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FIRST ORDER PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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Abstract. The phase space for nonlinear hyperbolic functional differential equations with unbounded delay is constructed. The set of axioms for generalized solutions of initial problems is presented. A theorem on the existence and continuous dependence upon initial data is given. The Cauchy problem is transformed into a system of integral functional equations. The existence of solutions of this system is proved by the method of successive approximations and by using theorems on integral inequalities. Examples of phase spaces are given.

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1. Introduction

For any metric spaces U and V we denote by $C(U, V)$ the class of all continuous functions from U to V. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let us denote by $M_{k\times n}$ the set of all $k \times n$ matrices with real elements. For $x \in R^n$, $Y \in M_{k\times n}$, where $x = (x_1, \dots, x_n)$, $Y = [y_{ij}]_{i=1,\dots,k,\ j=1,\dots,n}$, we define the norms

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad \|Y\| = \max\Big\{ \sum_{j=1}^{n} |y_{ij}| : 1 \le i \le k \Big\}.$$

We will denote by $L([0, c], R_+)$, $c > 0$, $R_+ = [0, +\infty)$, the class of all functions $\gamma : [0, c] \to R_+$, which are integrable on $[0, c]$. Let $B = (-\infty, 0] \times [-r, r]$ where $r = (r_1, \dots, r_n) \in R^n_+$, $R_+ = [0, +\infty)$. For a function $z : (-\infty, a] \times R^n \to R$, $a > 0$, and for a point $(t, x) \in (-\infty, a] \times R^n$ we define a function $z_{(t,x)} : B \to R$ as follows: $z_{(t,x)}(s, y) = z(t + s, x + y)$, $(s, y) \in B$. Suppose that the functions $\psi_0 : [0, a] \to R$ and $\psi' = (\psi_1, \dots, \psi_n) : [0, a] \times R^n \to R^n$ are given. The requirement on ψ_0 is that $\psi_0(t) \le t$ for $t \in [0, a]$. For $(t, x) \in R^{1+n}$ we write $\psi(t, x) = (\psi_0(t), \psi'(t, x))$.

The phase space X for equations with unbounded delay is a linear space with the norm $\|\cdot\|_X$ consisting of functions mappings the set B into R. Write $\Omega = [0, a] \times R^n \times X \times R^n$ and suppose that the functions

$$f : \Omega \to R \quad \text{and} \quad \varphi : (-\infty, 0] \times R^n \to R$$

are given. We consider the nonlinear functional equation

$$\partial_t z(t, x) = f\big(t, x, z_{\psi(t,x)}, \partial_x z(t, x)\big), \tag{1}$$

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with the initial condition

$$z(t,x) = \varphi(t,x), \quad (t,x) \in (-\infty, 0] \times R^n, \tag{2}$$

where $\partial_x z(t,x) = (\partial_{x_1} z(t,x), \ldots, \partial_{x_n} z(t,x))$. Note that the symbol $z_{\psi(t,x)}$ denotes the restriction of z to the set $(-\infty, \psi_0(t)] \times [\psi'(t,x) - r, \psi'(t,x) + r]$ and this restriction is shifted to the set B.

We consider weak solutions of problem (1), (2). A function $\bar{z} : (-\infty, c] \times R^n \to R$, $0 < c \le a$, is a solution to the above problem if

(i) $\bar{z}_{\psi(t,x)} \in X$ for $(t,x) \in [0,c] \times R^n$ and $\partial_x \bar{z}$ exists on $[0,c] \times R^n$,

(ii) the function $\bar{z}(\cdot, x) : [0,c] \to R$ is absolutely continuous on $[0,c]$ for each $x \in R^n$,

(iii) for each $x \in R^n$ equation (1) is satisfied for almost all $t \in [0,c]$ and condition (2) holds.

Numerous papers have been published concerning functional differential equations of the form

$$\partial_t z(t,x) = F\big(t, x, z_{(t,x)}, \partial_x z(t,x)\big),$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, and for adequate weakly coupled systems. The following questions have been considered: functional differential inequalities generated by initial or mixed problems ([3]–[5]), uniqueness of classical or generalized solutions ([12], [24], [25]), existence theory of classical and different classes of weak solutions ([2], [6], [7], [9], [11], [16], [18], [20], [22], [23], [26]–[28]), approximate solutions of initial or mixed problems ([17], [30]), difference inequalities and applications ([13], [19]). All these problems have the property that the initial or boundary functions are given on bounded sets. The theory of hyperbolic functional differential problems has been developed in the monograph [14].

The paper [8] initiated the investigation of nonlinear hyperbolic functional differential equations with unbounded delay. The main assumptions in the existence theorem concern the space X and the space $C([0,a] \times R^n, R)$ and their suitable subspaces. The assumptions are formulated in the form of inequalities for norms in some functional spaces. They are strictly connected with initial problems and are not applicable to initial boundary value problems.

The purpose of this paper is to give sufficient conditions for the existence of generalized solutions to problem (1), (2). The Cauchy problem is transformed into a system of integral functional equations. The method of bicharacteristics is used. It consists of linearization of the right-hand side of equation (1) with respect to the last variable. In the second step a quasilinear system is constructed for unknown functions and for their spatial derivatives. The system obtained in this way is equivalent to a system of integral functional equations of the Volterra type. The classical solution of this system generates a generalized solution of the original problem. A result on continuous dependence is proved by using the method of integral inequalities. Note that our results are new also in the case where the set B is bounded. It is important in our considerations that all the assumptions on the phase space can be adapted to initial boundary value problems.

Existence results for a class of parabolic differential integral equations with unbounded delay can be found in [1]. The theory of ordinary functional differential equations with unbounded delay is studied in [10], [15]. For further bibliography concerning functional partial differential equations see the monographs [14], [21], [29].

2. Definitions and Fundamental Axioms

Assume that $c > 0$, $w : (-\infty, c] \times [-r, r] \to R$ and $t \in (-\infty, c]$. We define a function $w_{(t)} : B \to R$ by $w_{(t)}(s, y) = w(t + s, y)$, $(s, y) \in B$. For each $t \in (-\infty, c]$ the function $w_{(t)}$ is the restriction of w to the set $(-\infty, t] \times [-r, r]$ and this restriction is shifted to the set B. If $w : (-\infty, c] \times [-r, r] \to R$, $c > 0$, and $w|_{[0,c]\times[-r,r]}$ is continuous, then we write

$$\|w\|_{[0,t]} = \max\big\{|w(s, y)| : (s, y) \in [0, t] \times [-r, r]\big\},$$

where $t \in [0, c]$.

Assumption H$[X]$. Suppose that $(X, \|\cdot\|_X)$ is a Banach space and

1) there is $\chi \in R_+$ independent of w such that for each function $w \in X$ we have

$$|w(0, x)| \le \chi\|w\|_X, \quad x \in [-r, r], \tag{3}$$

2) if $w : (-\infty, c] \times [-r, r] \to R$, $c > 0$, is a function such that $w_{(0)} \in X$ and $w|_{[0,c]\times[-r,r]}$ is continuous, then $w_{(t)} \in X$ for $t \in [0, c]$ and

(i) the function $t \to w_{(t)}$ is continuous on $[0, c]$,

(ii) there are $K, K_0 \in R_+$ independent of w such that

$$\|w_{(t)}\|_X \le K\|w\|_{[0,t]} + K_0\|w_{(0)}\|_X, \quad t \in [0, c]. \tag{4}$$

Now we give examples of phase spaces

Example 1. Let X be the class of all functions $w : B \to R$ which are uniformly continuous and bounded on B. For $w \in X$ we put

$$\|w\|_X = \sup\big\{|w(s, y)| : (s, y) \in B\big\}. \tag{5}$$

Then Assumption H$[X]$ is satisfied with all the constants equal to 1.

Example 2. Let X be the class of all functions $w : B \to R_+$ such that $w \in C(B, R)$ and there exists the limit $\lim_{t\to-\infty} w(t, x) = w_0(x)$ uniformly with respect to $x \in [-r, r]$. The norm in the space X is defined by (5). Then Assumption H$[X]$ is satisfied with all the constants equal to 1.

Example 3. Let $\gamma : (-\infty, 0] \to (0, \infty)$ be a continuous function. Assume also that γ is nonincreasing on $(-\infty, 0]$. Let X be the space of all continuous functions $w : B \to R$ such that

$$\lim_{t\to-\infty} \frac{|w(t, x)|}{\gamma(t)} = 0, \quad x \in [-r, r].$$

Put

$$\|w\|_X = \sup\left\{\frac{|w(t, x)|}{\gamma(t)} : \ (t, x) \in B\right\}.$$

Then Assumption H$[X]$ is satisfied with $K = \frac{1}{\gamma(0)}$, $K_0 = 1$, $\chi = \gamma(0)$.

Example 4. Let $p \geq 1$ be fixed. Denote by V the class of all $w : B \to R$ such that

(i) w is continuous on $\{0\} \times [-r, r]$ and

$$\int_{-\infty}^{0} |w(\tau, x)|^p d\tau < +\infty \quad \text{for} \quad x \in [-r, r],$$

(ii) for each $t \in (-\infty, 0]$ the function $w(t, \cdot) : [-r, r] \to R$ is continuous.

We define the norm in the space V by

$$\|w\|_V = \max\big\{|w(t,x)| : (t,x) \in \{0\} \times [-r,r]\big\} + \sup\Big\{\Big(\int_{-\infty}^{0} |w(\tau,x)|^p d\tau\Big)^{1/p} : x \in [-r,r]\Big\}. \tag{6}$$

Write $X = \bar{V}$ where $\bar{V}$ is the closure of V with the norm (6). Then Assumption H$[X]$ is satisfied with $K = 1 + c^{1/p}$, $K_0 = 1$, $\chi = 1$.

Example 5. Denote by V the set of all functions $w : B \to R$ which are bounded and

(i) w is continuous on $\{0\} \times [-r, r]$ and

$$I(x) = \sup\Big\{\int_{-m}^{-(m-1)} |w(\tau, x)| d\tau : m \in \mathbf{N}\Big\} < +\infty,$$

where $x \in [-r, r]$ and $\mathbf{N}$ is the set of natural numbers;

(ii) for each $t \in (-\infty, 0]$ the function $w(t, \cdot) : [-r, r] \to R$ is continuous.

The norm in the space V is defined by

$$\|w\|_V = \max\big\{|w(t,x)| : (t,x) \in \{0\} \times [-r,r]\big\} + \sup\big\{I(x) : x \in [-r,r]\big\}.$$

Put $X = \bar{V}$ where $\bar{V}$ is the closure on V with the above given norm. Then Assumption H$[X]$ is satisfied with $K = 1 + c$, $K_0 = 2$, $\chi = 1$.

If $z : (-\infty, c] \times R^n \to R$, $c > 0$, is a function such that $z|_{[0,c] \times R^n}$ is continuous and $(t, x) \in [0, c] \times R^n$, then we put

$$\|z\|_{[0,t;x]} = \max\big\{|z(s,y)| : (s,y) \in [0,t] \times [x-r, x+r]\big\}.$$

Suppose additionally that the function $z|_{[0,c] \times R^n}$ satisfies the Lipschitz condition with respect to x. Then we write

$$\text{Lip}[z]|_{[0,t;R^n]} = \sup\Big\{\frac{|z(s,y) - z(s,\bar{y})|}{\|y - \bar{y}\|} : (s,y), (s,\bar{y}) \in [0,t] \times R^n, y \neq \bar{y}\Big\}.$$

Lemma 1. *Suppose that Assumption $H[X]$ is satisfied and $z : (-\infty, c] \times R^n \to R$, $0 < c \leq a$. If $z_{(0,x)} \in X$ for $x \in R^n$ and $z|_{[0,c] \times R^n}$ is continuous, then $z_{(t,x)} \in X$ for $(t, x) \in (0, c] \times R^n$ and*

$$\|z_{(t,x)}\|_X \leq K \|z\|_{[0,t;x]} + K_0 \|z_{(0,x)}\|_X. \tag{7}$$

If we assume additionally that the function $z|_{[0,c] \times R^n}$ satisfies the Lipschitz condition with respect to x, then

$$\|z_{(t,x)} - z_{(t,\bar{x})}\|_X \leq K \,\text{Lip}[z]|_{[0,t;R^n]} \|x - \bar{x}\| + K_0 \|z_{(0,x)} - z_{(0,\bar{x})}\|_X, \tag{8}$$

where $(t,x),(t,\bar{s}) \in [0,c] \times R^n$.

Proof. Inequality (7) is a consequence of (4) for $w : (-\infty,c] \times [-r,r] \to R$ given by $w(s,y) = z(s,x+y)$ with fixed $x \in R^n$. We prove (8). Suppose that $(t,x),(t,\bar{x}) \in [0,c] \times R^n$ and the function $\tilde{z} : (-\infty,c] \times R^n \to R$ is defined by $\tilde{z}(s,y) = z(s,y+\bar{x}-x)$, $(s,y) \in (-\infty,c] \times R^n$. Then $\tilde{z}_{(t,x)} = z_{(t,\bar{x})}$ and

$$\|z_{(t,x)} - z_{(t,\bar{x})}\|_X = \|(z-\tilde{z})_{(t,x)}\| \le K\|z-\tilde{z}\|_{[0,t;x]} + K_0\|(z-\tilde{z})_{(0,x)}\|_X$$
$$\le K\,\mathrm{Lip}[z]|_{[0,t;R^n]}\|x-\bar{x}\| + K_0\|z_{(0,x)} - z_{(0,\bar{x})}\|_X,$$

which proves (8). □

Our basic assumption on the initial functions is the following.

Assumption H$[\varphi]$. Suppose that $\varphi : (-\infty,0] \times R^n \to R$ and there exist the derivatives $(\partial_{x_1}\varphi,\dots,\partial_{x_n}\varphi) = \partial_x\varphi$ on $(-\infty,0] \times R^n$ and

1) $\varphi_{(0,x)} \in X$ and $(\partial_{x_i}\varphi)_{(0,x)} \in X$, $1 \le i \le n$, for $x \in R^n$,

2) there are $b_0,b_1,c_0,c_1 \in R_+$ such that

$$\|\varphi_{(0,x)}\|_X \le b_0, \quad \|\varphi_{(0,x)} - \varphi_{(0,\bar{x})}\|_X \le b_1\|x-\bar{x}\|,$$
$$\|(\partial_x\varphi)_{(0,x)}\|_X \le c_0, \quad \|(\partial_x\varphi)_{(0,x)} - (\partial_x\varphi)_{(0,\bar{x})}\|_X \le c_1\|x-\bar{x}\|,$$

where $x,\bar{x} \in R^n$ and

$$\|(\partial_x\varphi)_{(0,x)}\|_X = \sum_{i=1}^{n} \|(\partial_{x_i}\varphi)_{(0,x)}\|_X,$$
$$\left\|(\partial_x\varphi)_{(0,x)} - (\partial_x\varphi)_{(0,\bar{x})}\right\|_X = \sum_{i=1}^{n} \left\|(\partial_{x_i}\varphi)_{(0,x)} - (\partial_{x_i}\varphi)_{(0,\bar{x})}\right\|_X.$$

Let us denote by $I[X]$ the class of all initial functions $\varphi : (-\infty,0] \times R^n \to R$ satisfying Assumption H$[\varphi]$. We define some function spaces. Let $\varphi \in I[X]$ and let $0 < c \le a$, $d = (d_0,d_1,d_2) \in R^3_+$, $\lambda = (\lambda_0,\lambda_1)$, where $\lambda_0,\lambda_1 \in L([0,c],R_+)$. Let us denote by $C^{1.L}_{\varphi.c}[d,\lambda]$ the class of all functions $z : (-\infty,c] \times R^n \to R$ such that

(i) $z(t,x) = \varphi(t,x)$ for $(t,x) \in (-\infty,0] \times R^n$;

(ii) the derivatives $(\partial_{x_1}z,\dots,\partial_{x_n}z) = \partial_x z$ exist on $[0,c] \times R^n$ and

$$|z(t,x)| \le d_0, \quad |z(t,x) - z(\bar{t},x)| \le \left|\int_t^{\bar{t}} \lambda_0(\tau)d\tau\right|,$$

and

$$\|\partial_x z(t,x)\| \le d_1, \quad \left\|\partial_x z(t,x) - \partial_x z(\bar{t},\bar{x})\right\| \le \left|\int_t^{\bar{t}} \lambda_1(\tau)d\tau\right| + d_2\|x-\bar{x}\|$$

on $[0,c] \times R^n$.

Let $\varphi \in I[X]$ be given and $p = (p_0,p_1) \in R^2_+$, $\mu \in L([0,c],R_+)$, $0 < c \le a$. We will denote by $C^L_{\partial\varphi.c}[p,\mu]$ the class of all functions $u : (-\infty,c] \times R^n \to R^n$ such that

(i) $u(t,x) = \partial_x\varphi(t,x)$ for $(t,x) \in (-\infty,0] \times R^n$;

(ii) for $(t,x) \in [0,c] \times R^n$ we have

$$\|u(t,x)\| \le p_0, \quad \big\|u(t,x) - u(\bar t,\bar x)\big\| \le \left| \int_t^{\bar t} \mu(\tau)d\tau \right| + p_1\|x-\bar x\|].$$

We will prove that under suitable assumptions on f and ψ and for sufficiently small c, $0 < c \le a$, there exists a solution $\bar z$ to problem (1), (2) such that $\bar z \in C^{1.L}_{\varphi.c}[d,\lambda]$ and $\partial_x \bar z \in C^{L}_{\partial\varphi.c}[p,\mu]$.

Let us fix our notations on vectors and matrices. The product of two matrices is denoted by $\diamond$. If $Y \in M_{k\times k}$ then Y^T is the transposed matrix. We use the symbol $\circ$ to denote the scalar product in R^n. If $y = (y_1,\dots,y_n) \in R^n$, $w = (w_1,\dots,w_n)$ and $w_i \in X$ for $1 \le i \le n$ then $y \circ w$ is the function defined by $y \circ w = y_1w_1 + \ldots + y_nw_n$. In the sequel we will need the following lemma.

Lemma 2. *Suppose that Assumption $H[X]$ is satisfied and $\varphi \in I[X]$, $z \in C^{1.L}_{\varphi.c}[d,\lambda]$, where $0 < c \le a$. Then*

$$\big\|\varphi_{(0,\bar y)} - \varphi_{(0,y)} - (\partial_x\varphi)_{(0,y)} \circ (\bar y - y)\big\|_X \le c_1\|y-\bar y\|^2, \tag{9}$$

where $y, \bar y \in R^n$, and

$$\big\|z_{(t,\bar y)} - z_{(t,y)} - (\partial_x z)_{(t,y)} \circ (\bar y - y)\big\|_X \le (Kd_2 + K_0c_1)\|y-\bar y\|^2, \tag{10}$$

where $(t,y), (t,\bar y) \in [0,c] \times R^n$.

Proof. Suppose that $y, \bar y \in R^n$ and the function $\bar\varphi : (-\infty,0] \times R^n \to R$ is defined by $\bar\varphi(\tau,x) = \varphi(\tau, \bar y - y + x)$, $(\tau,x) \in (-\infty,0] \times R^n$. Write $\beta = \bar\varphi - \varphi - (\partial_x\varphi) \circ (\bar y - y)$. Then inequality (9) is equivalent to $\|\beta_{(0,y)}\|_X \le c_1\|\bar y - y\|^2$. It follows that there is $s \in [0,1]$ such that

$$\beta_{(0,y)} = \big[(\partial_x\varphi)_{(0,y+s(\bar y - y))} - (\partial_x\varphi)_{(0,y)}\big] \circ (\bar y - y).$$

Then we obtain (9) from condition 2) of Assumption H$[\varphi]$.

We prove inequality (10). Let $\bar z : (-\infty,c] \times R^n \to R$ be the function given by

$$\bar z(\tau,x) = z(\tau, \bar y - y + x), \quad (\tau,x) \in (-\infty,c] \times R^n$$

and $v = \bar z - z - (\partial_x z) \circ (\bar y - y)$. Assertion (10) is equivalent to

$$\|v_{(t,y)}\|_X \le (Kd_2 + K_0c_1)\|y-\bar y\|^2.$$

According to Lemma 1, we have

$$\|v_{(t,y)}\|_X \le K\|v\|_{[0,t;y]} + K_0\|v_{(0,y)}\|_X \le Kd_2\|y-\bar y\|^2$$
$$+K_0\big\|\varphi_{(0,\bar y)} - \varphi_{(0,y)} - (\partial_x\varphi)_{(0,y)} \circ (\bar y - y)\big\| \le (Kd_2 + K_0c_1)\|y-\bar y\|^2,$$

which completes the proof of (10). □

3. Bicharacteristics of Nonlinear Functional Differential Equations

We will need the following assumptions.

Assumption H$[\partial_q f]$. Suppose that $f : \Omega \to R$ and

1) the derivatives

$$\big(\partial_{q_1} f(P), \dots, \partial_{q_n} f(P)\big) = \partial_q f(P)$$

exist for $P = (t, x, w, q) \in \Omega$ and the function $\partial_q f(\cdot, x, w, q) : [0, a] \to R^n$ is measurable for every $(x, w, q) \in R^n \times X \times R^n$ and there is a function $\alpha \in L([0, a], R_+)$ such that

$$\big\|\partial_q f(t, x, w, q)\big\| \le \alpha(t) \quad \text{on} \quad \Omega,$$

2) there is a function $\gamma \in L([0, a], R_+)$ such that

$$\big\|\partial_q f(t, x, w, q) - \partial_q f(t, \bar{x}, \bar{w}, \bar{q})\big\| \le \gamma(t)\big[\|x - \bar{x}\| + \|w - \bar{w}\|_X + \|q - \bar{q}\|\big]$$

on Ω.

Assumption H$[\psi]$. Suppose that the functions $\psi_0 : [0, a] \to R$ and $\psi' = (\psi_1, \dots, \psi_n) : [0, a] \times R^n \to R^n$ are continuous and $\psi_0(t) \le t$ for $t \in (0, a]$. Assume that the partial derivatives

$$\big[\partial_{x_j} \psi_i(t, x)\big]_{i,j=1,\dots,n} = \partial_x \psi'(t, x)$$

exist on $[0, a] \times R^n$ and there are $s_0, s_1 \in R_+$ such that

$$\big\|\partial_x \psi'(t, x)\big\| \le s_0, \quad \big\|\partial_x \psi'(t, x) - \partial_x \psi'(t, \bar{x})\big\| \le s_1 \|x - \bar{x}\| \text{ on } [0, a] \times R^n.$$

Suppose that Assumptions H$[X]$, H$[\partial_q f]$, H$[\psi]$ are satisfied and $\varphi \in I[X]$, $z \in C^{1.L}_{\varphi.c}[d, \lambda]$, $u \in C^{L}_{\partial\varphi.c}[p, \mu]$. Consider the Cauchy problem

$$\eta'(\tau) = -\partial_q f\big(\tau, \eta(\tau), z_{\psi(\tau, \eta(\tau))}, u(\tau, \eta(\tau))\big), \quad \eta(t) = x, \tag{11}$$

where $(t, x) \in [0, c] \times R^n$. Let us denote by $g[z, u](\cdot, t, x)$ the solution to (11). The function $g[z, u]$ is the bicharacteristic of equation (1) corresponding to z and u. We prove a theorem on the existence and uniqueness and on the regularity of bicharacteristics. For functions $\varphi \in I[X]$ and $z \in C^{1.L}_{\varphi.c}[d, \lambda]$, $u \in C^{L}_{\partial\varphi.c}[p, \mu]$ we write

$$\|\varphi\|_{X, R^n} = \sup\big\{\|\varphi_{(0,x)}\|_X : x \in R^n\big\}$$

and

$$\|z\|_t = \sup\big\{|z(s, y)| : (s, y) \in [0, t] \times R^n\big\},$$
$$\|u\|_t = \sup\big\{\|u(s, y)\| : (s, y) \in [0, t] \times R^n\big\},$$

where $t \in [0, c]$.

Theorem 1. *Suppose that Assumptions $H[X]$, $H[\partial_q f]$, $H[\psi]$ are satisfied and $\varphi, \bar{\varphi} \in I[X]$, $z \in C^{1.L}_{\varphi.c}[d, \lambda]$, $\bar{z} \in C^{1.L}_{\bar{\varphi}.c}[d, \lambda]$, $u \in C^{L}_{\partial\varphi.c}[p, \mu]$, $\bar{u} \in C^{L}_{\partial\bar{\varphi}.c}[p, \mu]$,* *where* $0 < c \le a$. *Then the solutions* $g[z,u](\cdot,t,x)$ *and* $g[\bar{z},\bar{u}](\cdot,t,x)$ *exist on* $[0,c]$, *they are unique and we have the estimates*

$$\big\|g[z,u](\tau,t,x) - g[z,u](\tau,\bar{t},\bar{x})\big\| \le \Theta(\tau,t)\Big[\Big|\int_{\bar{t}}^{t}\alpha(\xi)d\xi\Big| + \|x-\bar{x}\|\Big] \tag{12}$$

for $(t,x),(\bar{t},\bar{x}) \in [0,c]\times R^n$, $\tau\in[0,c]$, *where*

$$\Theta(\tau,t) = \exp\Big[A\Big|\int_t^{\tau}\gamma(\xi)d\xi\Big|\Big], \quad A = 1 + p_1 + s_0(Kd_1 + K_0 b_1),$$

and

$$\begin{aligned}&\big\|g[z,u](\tau,t,x) - g[\bar{z},\bar{u}](\tau,t,x)\big\|\\ &\le \Theta(\tau,t)\Big|\int_t^{\tau}\gamma(\xi)\Big|\big[K\|z-\bar{z}\|_\xi + \|u-\bar{u}\|_\xi + K_0\|\varphi-\bar{\varphi}\|_{X,R^n}\big]d\xi,\end{aligned} \tag{13}$$

where $(\tau,t,x)\in[0,c]\times[0,c]\times R^n$.

Proof. We begin by proving that problem (11) has exactly one solution. It follows from Assumptions H$[\psi]$, H$[\partial_q f]$ and Lemma 1 that the following Lipschitz condition is satisfied:

$$\big\|\partial_q f(\tau,y,z_{\psi(\tau,y)},u(\tau,y)) - \partial_q f(\tau,\bar{y},z_{\psi(\tau,\bar{y})},u(\tau,\bar{y}))\big\| \le \gamma(\tau)A\|y-\bar{y}\|,$$

where $\tau\in[0,c]$, $y,\bar{y}\in R^n$. It follows that there exists exactly one Carathéodory solution to problem (11) and the solution is defined on the interval $[0,c]$.

Now we prove estimate (12). We transform (11) into an integral equation. Write

$$P[z,u](\xi,t,x) = \big(\xi, g[z,u](\xi,t,x), z_{\psi(\xi,g[z,u](\xi,t,x))}, u(\xi,g[z,u](\xi,t,x))\big).$$

It follows from Assumptions H$[\psi]$, H$[\partial_q f]$ and Lemma 1 that

$$\begin{aligned}\big\|g[z,u](\tau,t,x) - g[z,u](\tau,\bar{t},\bar{x})\big\| &\le \|x-\bar{x}\| + \Big|\int_t^{\bar{t}}\alpha(\xi)d\xi\Big|\\ &+\Big|\int_{\tau}^{t}\big\|\partial_q f(P[z,u](\xi,t,x)) - \partial_q f(P[z,u](\xi,\bar{t},\bar{x}))\big\|d\xi\Big|\\ \le \|x-\bar{x}\| + \Big|\int_t^{\bar{t}}\alpha(\xi)d\xi\Big| &+ A\Big|\int_t^{\tau}\gamma(\xi)\big\|g[z,u](\xi,t,x) - g[z,u](\xi,\bar{t},\bar{x})\big\|d\xi\Big|,\end{aligned}$$

where $(t,x),(\bar{t},\bar{x})\in[0,c]\times R^n$, $\tau\in[0,c]$. Now we obtain (12) from the Gronwall inequality. Our next aim is to prove (13). For $z\in C^{1.L}_{\varphi.c}[d,\lambda]$, $\bar{z}\in C^{1.L}_{\bar{\varphi}.c}[d,\lambda]$ and $u\in C^{L}_{\partial\varphi.c}[p,\mu]$ $\bar{u}\in C^{L}_{\partial\bar{\varphi}.c}[p,\mu]$ we have

$$\begin{aligned}&\big\|g[z,u](\tau,t,x) - g[\bar{z},\bar{u}](\tau,t,x)\big\|\\ &\le \Big|\int_t^{\tau}\big\|\partial_q f(P[z,u](\xi,t,x)) - \partial_q f(P[\bar{z},\bar{u}](\xi,t,x))\big\|d\xi\Big|.\end{aligned} \tag{14}$$

It follows from Assumption H$[X]$ and Lemma 1 that

$$\big\|z_{\psi(\xi,g[z,u](\xi,t,x))} - \bar{z}_{\psi(\xi,g[\bar{z},\bar{u}](\xi,t,x))}\big\|_X \le \big\|z_{\psi(\xi,g[z,u](\xi,t,x))} - z_{\psi(\xi,g[\bar{z},\bar{u}](\xi,t,x))}\big\|_X$$

$$+\big\|z_{\psi(\xi,g[\bar z,\bar u](\xi,t,x))}-\bar z_{(\xi,g[\bar z,\bar u](\xi,t,x))}\big\|_X \le s_0(Kd_1+K_0b_1)\big\|g[z,u](\xi,t,x)$$
$$-g[\bar z,\bar u](\xi,t,x)\big\|+K\|z-\bar z\|_\xi+K_0\|\varphi-\bar\varphi\|_{X,R^n},$$

where $(\xi,t,x)\in[0,c]\times[0,c]\times R^n$. In a similar way we obtain

$$\big\|u(\xi,g[z,u](\xi,t,x))-\bar u(\xi,g[\bar z,\bar u](\xi,t,x))\big\|$$
$$\le \|u-\bar u\big\|_\xi+p_1\|g[z,u](\xi,t,x)-g[\bar z,\bar u](\xi,t,x)\big\|,$$

where $(\xi,t,x)\in[0,c]\times[0,c]\times R^n$. The above estimates and (14) imply the integral inequality

$$\big\|g[z,u](\tau,t,x)-g[\bar z,\bar u](\tau,t,x)\big\|$$
$$\le \left|\int_t^\tau \gamma(\xi)\big[K\|z-\bar z\|_\xi+\|u-\bar u\|_\xi+K_0\|\varphi-\bar\varphi\|_{X,R^n}\big]d\xi\right|$$
$$+A\left|\int_t^\tau \gamma(\xi)\big\|g[z,u](\xi,t,x)-g[\bar z,\bar u](\xi,t,x)\big\|d\xi\right|,$$

where $(\xi,t,x)\in[0,c]\times[0,c]\times R^n$. Now we obtain (13) from the Gronwall inequality. □

4. Functional integral equations

Let us denote by $CL(X,R)$ the set of all linear and continuous operators from X to R. The norm in the space $CL(X,R)$ will be denoted by $\|\cdot\|_*$. We formulate further assumptions on f.

Assumption H$[f]$. Suppose that Assumption H$[\partial_q f]$ is satisfied and
1) there is $\tilde\gamma\in L([0,a],R_+)$ such that $|f(t,x,w,q)|\le\tilde\gamma(t)$ on Ω,
2) for every $P=(t,x,w,q)\in\Omega$ there exist the derivatives

$$\big(\partial_{x_1}f(P),\dots,\partial_{x_n}f(P)\big)=\partial_x f(P)$$

and the Fréchet derivative $\partial_w f(P)\in CL(X,R)$ and the estimates

$$\|\partial_x f(P)\|,\ \|\partial_w f(P)\|_*\le\alpha(t)$$

are satisfied on Ω,
3) the Lipschitz conditions

$$\big\|\partial_x f(t,x,w,q)-\partial_x f(t,\bar x,\bar w,\bar q)\big\|\le\gamma(t)\big[\|x-\bar x\|+\|w-\bar w\|_X+\|y-\bar y\|\big],$$
$$\big\|\partial_w f(t,x,w,q)-\partial_w f(t,\bar x,\bar w,\bar q)\big\|_*\le\gamma(t)\big[\|x-\bar x\|+\|w-\bar w\|_X+\|y-\bar y\|\big],$$

are satisfied on Ω.

Remark 1. To simplify the formulation of the existence result we have assumed the same estimate for the derivatives $\partial_x f$, $\partial_q f$ and $\partial_w f$. We have also assumed the Lipschitz condition for these derivatives with the same coefficient.

If Assumption H$[f]$ if satisfied and $P=(t,x,w,q)\in\Omega$, $\tilde w=(\tilde w_1,\dots,\tilde w_n)$ and $\tilde w_i\in C(X,R)$ for $1\le i\le n$, then we write

$$\partial_w f(P)*\tilde w=\big(\partial_w f(P)\tilde w_1,\dots,\partial_w f(P)\tilde w_n\big).$$

Now we formulate the system of integral functional equations corresponding to problem (1), (2). Suppose that $\varphi \in I[X]$ and $z \in C^{1.L}_{\varphi.c}[d,\lambda]$, $u \in C^{L}_{\partial\varphi.c}[p,\mu]$. Write

$$F[z,u](t,x) = \varphi\big(0, g[z,u](0,t,x)\big) + \int_0^t \big[f(P[z,u](\xi,t,x)) - \partial_q f(P[z,u](\xi,t,x)) \circ u(\xi, g[z,u](\xi,t,x))\big] d\xi \tag{15}$$

and

$$G[z,u](t,x) = \partial_x\varphi\big(0, g[z,u](0,t,x)\big) + \int_0^t \Big[\partial_x f(P[z,u](\xi,t,x)) + \big[\partial_w f(P[z,u](\xi,t,x)) * u_{\psi(\xi,g[z,u](\xi,t,x))}\big] \diamond \partial_x\psi'(\xi, g[z,u](\xi,t,x))\Big] d\xi, \tag{16}$$

where $u_{\psi(s,y)} = ((u_1)_{\psi(s,y)}, \dots, (u_n)_{\psi(s,y)})$ for $(s,y) \in [0,c] \times R^n$ and $G[z,u] = (G_1[z,u], \dots, G_n[z,u])$.

We will consider the following system of functional integral equations:

$$z = F[z,u], \quad u = G[z,u], \quad \text{and} \quad z = \varphi,\ u = \partial_x\varphi \quad \text{on} \quad (-\infty,0] \times R^n. \tag{17}$$

The proof of the existence of a solution of problem (17) is based on the following method of successive approximations. Suppose that $\varphi \in I[X]$ and that Assumptions H$[X]$, H$[f]$ and H$[\psi]$ are satisfied. We define the sequence $\{z^{(m)}, u^{(m)}\}$ in the following way. Write

$$z^{(0)}(t,x) = \varphi(t,x) \text{ on } (-\infty,0] \times R^n, \quad z^{(0)}(t,x) = \varphi(0,x) \text{ on } (0,c] \times R^n \tag{18}$$

and

$$u^{(0)}(t,x) = \partial_x\varphi(t,x) \text{ on } (-\infty,0] \times R^n, \quad u^{(0)}(t,x) = \partial_x\varphi(0,x) \text{ on } (0,c] \times R^n. \tag{19}$$

Then $z^{(0)} \in C^{1.L}_{\varphi.c}[d,\lambda]$ and $u^{(0)} \in C^{L}_{\partial\varphi.c}[p,\mu]$. Suppose now that $z^{(m)} \in C^{1.L}_{\varphi.c}[d,\lambda]$ and $u^{(m)} \in C^{L}_{\partial\varphi.c}[p,\mu]$ are known functions. Then

(i) $u^{(m+1)}$ is a solution of the problem

$$u = G^{(m)}[u], \quad u(t,x) = \partial_x\varphi(t,x) \quad \text{on} \quad (-\infty,0] \times R^n, \tag{20}$$

where $G^{(m)} = (G^{(m)}_1, \dots, G^{(m)}_n)$ and

$$G^{(m)}[u](t,x) = \partial_x\varphi\big(0, g[z^{(m)},u](0,t,x)\big) + \int_0^t \Big\{\partial_x f(P[z^{(m)},u](\xi,t,x)) + \big[\partial_w f(P[z^{(m)},u](\xi,t,x)) * (u^{(m)})_{\psi(\xi,g[z^{(m)},u](\xi,t,x))}\big] \diamond \partial_x\psi'(\xi, g[z^{(m)},u](\xi,t,x))\Big\} d\xi; \tag{21}$$

(ii) the function $z^{(m+1)}$ is given by

$$\begin{aligned} z^{(m+1)}(t,x) &= F[z^{(m)}, u^{(m+1)}](t,x) \quad \text{on} \quad [0,c] \times R^n, \\ z^{(m+1)}(t,x) &= \varphi(t,x) \quad \text{on} \quad (-\infty,0] \times R^n. \end{aligned} \tag{22}$$

The problem of the existence of the sequence $\{z^{(m)}, u^{(m)}\}$ is the main difficulty in our method. We prove that this sequence exists provided $c \in (0, a]$ is sufficiently small.

5. The existence of the sequence of successive approximations

We begin with the construction of function spaces $C^{1.L}_{\varphi.c}[d, \lambda]$ and $C^{L}_{\partial\varphi.c}[p, \mu]$, where $\varphi \in I[X]$ and $0 < c \le a$. Let us denote by $\Gamma, \tilde{\Gamma} : [0, a] \to R_+$ the functions given by

$$\Gamma(t) = \Theta(0,t)\Big[\chi c_1 + \tilde{b}\int_0^t \gamma(\xi)d\xi + \bar{b}\int_0^t \alpha(\xi)d\xi\Big],$$

$$\tilde{\Gamma}(t) = \Theta(0,t)\Big[\chi b_1 + (A + p_1)\int_0^t \alpha(\xi)d\xi + p_0 A\int_0^t \gamma(\xi)d\xi\Big],$$

where

$$\tilde{b} = A\big[1 + s_0(Kp_0 + K_0c_0)\big], \;\; \bar{b} = s_1(Kp_0 + K_0c_0) + s_0^2(Kp_1 + K_0c_1).$$

Write

$$\tilde{c} = \Gamma(c) + 1 + s_0(Kp_0 + K_0c_0),$$

$$\lambda_0(t) = [\tilde{\Gamma}(a) + p_0]\alpha(t) + \tilde{\gamma}(t), \;\; \lambda_1(t) = \mu(t) = \tilde{c}\alpha(t).$$

Assumption $\mathbf{H}_0[c]$. Suppose that

$$p_0 \ge \chi c_0 + \big[1 + s_0(Kp_0 + K_0c_0)\big]\int_0^c \alpha(\tau)d\tau, \quad p_1 \ge \Gamma(c),$$

$$d_0 \ge \chi b_0 + \int_0^c \big[\tilde{\gamma}(s) + p_0\alpha(s)\big]ds, \quad d_1 = p_0, \quad , d_2 = p_1.$$

Remark 2. If we assume that $p_0 > \chi c_0$, $p_1 > \chi c_1$ and $d_0 > \chi b_0$, then there is $c \in [0, a]$ such that Assumption $H_0[c]$ is satisfied.

Theorem 2. *Suppose that $\varphi \in I[X]$ and that Assumptions $H[X]$, $H[\psi]$, $H[f]$ and $H_0[c]$ are satisfied. Then there are $d \in R^3_+$, $p \in R^2_+$, $\lambda_0, \lambda_1, \mu \in L([0,c], R_+)$ such that for $m \ge 0$ we have*

(I_m) $z^{(m)}$ and $u^{(m)}$ are defined on $(-\infty, c] \times R^n$ and $z^{(m)} \in C^{1.L}_{\varphi.c}[d, \lambda]$, $u^{(m)} \in C^{L}_{\partial\varphi.c}[p, \mu]$,

(II_m) $\partial_x z^{(m)}(t,x) = u^{(m)}(t,x)$ on $[0, c] \times R^n$.

Proof. We will prove (I_m) and (II_m) by induction. It follows from (18), (19) that conditions (I_0) and (II_0) are satisfied. Supposing now that conditions (I_m) and (II_m) hold for a given $m \ge 0$, we will prove that there exists a solution $u^{(m+1)} \in C^{L}_{\partial\varphi.c}[p, \mu]$ to problem (20) and that the function $z^{(m+1)}$ given by (22) is an element of the space $C^{1.L}_{\varphi.c}[d, \lambda]$. We have divided the proof into a sequence of steps.

I. We first show that

$$G^{(m)} : C^{L}_{\partial\varphi.c}[p, \mu] \to C^{L}_{\partial\varphi.c}[p, \mu]. \tag{23}$$

Write

$$w_i^{(m)}[u;\tau,t,x] = \sum_{j=1}^{n} \partial_{x_i}\psi_j\big(\tau, g[z^{(m)},u](\tau,t,x)\big)\big(u_j^{(m)}\big)_{\psi(\tau,g[z^{(m)},u](\tau,t,x))},\ 1 \le i \le n.$$

It follows from Assumptions H$[X]$ and H$[\psi]$ that for $u \in C^L_{\partial\varphi.c}[p,\mu]$ and $y = g[z^{(m)},u](\tau,t,x)$ we have

$$\sum_{i=1}^{n} \big\|w_i^{(m)}[u;\tau,t,x]\big\|_X$$
$$\le s_0 \sum_{j=1}^{n} \big\|(u_j^{(m)})_{\psi(\tau,y)}\big\|_X \le s_0 \sum_{j=1}^{n} \Big[K\|u_j^{(m)}\|_{[0,\tau;y]} + K_0\|(u_j^{(m)})_{(0,y)}\|_X\Big]$$
$$\le s_0(Kp_0 + K_0c_0), \quad 1 \le i \le n. \tag{24}$$

Therefore it follows that

$$\big\|G^{(m)}[u](t,x)\big\| \le \chi c_0 + \big[1 + s_0(Kp_0 + K_0c_0)\big] \int_0^t \alpha(\tau)d\tau \le p_0 \tag{25}$$

for $(t,x) \in [0,c] \times R^n$.

Our next goal is to estimate the number $\|G^{(m)}[u](t,x) - G^{(m)}[u](\bar{t},\bar{x})\|$. It is easily seen that for $y, \bar{y} \in R^n$ we have

$$\big\|\partial_x\varphi(0,y) - \partial_x\varphi(0,\bar{y})\big\| \le \chi c_1\|y - \bar{y}\|.$$

Using Theorem 1 and the above inequality, we get

$$\big\|\partial_x\varphi(0, g[z^{(m)},u](0,t,x)) - \partial_x\varphi(0, g[z^{(m)},u](0,\bar{t},\bar{x}))\big\|$$
$$\le \chi c_1\Theta(0,c)\bigg[\|x - \bar{x}\| + \bigg|\int_t^{\bar{t}} \alpha(\tau)d\tau\bigg|\bigg] \quad \text{on} \quad [0,c] \times R^n. \tag{26}$$

From Assumptions H$[X]$ and H$[f]$ it follows that the terms

$$\big\|\partial_x f\big(P[z^{(m)},u](\xi,t,x)\big) - \partial_x f\big(P[z^{(m)},u](\xi,\bar{t},\bar{x})\big)\big\|,$$
$$\big\|\partial_w f\big(P[z^{(m)},u](\xi,t,x)\big) - \partial_w f\big(P[z^{(m)},u](\xi,\bar{t},\bar{x})\big)\big\|_*$$

can be estimated from above by

$$A\gamma(\xi)\big\|g[z^{(m)},u](\xi,t,x) - g[z^{(m)},u](\xi,\bar{t},\bar{x})\big\|.$$

We conclude from Assumption H$[\psi]$ and Lemma 1 that

$$\sum_{i=1}^{n} \big\|w_i^{(m)}[u;\xi,t,x] - w_i^{(m)}[u;\xi,\bar{t},\bar{x}]\big\|_X \le \bar{b}\big\|g[z^{(m)},u](\xi,t,x) - g[z^{(m)},u](\xi,\bar{t},\bar{x})\big\|.$$

The above estimates and Theorem 1 imply

$$\big\|G^{(m)}[u](t,x) - G^{(m)}[u](\bar{t},\bar{x})\big\|$$
$$\le \Gamma(c)\bigg[\bigg|\int_t^{\bar{t}} \alpha(s)ds\bigg| + \|x - \bar{x}\|\bigg] + \big[1 + s_0(Kp_0 + K_0b_0)\big]\bigg|\int_t^{\bar{t}} \alpha(s)ds\bigg|$$

on $[0,c] \times R^n$. By the above inequality and (25) we obtain (23).

II. Our next claim is that

$$\big\|G^{(m)}[u](t,x) - G^{(m)}[\tilde{u}](t,x)\big\| \leq \tilde{c}\int_0^t \gamma(\tau)\|u-\tilde{u}\|_\tau d\tau, \tag{27}$$

where $u,\tilde{u} \in C^L_{\partial\varphi.c}[p,\mu]$ and $(t,x) \in [0,c]\times R^n$. The proof starts with the observation that

$$\begin{aligned}
&\big\|G^{(m)}[u](t,x) - G^{(m)}[\tilde{u}](t,x)\big\| \\
&\leq \big\|\partial_x\varphi(0,g[z^{(m)},u](0,t,x)) - \partial_x\varphi(0,g[z^{(m)},\tilde{u}](0,t,x))\big\| \\
&+\int_0^t \big\|\partial_x f(P[z^{(m)},u](\tau,t,x)) - \partial_x f(P[z^{(m)},\tilde{u}](\tau,t,x))\big\|d\tau \\
&+\int_0^t \big\|\partial_w f(P[z^{(m)},u](\tau,t,x)) - \partial_w f(P[z^{(m)},\tilde{u}](\tau,t,x))\big\|_* \sum_{i=1}^n \big\|w_i^{(m)}[u;\tau,t,x]\big\|_X d\tau \\
&+\int_0^t \big\|\partial_w f(P[z^{(m)},u](\tau,t,x))\big\|_* \sum_{i=1}^n \big\|w_i^{(m)}[u;\tau,t,x] - w_i^{(m)}[\tilde{u};\tau,t,x]\big\|_X d\tau.
\end{aligned}$$

It follows from Assumption H$[f]$ that

$$\begin{aligned}
&\int_0^t \big\|\partial_x f(P[z^{(m)},u](\tau,t,x)) - \partial_x f(P[z^{(m)},\tilde{u}](\tau,t,x))\big\|d\tau \\
&\leq \int_0^t \gamma(\tau)\big[\|u-\tilde{u}\|_\tau + A\|g[z^{(m)},u](\tau,t,x) - g[z^{(m)},\tilde{u}](\tau,t,x)\|\big]d\tau
\end{aligned}$$

and the same estimate holds for the Fréchet derivative $\partial_w f$. We conclude from Assumption H$[X]$ and H$[\psi]$ that

$$\begin{aligned}
&\sum_{i=1}^n \big\|w_i^{(m)}[u;\tau,t,x] - w_i^{(m)}[\tilde{u};\tau,t,x]\big\|_X d\tau \\
&\leq \big\|\partial_x\psi'(\tau,g[z^{(m)},u](\tau,t,x)) - \partial_x\psi'(\tau,g[z^{(m)},\tilde{u}](\tau,t,x))\big\| \\
&\times \sum_{j=1}^n \big\|(u_j^{(m)})_{\psi(\tau,g[z^{(m)},u](\tau,t,x))}\big\|_X + \big\|\partial_x\psi'(\tau,g[z^{(m)},\tilde{u}](\tau,t,x))\big\| \\
&\times \sum_{j=1}^n \big\|(u_j^{(m)})_{\psi(\tau,g[z^{(m)},u](\tau,t,x))} - (u_j^{(m)})_{\psi(\tau,g[z^{(m)},\tilde{u}](\tau,t,x))}\big\|_X \\
&\leq \bar{b}\,\big\|g[z^{(m)},u](\tau,t,x) - g[z^{(m)},\tilde{u}](\tau,t,x)\big\|.
\end{aligned}$$

The above estimates and (24), (26) imply

$$\begin{aligned}
\big\|G^{(m)}[u](t,x) - G^{(m)}[\tilde{u}](t,x)\big\| &\leq c_1\chi\big\|g[z^{(m)},u](0,t,x) - g[z^{(m)},\tilde{u}](0,t,x)\big\| \\
&+\big[1+s_0(Kp_0+K_0c_0)\big]\int_0^t \gamma(\tau)\|u-\tilde{u}\|_\tau d\tau \\
&+\int_0^t \big[\tilde{b}\gamma(\tau) + \bar{b}\alpha(\tau)\big]\big\|g[z^{(m)},u](\tau,t,x) - g[z^{(m)},\tilde{u}](\tau,t,x)\big\|d\tau.
\end{aligned}$$

Then using Theorem 1 we get (27).

III. Write

$$[|u-\tilde{u}|] = \sup\left\{\|u-\tilde{u}\|_t \exp\left[-2\tilde{c}\int_0^t \gamma(\tau)d\tau\right] : t\in[0,c]\right\},$$

where $u,\tilde{u}\in C^L_{\partial\varphi.c}[p,\mu]$. We claim that

$$\big[|G^{(m)}[u]-G^{(m)}[\tilde{u}]|\big] \le \frac{1}{2}[|u-\tilde{u}|]. \tag{28}$$

According to (27) we have

$$\begin{aligned}\big\|G^{(m)}[u](t,x)-G^{(m)}[\tilde{u}](t,x)\big\| &\le [|u-\tilde{u}|]\tilde{c}\int_0^t \exp\left[2\tilde{c}\int_0^\tau \gamma(\xi)d\xi\right]d\tau\\ &\le \frac{1}{2}[|u-\tilde{u}|]\exp\left[2\tilde{c}\int_0^t \gamma(\xi)d\xi\right], \quad (t,x)\in[0,c]\times R^n,\end{aligned}$$

and inequality (28) follows. From the Banach fixed point theorem it follows that there is exactly one $u^{(m+1)}\in C^L_{\partial\varphi.c}[p,\mu]$ satisfying (20).

IV. We next claim that the function $z^{(m+1)}$ given by (22) satisfies condition (II_{m+1}). It is sufficient to show that the function

$$\Delta(t,x,\bar{x}) = z^{(m+1)}(t,\bar{x}) - z^{(m+1)}(t,x) - u^{(m+1)}(t,x)\circ(\bar{x}-x)$$

satisfies the condition

$$|\Delta(t,x,\bar{x})| \le \tilde{C}\|x-\bar{x}\|^2 \tag{29}$$

with a constant $\tilde{C}\in R_+$ independent of $(t,x),(t,\bar{x})\in[0,c]\times R^n$. Write

$$g^{(m)}(\tau,t,x) = g[z^{(m)},u^{(m+1)}](\tau,t,x)$$

and

$$\begin{aligned}&\Lambda^{(m)}(\tau,t,x,\bar{x})\\ &= \int_\tau^t \big[\partial_q f(P[z^{(m)},u^{(m+1)}](\xi,t,\bar{x})) - \partial_q f(P[z^{(m)},u^{(m+1)}](\xi,t,x))\big]d\xi.\end{aligned}$$

Then we have

$$\Lambda^{(m)}(\tau,t,x,\bar{x}) = g^{(m)}(\tau,t,\bar{x}) - g^{(m)}(\tau,t,x) - (\bar{x}-x).$$

According to (20), (22) we have

$$\begin{aligned}\Delta(t,x,\bar{x}) =& F[z^{(m)},u^{(m+1)}](t,\bar{x}) - F[z^{(m)},u^{(m+1)}](t,x)\\ &- G^{(m)}[u^{(m+1)}](t,x)\circ(\bar{x}-x).\end{aligned}$$

Throughout the proof, $Q^{(m)}$ denotes the intermediate point

$$\begin{aligned}&Q^{(m)}(s,\tau,t,x,\bar{x})\\ &= \big(\tau, y^{(m)}+s[\bar{y}^{(m)}-y^{(m)}], Z^{(m)}+s[\bar{Z}^{(m)}-Z^{(m)}], U^{(m)}+s[\bar{U}^{(m)}-U^{(m)}]\big),\end{aligned}$$

where $0\le s\le 1$ and

$$y^{(m)} = g^{(m)}(\tau,t,x), \quad \bar{y}^{(m)} = g^{(m)}(\tau,t,\bar{x}),$$

$$Z^{(m)} = z^{(m)}_{\psi(\tau,g^{(m)}(\tau,t,x))}, \quad \bar{Z}^{(m)} = z^{(m)}_{\psi(\tau,g^{(m)}(\tau,t,\bar{x}))},$$
$$U^{(m)} = u^{(m+1)}(\tau, g^{(m)}(\tau,t,x)), \quad \bar{U}^{(m)} = u^{(m+1)}(\tau, g^{(m)}(\tau,t,\bar{x})).$$

The proof of (29) is based on the following observation. Write

$$\begin{aligned}
\bar{\Delta}(t,x,\bar{x}) &= \varphi(0, g^{(m)}(0,t,\bar{x})) - \varphi(0, g^{(m)}(0,t,x)) \\
&-\partial_x\varphi(0, g^{(m)}(0,t,x)) \circ \big[g^{(m)}(0,t,\bar{x}) - g^{(m)}(0,t,x)\big] \\
&+\int_0^t \int_0^1 \Big[\partial_x f\big(Q^{(m)}(s,\xi,t,x,\bar{x})\big) - \partial_x f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big)\Big] ds \\
&\quad \circ \big[g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x)\big] d\xi \\
&+\int_0^t \int_0^1 \Big[\partial_w f\big(Q^{(m)}(s,\xi,t,x,\bar{x})\big) - \partial_w f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big)\Big] ds \\
&\quad \times \big[z^{(m)}_{\psi(\xi,g^{(m)}(\xi,t,\bar{x}))} - z^{(m)}_{\psi(\xi,g^{(m)}(\xi,t,x))}\big] d\xi \\
&+\int_0^t \int_0^1 \Big[\partial_q f\big(Q^{(m)}(s,\xi,t,x,\bar{x})\big) - \partial_q f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big)\Big] ds \\
&\quad \circ \big[u^{(m+1)}(\xi, g^{(m)}(\xi,t,\bar{x})) - u^{(m+1)}(\xi, g^{(m)}(\xi,t,x))\big] d\xi \\
&+\int_0^t \partial_w f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big)\Big[z^{(m)}_{\psi(\xi,g^{(m)}(\xi,t,\bar{x}))} - z^{(m)}_{\psi(\xi,g^{(m)}(\xi,t,x))} - \\
&\big[(u^{(m)})_{\psi(\xi,g^{(m)}(\xi,t,x))} \diamond \partial_x\psi'(\xi, g^{(m)}(\xi,t,x))\big] \circ \big[g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x)\big]\Big] d\xi
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Delta} &= \partial_x\varphi(0, g^{(m)}(0,t,x)) \circ \Lambda^{(m)}(0,t,x,\bar{x}) \\
&+\int_0^t \partial_x f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big) \circ \Lambda^{(m)}(\xi,t,x,\bar{x}) d\xi \\
&+\int_0^t \Big[\partial_w f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big) * (u^{(m)})_{\psi(\xi,g^{(m)}(\xi,t,x))}\Big] \diamond \partial_x\psi'(\xi, g^{(m)}(\xi,t,x)) \\
&\quad \circ \Lambda^{(m)}(\xi,t,x,\bar{x}) d\xi \\
&-\int_0^t \Big[\partial_q f\big(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})\big) - \partial_q f(P[z^{(m)},u^{(m+1)}](s,t,x))\Big] ds \\
&\quad \circ u^{(m+1)}(\xi, g^{(m)}(\xi,t,x)) d\xi.
\end{aligned}$$

By applying the Hadamard mean value theorem to the difference

$$f\big(P[z^{(m)},u^{(m+1)}](\xi,t,\bar{x})\big) - f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big)$$

we get

$$\Delta(t,x,\bar{x}) = \bar{\Delta}(t,x,\bar{x}) + \tilde{\Delta}(t,x,\bar{x}) \quad \text{on} \quad [0,c] \times R^n.$$

V. We prove that

$$\tilde{\Delta}(t,x,\bar{x}) = 0 \quad \text{for} \quad (t,x),(t,\bar{x}) \in [0,c] \times R^n. \tag{30}$$

It follows easily that

$$\tilde{\Delta}(t,x,\bar{x}) = \int_0^t \Big[\partial_q f\big(P[z^{(m)},u^{(m+1)}](\xi,t,\bar{x})\big) - \partial_q f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big)\Big] \circ \big[W^{(m)}(\tau,t,x) - u^{(m+1)}(\tau,g^{(m)}(\tau,t,x))\big]d\tau,$$

where

$$W^{(m)}(\tau,t,x) = \partial_x\varphi(0,g^{(m)}(0,t,x)) + \int_0^\tau \partial_x f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big)d\xi + \int_0^\tau \Big[\partial_w f\big(P[z^{(m)},u^{(m+1)}](\xi,t,x)\big) * (u^{(m)})_{\psi(\xi,g^{(m)}(\xi,t,x))}\Big] \diamond \partial_x\psi'\big(\xi,g^{(m)}(\xi,t,x)\big)d\xi.$$

The equality

$$g^{(m)}(\xi,\tau,g^{(m)}(\tau,t,x)) = g^{(m)}(\xi,t,x), \quad (t,x)\in[0,c]\times R^n, \quad \xi,\tau\in[0,c],$$

which is a consequence of Assumption H$[\partial_q f]$, implies that

$$u^{(m+1)}(\tau,g^{(m)}(\tau,t,x)) = W^{(m)}(\tau,t,x), \quad (t,x)\in[0,c]\times R^n, \quad \tau\in[0,c],$$

which completes the proof of (30). Thus we have proved that $\Delta(t,x,\bar{x}) = \bar{\Delta}(t,x,\bar{x})$ on $[0,c]\times R^n$.

VI. It remains to prove (29) for $\bar{\Delta}$. It follows from Assumption H$[f]$ and from Lemma 1 that

$$\big\|\partial_x f(Q^{(m)}(s,\xi,t,x,\bar{x})) - \partial_x f(P[z^{(m)},u^{(m+1)}](\xi,t,x))\big\| \le A\gamma(\xi)\big\|g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x)\big\|$$

and the same estimate holds for the derivative $\partial_q f$ and the Fréchet derivative $\partial_w f$. According to Assumptions H$[\psi]$, H$[\varphi]$ and Lemma 1, we have that

$$\big\|z^{(m)}_{\psi(s,g^{(m)}(\xi,t,\bar{x}))} - z^{(m)}_{\psi(s,g^{(m)}(\xi,t,x))}\big\| \le s_0(Kd_1+K_0d_1)\big\|g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x)\big\|.$$

An easy computation shows that

$$\big\|\varphi(0,g^{(m)}(0,t,\bar{x})) - \varphi(0,g^{(m)}(0,t,x)) - \partial_x\varphi(0,g^{(m)}(0,t,x)) \circ[g^{(m)}(0,t,\bar{x}) - g^{(m)}(0,t,x)]\big\| \le \chi c_1\big\|g^{(m)}(0,t,\bar{x}) - g^{(m)}(0,t,x)\big\|^2.$$

Since $\partial_x z^{(m)} = u^{(m)}$, we have that there is $\bar{C}\in R_+$ such that

$$\big\|z^{(m)}_{\psi(\xi,g^{(m)}(\xi,t,\bar{x}))} - z^{(m)}_{\psi(\xi,g^{(m)}(\xi,t,x))} - \big[(u^{(m)})_{\psi(\xi,g^{(m)}(\xi,t,x))} \diamond \partial_x\psi'(\xi,g^{(m)}(\xi,t,x))\big]\circ \big[g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x)\big]\big\| \le \bar{C}\big\|g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x)\big\|^2.$$

It follows from the above estimates and the definition of $\bar{\Delta}$ that there is $C_0\in R_+$ such that

$$|\Delta(t,x,\bar{x})| \le \chi c_1\big\|g^{(m)}(0,t,\bar{x}) - g^{(m)}(0,t,x)\big\|^2 + C_0\int_0^t [\gamma(\xi)+\alpha(\xi)]\,\big\|g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x)\big\|^2 d\xi \quad \text{on} \quad [0,c]\times R^n.$$

Thus it follows from Theorem 1 that there is $\tilde{C}\in R_+$ such that estimate (29) holds and consequently $\partial_x z^{(m+1)} = u^{(m+1)}$.

VII. Now we prove that $z^{(m+1)} \in C^{1.L}_{\varphi.c}[c,\lambda]$. From (II_{m+1}) it follows that $\|\partial_x z^{(m+1)}(t,x)\| \le d_1$ and

$$\left\|\partial_x z^{(m+1)}(t,x) - \partial_x z^{(m+1)}(\bar t,\bar x)\right\| \le \left|\int_t^{\bar t} \lambda_1(\xi)d\xi\right| + d_2\|x-\bar x\|$$

on $[0,c]\times R^n$. Assumptions H[ψ] and H[f] imply

$$|z^{(m+1)}(t,x)| \le \chi b_0 + \int_0^t [\tilde\gamma(\xi) + p_0\alpha(\xi)]d\xi \le d_0$$

and

$$\left|z^{(m+1)}(t,x) - z^{(m+1)}(\bar t,x)\right| \le \tilde\Gamma(c)\left|\int_t^{\bar t}\alpha(\xi)d\xi\right| + \left|\int_t^{\bar t}\Big[\tilde\gamma(\xi)+p_0\alpha(\xi)\Big]d\xi\right|$$

$$= \left|\int_t^{\bar t}\Big[(\tilde\Gamma(c)+p_0)\alpha(\xi)+\tilde\gamma(\xi)\Big]d\xi\right| = \left|\int_t^{\bar t}\lambda_0(\xi)d\xi\right|.$$

This completes the proof of the theorem. □

6. Convergence of the sequence $\{z^{(m)}, u^{(m)}\}$

We prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent if the constant $c\in[0,a]$ is sufficiently small. Write

$$\tilde\Theta(t) = \exp\left[\tilde c\int_0^t\gamma(\xi)d\xi\right]\cdot\left[1+\int_0^t[\tilde\Gamma(c)\gamma(\xi)+p_0\gamma(\xi)+2\alpha(\xi)]d\xi\right],$$

$$\eta_0(t) = s_0K\tilde\Theta(a)\alpha(t), \qquad \eta(t) = \gamma(t)\big[K\tilde c\tilde\Theta(a)+K\tilde\Gamma(a)+p_0\big]+\alpha(t).$$

Assumption H[c]. Suppose that Assumption $\mathrm{H}_0[c]$ is satisfied and c is such a small constant that

$$q = \max\left\{\int_0^c\eta(\xi)d\xi, \int_0^c\eta_0(\xi)d\xi\right\} < 1. \tag{31}$$

Theorem 3. *If Assumptions H[X], H[ψ], H[f] and H[c] are satisfied and $\varphi\in I[X]$, then the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on $[0,c]\times R^n$.*

Proof. An easy computation shows that the integral inequality

$$\left\|u^{(m+1)}-u^{(m)}\right\|_t \le \tilde c\int_0^t\gamma(\xi)\left\|u^{(m+1)}-u^{(m)}\right\|_\xi d\xi$$

$$+K\int_0^t\Big[\tilde c\gamma(\xi)\left\|z^{(m)}-z^{(m-1)}\right\|_\xi + s_0\alpha(\xi)\left\|u^{(m)}-u^{(m-1)}\right\|_\xi\Big]d\xi, \quad t\in[0,c],$$

is satisfied. This gives

$$\left\|u^{(m+1)}-u^{(m)}\right\|_t \le K\int_0^t\Big[\tilde c\gamma(\xi)\left\|z^{(m)}-z^{(m-1)}\right\|_\xi + s_0\alpha(\xi)\left\|u^{(m)}-u^{(m-1)}\right\|_\xi\Big]d\xi$$

$$\times\exp\left[\tilde c\int_0^t\gamma(\xi)d\xi\right], \quad m\ge 0, \quad t\in[0,c]. \tag{32}$$

It is easily seen that

$$\big\|z^{(m+1)}-z^{(m)}\big\|_t \le \int_0^t \big[\tilde\Gamma(c)\gamma(\xi)+p_0\gamma(\xi)+\alpha(\xi)\big] \times\Big[K\big\|z^{(m)}-z^{(m-1)}\big\|_\xi+\big\|u^{(m+1)}-u^{(m)}\big\|_\xi\Big]d\xi + \int_0^t \alpha(\xi)\big\|u^{(m+1)}-u^{(m)}\big\|_\xi d\xi, \quad m\ge 0, \quad t\in[0,c]. \tag{33}$$

Combining (32) and (33) we deduce that

$$\big\|z^{(m+1)}-z^{(m)}\big\|_t \le K\int_0^t \big[\tilde\Gamma(c)\gamma(\xi)+p_0\gamma(\xi)+\alpha(\xi)\big]\big\|z^{(m)}-z^{(m-1)}\big\|_\xi d\xi + K\Big\{\tilde\Theta(t)-\exp\Big[\tilde c\int_0^t\gamma(\xi)d\xi\Big]\Big\} \times\int_0^t\Big[\tilde c\gamma(\xi)\big\|z^{(m)}-z^{(m-1)}\big\|_\xi+s_0\alpha(\xi)\big\|u^{(m)}-u^{(m-1)}\big\|_\xi\Big]d\xi, \tag{34}$$

where $t\in[0,c],\ m\ge 1$. Adding inequalities (32) and (34), we conclude that

$$\big\|u^{(m+1)}-u^{(m)}\big\|_t+\big\|z^{(m+1)}-z^{(m)}\big\|_t \le \int_0^t \eta_0(\xi)\big\|u^{(m)}-u^{(m-1)}\big\|_\xi d\xi + \int_0^t \eta(\xi)\big\|z^{(m)}-z^{(m-1)}\big\|_\xi d\xi, \quad t\in[0,c], \quad m\ge 1,$$

and consequently

$$\big\|u^{(m+1)}-u^{(m)}\big\|_t+\big\|z^{(m+1)}-z^{(m)}\big\|_t \le q\Big[\big\|u^{(m)}-u^{(m-1)}\big\|_t+\big\|z^{(m)}-z^{(m-1)}\big\|_t\Big], \quad t\in[0,c], \quad m\ge 1.$$

There is $\bar c\in R_+$ such that

$$\big\|u^{(1)}-u^{(0)}\big\|_t+\big\|z^{(1)}-z^{(0)}\big\|_t\le\bar c \quad \text{for} \quad t\in[0,c].$$

Finally, the convergence of the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ follows from condition (31). This is our claim. □

7. The Main Theorem

We state the main result on the existence and continuous dependence of solutions on initial functions. For a function $\varphi\in I[X]$ we write

$$\|\varphi(0,\cdot)\|=\sup\big\{|\varphi(0,x)|: x\in R^n\big\}, \quad \|\partial_x\varphi(0,\cdot)\|=\sup\big\{\|\partial_x\varphi(0,x)\|: x\in R^n\big\}.$$

Theorem 4. *If Assumptions $H[X]$, $H[\psi]$, $H[f]$ and $H[c]$ are satisfied and $\varphi\in I[X]$, then there is a solution $v:(-\infty,c]\times R^n\to R$ to problem* (1), (2)*. Moreover, $v\in C^{1.L}_{\varphi.c}[d,\lambda]$ and $\partial_x v\in C^{L}_{\partial\varphi.c}[p,\mu]$. If $\bar\varphi\in I[X]$ and $\bar v\in C^{1.L}_{\bar\varphi.c}[d,\lambda]$ is a solution of equation* (1) *with the initial condition $z(t,x)=\bar\varphi(t,x)$ for $(t,x)\in(-\infty,0]\times R^n$, then there are $Q,\Lambda\in C([0,c],R_+)$ such that*

$$\|v-\bar v\|_t+\|\partial_x v-\partial_x\bar v\|_t\le\Lambda(t)\Big[\big\|\varphi(0,\cdot)-\bar\varphi(0,\cdot)\big\|+\big\|\partial_x\varphi(0,\cdot)-\partial_x\bar\varphi(0,\cdot)\big\|$$

$$+Q(t)\big(\|\varphi-\bar\varphi\|_{X,R^n}+\|\partial_x\varphi-\partial_x\bar\varphi\|_{X,R^n}\big)\Big],\quad t\in[0,c]. \tag{35}$$

Proof. It follows from Theorem 3 that there are functions $v\in C^{1.L}_{\varphi.c}[d,\lambda]$ and $\bar u\in C^{L}_{\partial\varphi.c}[d,\mu]$ such that

$$v(t,x)=\lim_{m\to\infty}z^{(m)}(t,x)\ \text{ and }\ \bar u(t,x)=\lim_{m\to\infty}u^{(m)}(t,x)$$

uniformly on $[0,c]\times R^n$. Furthermore, we have that $\partial_x v$ exists on $[0,c]\times R^n$ and $\partial_x v(t,x)=\bar u(t,x)$ for $(t,x)\in[0,c]\times R^n$. The passage to the limit in (22) implies that

$$v(t,x)=\varphi\big(0,g[v,\partial_x v](0,t,x)\big)+\int_0^t\Big[f\big(P[v,\partial_x v](\xi,t,x)\big)$$
$$-\partial_q f\big(P[v,\partial_x v](\xi,t,x)\big)\circ\partial_x v(\xi,g[v,\partial_x v](\xi,t,x))\Big]d\xi,\quad (t,x)\in[0,c]\times R^n. \tag{36}$$

For $x\in R^n$ let us put $y=g[v,\partial_x v](0,t,x)$. It follows that $g[v,\partial_x v](\tau,t,x)=g[v,\partial_x v](\tau,0,y]$ for $t,\tau\in[0,c]$ and $x=g[v,\partial_x v](t,0,y)$. Write $\tilde g(t,0,y)=g[v,\partial_x v](t,0,y)$, $(t,y)\in[0,c]\times R^n$. Then relation (36) is equivalent to

$$v(t,\tilde g(t,0,y))=\varphi(0,y)+\int_0^t\Big[f\big(\xi,\tilde g(\xi,0,y),v_{\psi(\xi,\tilde g(\xi,0,y))},\partial_x v(\xi,\tilde g(\xi,0,y))\big)$$
$$-\partial_q f\big(\xi,\tilde g(\xi,0,y),v_{\psi(\xi,\tilde g(\xi,0,y))},\partial_x v(\xi,\tilde g(\xi,0,y))\big)\circ\partial_x v(\xi,\tilde g(\xi,0,y))\Big]d\xi,$$

where $(t,x)\in[0,c]\times R^n$. Differentiating the above relation with respect to t and making use of the inverse transformation $x=\tilde g(t,0,y)$, we see that v satisfies equation (1) for almost all $t\in[0,c]$ with fixed $x\in R^n$.

Now we prove inequality (35). It follows that the functions $(v,\partial_x v)$ satisfy the functional integral equations (17), and $(\bar v,\partial_x\bar v)$ is a solution of an adequate system with $\bar\varphi$ and $\partial_x\bar\varphi$ instead of φ and $\partial_x\varphi$. Analysis similar to that in the proof of Theorem 2 shows that

$$\|v-\bar v\|_t\le\|\varphi(0,\cdot)-\bar\varphi(0,\cdot)\|$$
$$+(\tilde\Gamma(t)+p_0)\int_0^t\gamma(\xi)\big[K\|v-\bar v\|_\xi+\|\partial_x v-\partial_x\bar v\|_\xi+K_0\|\varphi-\bar\varphi\|_{X,R^n}\big]d\xi$$
$$+\int_0^t\alpha(\xi)\big[K\|v-\bar v\|_\xi+\|\partial_x v-\partial_x\bar v\|_\xi+K_0\|\varphi-\bar\varphi\|_{X,R^n}\big]d\xi+\int_0^t\alpha(\xi)\|\partial_x v-\partial_x\bar v\|_\xi d\xi$$

and

$$\|\partial_x v-\partial_x\bar v\|_t\le\|\partial_x\varphi(0,\cdot)-\partial_x\bar\varphi(0,\cdot)\|$$
$$+\tilde c\int_0^t\gamma(\xi)\big[K\|v-\bar v\|_\xi+\|\partial_x v-\partial_x\bar v\|_\xi+K_0\|\varphi-\bar\varphi\|_{X,R^n}\big]d\xi$$
$$+s_0\int_0^t\alpha(\xi)\big[K\|\partial_x v-\partial_x\bar v\|_\xi+K_0\|\partial_x\varphi-\partial_x\bar\varphi\|_{X,R^n}\big]d\xi,$$

where $t \in [0, c]$. Write

$$\tilde{\Lambda} = \tilde{K}(\tilde{\Gamma}(a) + p_0 + \tilde{c})\gamma(t) + (\tilde{K} + 1 + K s_0)\alpha(t), \quad \tilde{K} = \max\{1, K\},$$

$$Q(t) = K_0(\tilde{\Gamma}(a) + p_0 + \tilde{c}) \int_0^t \gamma(\xi) d\xi + K_0 \bar{s} \int_0^t \alpha(\xi) d\xi, \quad \bar{s} = \max\{1, s_0\}.$$

According to the above estimates, we have the integral inequality

$$\|v - \bar{v}\|_t + \|\partial_x v - \partial_x \bar{v}\|_t \le \|\varphi(0, \cdot) - \bar{\varphi}(0, \cdot)\| + \|\partial_x \varphi(0, \cdot) - \partial_x \bar{\varphi}(0, \cdot)\|$$

$$+ Q(t)\big[\|\varphi - \bar{\varphi}\|_{X,R^n} + \|\partial_x \varphi - \partial_x \bar{\varphi}\|_{X,R^n}\big] + \int_0^t \tilde{\Lambda}(\xi)\big[\|v - \bar{v}\|_\xi + \|\partial_x v - \partial_x \bar{v}\|_\xi\big] d\xi$$

for $t \in [0, c]$ and we get (35) with

$$\Lambda(t) = \exp\left[\int_0^t \tilde{\Lambda}(\xi) d\xi\right], \quad t \in [0, c].$$

This is our claim. □

Remark 3. The results of the paper can be extended to nonlinear weakly coupled systems

$$\partial_t z_i(t, x) = f_i\big(t, x, z_{\psi(t,x)}, \partial_x z_i(t, x)\big), \quad i = 1, \ldots, k,$$

with the initial condition

$$z(t, x) = \varphi(t, x) \quad \text{for} \quad (t, x) \in (-\infty, 0] \times R^n,$$

where $z = (z_1, \ldots, z_k)$. These systems are of special hyperbolic type because each equation contains the vector of unknown functions $z = (z_1, \ldots, z_k)$ and first order derivatives of only one scalar function.

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References

1. M. Adimy, H. Bouzahir, and K. Ezzinbi, Existence for a class of partial functional differential equations with infinte delay. *Nonlinear Anal.* **46**(2001), 91–112.
2. P. Brandi and R. Ceppitelli, Existence, uniqueness and continuous dependence for a first order nonlinear partial differential equations in a hereditary structure. *Ann. Polon. Math.* **47**(1986), 121–136.
3. P. Brandi and C. Marcelli, Haar inequality in hereditary setting and applications. *Rend. Sem. Mat. Univ. Padova* **96**(1996), 177–199.
4. P. Brandi and C. Marcelli, On mixed problem for first order partial differential functional equations. *Atti. Sem. Mat. Fis. Univ. Modena* **46**(1998), 497–510.
5. S. Burys, On partial differential functional inequalities of the first order. *Univ. Iagell. Acta Math.* **16**(1974), 107–112.
6. T. Człapiński, On the existence of generalized solutions of nonlinear first order partial differential functional equations in two independent variables. *Czechoslovak Math. J.* **41**(1991), 490–506.

7. T. Człapiński, On the mixed problem for quasilinear partial differential-functional equations of the first order. *Z. Anal. Anwendungen* **16**(1997), 463–478.

8. T. Człapiński and Z. Kamont, Existence of weak solutions of nonlinear functiona differential equations with unbounded delay. *Appl. Anal.* (to appear).

9. M. El-Doma, Analysis of nonlinear integro differential equations arising in age dependent epidemic models. *Nonlinear Anal.* **11**(1978), 913–937.

10. Y. Hino, S. Murakami, and T. Naito, Functional Differential Equations with Infinite Delay. *Lecture Notes in Mathematics,* 1473. *Springer-Verlag, Berlin,* 1991.

11. D. Jaruszewska-Walczak, Existence of solutions of first-order partial differential functional equations. *Boll. Un. Mat. Ital. B (7)* **4**(1990), 57–82.

12. Z. Kamont and H. Leszczyński, Uniqueness result for the generalized entropy solutions to the Cauchy problem for first-order partial differential functional equations. *Z. Anal. Anwendungen.* **13**(1994), 477–491.

13. Z. Kamont, M. Kwapisz, and S. Zacharek, On difference functional inequalities related to some classes of partial differential functional equations. *Math. Nachr.* **146**(1990), 335–360.

14. Z. Kamont, Hyperbolic functional differential inequalities and applications. *Mathematics and its Applications,* 486. *Kluwer Academic Publishers, Dordrecht,* 1999.

15. V. Lakshmikantham, Wen Li-Zhi, and Zhang Bing Gen, Theory of Differential Equations with Unbounded Delay. *Mathematics and its Applications,* 298. *Kluwer Academic Publishers Group, Dordrecht,* 1994.

16. H. Leszczyński, On the existence and uniquencess of weak solutions of the Cauchy problem for weakly coupled systems of nonlinear partial differential functional equations of the first order. *Boll. Un. Mat. Ital. B (7)* **7**(1993), 323–354.

17. H. Leszczyński, Convergence results for unbounded solutions of first order nonlinear differential functional equations. *Ann. Polon. Math.* **54**(1996), 1–16.

18. R. Nagel and E. Sinestrari, Nonlinear hyperbolic Volterra integrodifferential equations. *Nonlinear Anal.* **27**(1996), 167–186.

19. K. Prządka, Difference methods for nonlinear partial differential functional equations of the first order. *Math. Nachr.* **138**(1988), 105–123.

20. E. Sinestrari and G. F. Webb, Nonlinear hyperbolic systems with nonlocal boundary conditions. *J. Math. Anal. Appl.* **121**(1987), 449–464.

21. A. L. Skubachevskii, Elliptic Functional-Differential Equations and Applications. *Operator Theory: Advances and Applications,* 91. *Birkhäuser Verlag, Basel,* 1997.

22. J. Szarski, Generalized Cauchy problem for differential functional equations with first order partial derivatives. *Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys.* **24**(1976), No. 8, 575–580.

23. J. Szarski, Cauchy problem for an infinite system of differential functional equations with first order partial derivatives. *Comment. Math. Special Issue* **1**(1978), 293–300.

24. K. Topolski, On the uniqueness of viscosity solutions for first order partial differential functional equations. *Ann. Polon. Math.* **59**(1994), 65–75.

25. K. Topolski, Classical methods for viscosity solutions of differential functional inequalities. *Nonlinear World* **4**(1997), 1–18.

26. J. Turo, A boundary value problem for hyperbolic systems of differential functional equations. *Nonlinear Anal.* **13**(1989), 7–18.

27. M. I. Umanaliev and J. A. Vied, On integral differential equations with first order partial derivatives. (Russian) *Differentsial'nye Uravneniya* **25**(1989), 465–477.

28. W. Walter, Functional differential equations of the Cauchy-Kovalevsky type. *Aequationes Math.* **28**(1985), 102–113.

29. J. Wu, Theory and Applications of Partial Functional Differential Equations. *Applied Mathematical Sciences,* 119. *Springer-Verlag, New York,* 1996.

30. B. Zubik-Kowal, The method of lines for first order partial differential functional equations. *Studia Sci. Math. Hungar.* **34**(1998), 413–428.

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