)-A(0,F_{2D},F_{3D})]\\
&=D\left(\frac{1}{g}\,f_1,0,0\right)-A\left(0,\frac{1}{g}\,F_{2D},\frac{1}{g}\,F_{3D}\right),\\
d)\ \ D(f_1,0,0)&=g[C(f_1,0,0)-B(0,f_{2C},f_{3C})]\\
&=C(gf_1,0,0)-B(0,gf_{2C},gf_{3C}),
\end{aligned}\tag{14}$$

where $g=\frac{\varkappa^2}{2\varkappa-1}$.

Corollary a.1. *On the boundary surface $z=0$ the displacements w in the problems $A(F_1,0,0)$ and $B(gF_1,0,0)$ or, which is the same, in the problems $A\left(\frac{1}{g}\,F_1,0,0)\right)$ and $B(F_1,0,0)$, appearing in Equality* (14a)*, are equal.*

Corollary a.2. *If on the plane part $z = 0$ of the boundary surface of the domain Ω the displacement is equal to zero $w = 0$, then the tangential stresses $Z_\rho|_{z=0}$ and $Z_\alpha|_{z=0}$ can be chosen so that on $z = 0$ we will have the defined normal stress $Z_z|_{z=0} = F_1(\rho,\alpha)$. Indeed, write (14a) as*

$$C(0,F_{2B},F_{3B}) = B(F_1,0,0) - A\left(\frac{1}{g}\,F_1,0,0\right)$$

taking into account that in the problem $B(F_1,0,0)$ we have $Z_\rho|_{z=0} = F_{2B}$, $Z_\alpha|_{z=0} = F_{3B}$ and that, according to Corollary a.1, the displacements $w|_{z=0}$ in the problems $B(F_1,0,0)$ and $A\left(\frac{1}{g}\,F_1,0,0\right)$ are equal, then since the tangential stresses $Z_\rho|_{z=0} = F_{2B}$ and $Z_\alpha|_{z=0} = F_{3B}$ in the problem $C(0,F_{2B},F_{3B})$ cause the stress $Z_z|_{z=0} = \left(1-\frac{1}{g}\right)F_1 = \left(\frac{\varkappa-1}{\varkappa}\right)^2 F_1 = k_1F_1$ in it, the tangential stresses $Z_\rho|_{z=0} = \frac{1}{k_1}F_{2B}$ and $Z_\alpha|_{z=0} = \frac{1}{k_1}F_{3B}$ will cause the stress $Z_z|_{z=0} = F_1(\rho,\alpha)$ in the problem $C\left(0,\frac{1}{k_1}\,F_{2B},\frac{1}{k_1}\,F_{2B}\right)$.

Remark 1. On the surface $z = 0$ of the domain Ω the problem of determination of tangential stresses $Z_\rho|_{z=0}$ and $Z_\alpha|_{z=0}$, such that they cause the defined stress $Z_z|_{z=0} = F_1(\rho,\alpha)$ with $w|_{z=0} = 0$, is reduced to the solution of the boundary value problem $B\left(\frac{1}{k_1}\,F_1,0,0\right)$, or to be more exact, to the determination of the tangential stresses $Z_\rho|_{z=0}$ and $Z_\alpha|_{z=0}$ in this problem.

Corollary b.1. *The displacements $w|_{z=0}$ in the problems $B(F_1,0,0)$ and $A\left(\frac{1}{g}\,F_1,0,0\right)$ or, which is the same, in the problems $B(gF_1,0,0)$ and $A(F_1,0,0)$, appearing in Equality (14b), are equal.*

Corollary b.2. *If on $z = 0$ we have $w = 0$, then the tangential displacements $u|_{z=0}$ and $v|_{z=0}$ can be chosen so that on $z = 0$ we will have the defined normal stress $Z_z|_{z=0} = F_1(\rho,\alpha)$. Indeed, if (14b) is represented as*

$$D(0,f_{2A},f_{3A}) = A(F_1,0,0) - B(gF_1,0,0)$$

and if we keep in mind that in the problem $A(F_1,0,0)$ we have $u|_{z=0} = f_{2A}$, $v|_{z=0} = f_{3A}$ and that, according to Corollary b.1, the displacements $w|_{z=0}$ in the problems $A(F_1,0,0)$ and $B(gF_1,0,0)$ are equal, then since the tangential displacements $u|_{z=0} = f_{2A}$ and $v|_{z=0} = f_{2A}$ in the problem $D(0,f_{2A},f_{3A})$ cause here the stress $Z_z|_{z=0} = (1-g)F_1 = -\frac{(\varkappa-1)^2}{2\varkappa-1}\,F_1 = k_2F_1$, the tangential stresses $u|_{z=0} = \frac{1}{k_2}\,f_{2A}$ and $v|_{z=0} = \frac{1}{k_2}\,f_{3A}$ will cause the stress $Z_z|_{z=0} = F_1(\rho,\alpha)$ in the problem $D\left(0,\frac{1}{k_2}f_{2A},\frac{1}{k_2}\,f_{3A}\right)$.

Remark 2. On the surface $z = 0$ of the domain Ω the problem of determination of tangential displacements $u|_{z=0}$ and $v|_{z=0}$, such that with $w|_{z=0} = 0$ they will cause the defined stress $Z_z|_{z=0} = F_1(\rho,\alpha)$, is reduced to the solution of the boundary value problem $A\left(\frac{1}{k_2}\,F_1,0,0\right)$, or, to be more precise, to the determination of the tangential displacements $u|_{z=0}$ and $v|_{z=0}$ in this problem.

Corollary c.1. *The stresses* $Z_z|_{z=0}$ *in the problems* $C(f_1,0,0)$ *and* $D\left(\frac{1}{g}\,f_1,0,0\right)$ *or, which is the same, in the problems* $C(gf_1,0,0)$ *and* $D(f_1,0,0)$*, appearing in* (14c)*, are equal.*

Corollary c.2. *The stresses* $Z_\rho|_{z=0}$ *and* $Z_\alpha|_{z=0}$ *can be applied to the free of stress plane part* $z=0$ *of the boundary surface of the domain* Ω *so that on* $z=0$ *we will have the defined normal displacement* $w|_{z=0}=f_1(\rho,\alpha)$ *(with* $Z_z|_{z=0}=0$*). Indeed, if* (14c) *is represented as*

$$A(0,F_{2D},F_{3D})=D(f_1,0,0)-C(gf_1,0,0)$$

and if we take into account that in the problem $D(f_1,0,0)$ *we have* $Z_\rho|_{z=0}=F_{2D}$*,* $Z_\alpha|_{z=0}=F_{3D}$ *and that ,according to Corollary .1, the stresses* $Z_z|_{z=0}$ *in the problems* $D(f_1,0,0)$ *and* $C(gf_1,0,0)$ *are equal, then since the tangential stresses* $Z_\rho|_{z=0}=F_{2D}$ *and* $Z_\alpha|_{z=0}=F_{3D}$ *in the problem* $A(0,F_{2D},F_{3D})$ *cause the displacement* $w|_{z=0}=(1-g)f_1=k_2f_1$ *in it , the tangential stresses* $Z_\rho|_{z=0}=\frac{1}{k_2}\,F_{2D}$ *and* $Z_\alpha|_{z=0}=\frac{1}{k_2}\,F_{3D}$ *will cause the displacement* $w|_{z=0}=f_1(\rho,\alpha)$ *in the problem* $A\left(0,\frac{1}{k_2}\,F_{2D},\frac{1}{k_2}\,F_{3D}\right)$.

Remark 3. The problem of determination of the tangential stresses $Z_\rho|_{z=0}$ and $Z_\alpha|_{z=0}$, on the surface $z=0$ of the domain Ω which with $Z_z|_{z=0}=0$ cause the defined displacement $w|_{z=0}=f_1(\rho,\alpha)$, is reduced to the solution of the boundary value problem $D\left(\frac{1}{k_2}\,f_1,0,0\right)$ or, to be more precise, to the determination of the tangential stresses $Z_\rho|_{z=0}$ and $Z_\alpha|_{z=0}$ in this problem.

Corollary d.1. *The stresses* $Z_z|_{z=0}$ *in the problems* $D(f_1,0,0)$ *and* $C(gf_1,0,0)$ *or, which is the same, in the problems* $D\left(\frac{1}{g}\,f_1,0,0\right)$ *and* $C(f_1,0,0)$*, appearing in* (14d) *are equal.*

Corollary d.2. *On the free of stress plane part* $z=0$ *of the boundary surface of the domain* Ω*, the displacements* $u|_{z=0}$ *and* $v|_{z=0}$ *can be applied so that on* $z=0$ *we will have the defined normal displacement* $w|_{z=0}=f_1(\rho,\alpha)$ *(with* $Z_z|_{z=0}=0$*). Indeed, if* (14d) *is represented as*

$$B(0,f_{2C},f_{3C})=C(f_1,0,0)-D\left(\frac{1}{g}\,f_1,0,0\right)$$

and if we keep in mind that in the problem $C(f_1,0,0)$ *we have* $u|_{z=0}=f_{2C}$*,* $v|_{z=0}=f_{3C}$ *and that according to Corollary* d.1*, the stresses* $Z_z|_{z=0}$ *in the problems* $C(f_1,0,0)$ *and* $D\left(\frac{1}{g}\,f_1,0,0\right)$ *are equal, then since the displacements* $u|_{z=0}=f_{2C}$ *and* $v|_{z=0}=f_{3C}$ *cause the displacement* $w|_{z=0}=\left(1-\frac{1}{g}\right)f_1=k_1f_1$ *in the problem* $B(0,f_{2C},f_{3C})$*, the displacements* $u|_{z=0}=\frac{1}{k_1}\,f_{2C}$ *and* $v|_{z=0}=\frac{1}{k_1}\,f_{3C}$ *will cause the displacement* $w|_{z=0}=f_1(\rho,\alpha)$ *in the problem* $B\left(0,\frac{1}{k_1}\,f_{2C},\frac{1}{k_1}\,f_{3C}\right)$.

Remark 4. On the surface $z = 0$ of the domain Ω the problem of determination of the displacements $u|_{z=0}$ and $v|_{z=0}$, which with $Z_z|_{z=0} = 0$ will cause the defined displacement $w|_{z=0} = f_1(\rho, \alpha)$, is reduced to the solution of the boundary value problem $C\left(\frac{1}{k_1} f_1, 0, 0\right)$ or, to be more precise, to the determination of the displacements $u|_{z=0}$ and $v|_{z=0}$ in this problem.

To illustrate the above-stated remarks, in particular, Remark 1, in the Cartesian coordinate system for a semi-infinite tetrahedral prism $\Omega = \{0 < x < x_1,\ 0 < y < y_1,\ 0 < z < \infty\}$, on the sides $x = 0$ and $x = x_1$ of which the condition $X_x = 0$, $v = 0$, $w = 0$ is satisfied while on the sides $y = 0$, $y = y_1$ the condition $Y_y = 0$, $w = 0$, $u = 0$ is fulfilled, consider the following problem: choose on $z = 0$ stresses Z_x and Z_y so that $Z_z|_{z=0} = a \sin\left(\frac{\pi}{x_1} x\right) \sin\left(\frac{\pi}{y_1} y\right)$ $(a = const)$, while $w|_{z=0} = 0$. The solution of this problem, according to Corollary $a.2$, is reduced to the solution of a boundary value problem where with $x = 0$ and $x = x_1$ we have $X_x = 0$, $v = 0$, $w = 0$, with $y = 0$ and $y = y_1$ we have $Y_y = 0$, $w = 0$, $u = 0$, and with $z = 0$ we have $Z_z|_{z=0} = \left(\frac{\varkappa}{\varkappa - 1}\right)^2 a \sin\left(\frac{\pi}{x_1} x\right) \sin\left(\frac{\pi}{y_1} y\right)$, $u = 0$, $v = 0$. After the solution of the latter problem (see Section 2) we shall have the required tangential stresses

$$Z_x|_{z=0} = -\frac{\varkappa a}{\varkappa - 1} \frac{y_1}{\sqrt{x_1^2 + y_1^2}} \cos\left(\frac{\pi}{x_1} x\right) \sin\left(\frac{\pi}{y_1} y\right),$$
$$Z_y|_{z=0} = -\frac{\varkappa a}{\varkappa - 1} \frac{x_1}{\sqrt{x_1^2 + y_1^2}} \sin\left(\frac{\pi}{x_1} x\right) \cos\left(\frac{\pi}{y_1} y\right).$$

Remark 5. Although in our opinion Theorem 1 is of a cognitive interest, the main aim of the given paper is neither the theorem itself, nor the solution of the boundary value problems $A(F_1, 0, 0)$, $B(F_1, 0, 0)$, $C(f_1, 0, 0)$ and $D(f_1, 0, 0)$. The main aim of the article is to state and solve some non-traditional boundary value problems of elasticity given in Remarks 1, 2, 3, 4, which follow from Theorem 1. In particular, the statement of the problem is: to choose stresses $Z_\rho|_{z=0}$ and $Z_\alpha|_{z=0}$ or displacements $u|_{z=0}$ and $v|_{z=0}$, so that both the displacement and stress along the normal to the plane $z = 0$, i.e., along the axis Oz, would satisfy certain conditions, or to be more precise, to obtain the desired stress $Z_z|_{z=0}$ for $w|_{z=0} = 0$ or the desired displacement $w|_{z=0}$ for $Z_z|_{z=0} = 0$.

4. Now let us go back to the general case when $F_1 \neq 0$, $F_2 \neq 0$, $F_3 \neq 0$, $f_1 \neq 0$, $f_2 \neq 0$, $f_3 \neq 0$ and write the following obvious equalities:

$$A(F_1, F_2, F_3) = A(0, F_2, F_3) + A(F_1, 0, 0),$$
$$A(0, F_2, F_3) = C(0, F_2, F_3) - A(F_{1C}, 0, 0).$$

If we introduce the notation $A(F_1, 0, 0) - A(F_{1C}, 0, 0) = A(F_{1AC}, 0, 0)$, then

$$A(F_1, F_2, F_3) = C(0, F_2, F_3) + A(F_{1AC}, 0, 0).$$

Quite similarly we can write

$$B(F_1, f_2, f_3) = D(0, f_2, f_3) + B(F_{1BD}, 0, 0),$$
$$C(f_1, F_2, F_3) = A(0, F_2, F_3) + C(f_{1CA}, 0, 0),$$
$$D(f_1, f_2, f_3) = B(0, f_2, f_3) + D(f_{1DB}, 0, 0).$$

The last four equalities and Theorem 1 result in Theorem 2.

Theorem 2.

$$\begin{aligned}
&a)\ \ A(F_1, F_2, F_3) = C(0, F_2, F_3) + g\left[B(F_{1AC}, 0, 0) - C(0, F_{2B}, F_{3B})\right],\\
&b)\ \ B(F_1, f_2, f_3) = D(0, f_2, f_3) + \frac{1}{g}\left[A(F_{1BD}, 0, 0) - D(0, f_{2A}, f_{3A})\right],\\
&c)\ \ C(f_1, F_2, F_3) = A(0, F_2, F_3) + \frac{1}{g}\left[D(f_{1CA}, 0, 0) - A(0, F_{2D}, F_{3D})\right],\\
&d)\ \ D(f_1, f_2, f_3) = B(0, f_2, f_3) + g\left[C(f_{1DB}, 0, 0) - B(0, f_{2C}, f_{3C})\right].
\end{aligned} \tag{15}$$

It should be noted that Lemma, Theorem 1 (with its 8 corollaries and Remarks 1, 2,3 and 4) and Theorem 2 besides a semi-infinite curvilinear prism also hold for a half-space. Indeed, the proof of Lemma and Theorems 1 and 2 can be extended to a half-space if we consider the half-space, say, in the Cartesian coordinate system and assume that the function ψ_{mn} is represented by formulas of double integral Fourier transform (in the case of a half-space we have $\psi_{mn} = \tilde{\psi}_{mn}$).

5. It follows from (15a) and (15d) that the solution of the boundary value problems $A(F_1, F_2, F_3)$ and $D(f_1, f_2, f_3)$ is reduced to a superposition of the solutions of the boundary value problems $B(F_1, f_2, f_3)$ and $C(f_1, F_2, F_3)$. In some cases this circumstance can simplify the solution of problems $A(F_1, F_2, F_3)$ and $D(f_1, f_2, f_3)$, since the solution of the boundary value problems $B(F_1, f_2, f_3)$ and $C(f_1, F_2, F_3)$ can be reduced to the solution of classical boundary value problems for the Laplace equation in the domain Ω. The above mentioned can be confirmed by an example which is more general than just a reduction of the problem $B(F_1, f_2, f_3)$ or the problem $C(f_1, F_2, F_3)$ to classical problems for the Laplace equation. In particular, a mixed boundary value problem of elasticity for a semi-infinite curvilinear prism when on a part of its side $z = 0$ conditions (3b) are given while on the other we have conditions (3c) can be reduced to the solution of classical mixed boundary value problems for the Laplace equation.

Consider a boundary value problem illustrating Remark 1, with the difference that on one part S_1 of the side $z = 0$ we assume that $Z_z = F_1(x, y)$, $u = 0$, $v = 0$ and on the other part S_2 of the side $z = 0$ we assume $w = 0$, $Z_x = 0$, $Z_y = 0$. In order to reduce this problem to mixed boundary value problems for the Laplace equation write the equilibrium equations for a semi-infinite elastic prism in the following form

$$\begin{aligned}
&a)\ \Delta e = 0,\\
&b)\ \Delta \omega_z = 0,\\
&c)\ \Delta\left(w + \frac{z}{2(\varkappa-1)}\,e\right) = 0,\\
&d)\ \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 w}{\partial x \partial z} - \left(\frac{\varkappa}{\varkappa-1}\frac{\partial e}{\partial x} - \frac{\partial \omega_z}{\partial y}\right),\\
&e)\ \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 w}{\partial y \partial z} - \left(\frac{\varkappa}{\varkappa-1}\frac{\partial e}{\partial y} + \frac{\partial \omega_z}{\partial x}\right).
\end{aligned} \tag{16}$$

The boundary conditions for the functions $G,\ K$ and w will have the following form

$$\begin{aligned}
&a)\ e = 0 \text{ with } x = 0,\ x = x_1,\ y = 0,\ y = y_1;\\
&b)\ \frac{\partial e}{\partial z} = 0 \text{ on } S_2 \text{ and } e = \frac{2(\varkappa-1)(1+\nu)}{\varkappa E}F_1(x,y) \text{ on } S_1;\\
&c)\ \frac{\partial \omega_z}{\partial x} = 0 \text{ with } x = 0 \text{ and } x = x_1,\ \frac{\partial \omega_z}{\partial y} = 0 \text{ with } y = 0 \text{ and} y = y_1;\\
&d)\ \omega_z = 0 \text{ on } S_1 \text{ and } \frac{\partial \omega_z}{\partial z} = 0 \text{ on } S_2;\\
&e)\ w = 0 \text{ with } x = 0,\ x = x_1,\ y = 0,\ y = y_1;\\
&f)\ w = 0 \text{ on } S_2 \text{ and } \frac{\partial w}{\partial z} = \frac{2(\varkappa-1)(1+\nu)}{\varkappa E}F_1(x,y) \text{ on } S_1.
\end{aligned} \tag{17}$$

It follows from (16) and (17) that for the functions e, ω_z and w we have mixed boundary value problems for the Laplace equation. Moreover (16b), (17c) and (17d) imply that in the entire domain Ω we have $\omega_z = 0$. If we assume that the functions e and w have been found then we can determine the displacements u and v integrating Equalities (16d) and (16e) with respect to z. The four functions of the variables x and y resulting from the double integration with respect to z of Equalities (16d) and (16e) and which are two pairs of conjugate harmonic functions, vanish by virtue of the boundary conditions for u and v on the sides $x = 0$, $x = x_1$, $y = 0$ and $y = y_1$.

Unfortunately Theorem 2 does not hold has not been proved for finite bodies. If Theorem 2 could be extended, say, onto the case of the rectangular parallelepiped $\Omega = \{0 < x < x_1, 0 < y < y_1, 0 < z < z_1\}$ ($z_1 = const$) with non-homogenous boundary conditions on the six sides, then it would be possible to give an analytical (precise) solution of all basic boundary value problems of elasticity for the rectangular parallelepiped. Indeed, the boundary value problem of elastic equilibrium of the rectangular parallelepiped with non-homogenous symmetry and antisymmetry conditions on the sides (that is: conditions (3b,c), similar conditions on $z = z_1$ and non-homogenous conditions (1), (2)) allows an analytical solution, since it can be reduced to the integration of Laplace equations with classical boundary conditions [5]. In particular, it would be possible to give an analytical solution of Lamé's problem (stated in 1852) on the elastic equilibrium of the rectangular parallelepiped with stresses defined on its surface [6].

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