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ON FACTORIZATION AND PARTIAL INDICES OF UNITARY MATRIX-FUNCTIONS OF ONE CLASS

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ABSTRACT. An effective factorization and partial indices are found for a class of unitary matrix functions.

Let R denote a normed ring of functions defined on the unit circle of a complex plane, say, a ring H^α of Hölder functions with a usual norm, $0 < \alpha < 1$, which can be decomposed into the direct sum of its subrings $R = R^+ + R_0^-$, where the elements R^+ are the boundary values of analytic functions defined within the unit circle, and the elements R_0^- are the boundary values of analytic functions defined outside the unit circle and vanishing at infinity. Also, let $R^- = R_0^- + \mathbb{C}$, where $\mathbb{C}$ denotes the ring of complex numbers. $M_q(R)$ will denote the ring of square $q \times q$ matrix-functions with entries from R. It is well known that the invertible matrix-function $G(t) \in M_q(R)$ is factored as $G(t) = G^+(t)D(t)G^-(t)$, where $G^\pm(t) \in M_q(R^\pm)$, $(G^\pm(t))^{-1} \in M_q(R^\pm)$, and $D(t) = \|d_{ij}(t)\|$ is a diagonal matrix with entries $d_{ii}(t) = t^{n_i}$. According to Muskhelishvili, integers $n_1, n_2, \dots, n_q$, are called partial indices $G(t)$ which can be used to determine the number of linearly independent solutions of the corresponding homogeneous singular integral equation [1].

By a polar decomposition of an arbitrary invertible matrix-function $G(t) = S(t)U(t)$ into a positive definite factor $S(t)$ and a unitary factor $U(t)$ and by using the factorization type for positive definite matrix-functions $(S(t))^2 = S^+(t)(S^+(t))^* = SU_1U_1^*S$ [2], [3] we see, in particular, that partial indices for positive matrices are equal to zero. Thus, taking into account the equality $G(t) = SU_1U_1^{-1}U = S^+U_2$, where $(S^+)^{\pm 1} \in M_q(R^+)$ and U_2 is a unitary matrix, we can say (at least formally) that the general problem of finding partial indices is reducible to the problem of finding such indices for unitary matrix-functions.

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Since partial indices are unstable in general [4], it is interesting to select classes of unitary matrix-functions with zero partial indices. In this connection we note the following

Proposition. Partial indices of a unitary matrix-function $U(t) \in M_q(R)$ with $\det U(t) = 1$ are equal to zero if and only if there exists a positive definite matrix-function $S(t)$ such that $S(t)U(t) \in M_q(R^+)$.

The sufficiency immediately follows from

$$U(t) = S^{-1}(t)S(t)U(t) = Y^+(t)^* Y^+(t)(S(t)U(t) \quad \text{and} \quad U = (U^*)^{-1}.$$

The necessity follows from the fact that if

$$U(t) = U^+(t)U^-(t) = (U^+(t)^*)^{-1}(U^-(t)^*)^{-1} = U_1^- U_1^+ = S_1 O_1 S_2 O_2,$$

where $U_1^- = S_1 O_1$ and $U_1^+ = S_2 O_2$ is a polar decomposition, then $S_1^{-2} U \in M_q(R_+)$.

The above proposition remains valid if there exists a factor $S(t)$ on the right side or if R^+ is replaced by R^-.

Using this proposition one can establish the following

Theorem. *Partial indices of a unitary matrix-function* $U(t) = \|u_{ij}(t)\|$ *with* $\det U = 1$ *of the form*

$$u_{ij}(t) = \alpha_{ij}^+(t), \quad u_{qj}(t) = \overline{\alpha_{qj}^+(t)} \quad \textit{for} \quad 1 \le i \le q-1, \ \ 1 \le j \le q, \tag{1}$$

where $\alpha_{ij}^+(t)$ *are polynomials, are equal to zero if and only if the condition*

$$\sum_{j=0}^{q} |\alpha_{qj}^+(0)|^2 \neq 0 \tag{2}$$

is fulfilled.

To prove the sufficiency of condition (2), for given $U(t)$ one should define a positive definite matrix-function $S(t) = \|s_{ij}(t)\|$ and $X^-(t) = \|x_{ij}^-(t)\| \in M_q(R^-)$ by the equation

$$S(t)U(t) = X^-(t). \tag{3}$$

Condition (1) implies that $s_{ij}(t) = \text{const}$, $1 \le i \le q-1$, $1 \le j \le q-1$, $\overline{s}_{qi}(t) \in R^+$, $1 \le i \le q-1$ are polynomials. We set $s_{ij} = \delta_{ij}$, $1 \le i \le q-1$, $1 \le j \le q-1$, where δ_{ij} is Kronecker's symbol and denote $\varphi_i^+ = \overline{s}_{qi}(t) = s_{iq}(t)$, $1 \le i \le q-1$, and $\varphi_q = s_{qq}(t)$. Equation (3) can now be rewritten as

$$\alpha_{jk}^+ + \varphi_j^+ \overline{\alpha}_{qk}^+ = x_{jk}^-, \quad 1 \le j \le q-1, \quad 1 \le k \le q, \tag{4}$$

$$\sum_{j=1}^{q-1} \alpha_{jk}^{+}\overline{\varphi}_j^{+} + \varphi_q \overline{\alpha}_{qk}^{+} = x_{qk}^{-}, \quad 1 \le k \le q. \tag{5}$$

Condition (2) implies $\alpha_{qj}^{+}(0) \neq 0$ for some $j = p$. Let F_p^{+} be some part of the series $(\alpha_{qp}^{+})^{-1}$ such that $F_p^{+}\alpha_{qp}^{+} = 1 + a_N t^N + a_{N+1}t^{N+1} + \cdots$ with sufficiently large N. When $k = p$, by (4) we obtain

$$\varphi_j^{+} = \mathbb{P}(\overline{F}_p^{+}\alpha_{jp}^{+}), \quad 1 \le j \le q-1, \tag{6}$$

where $\mathbb{P}$ is the projecting operator from R into R^{+}, $\mathbb{P}(R_0) = 0$. Setting

$$\varphi_q = 1 + \sum_{j=1}^{q-1} |\varphi_j^{+}|^2, \tag{7}$$

we see that $S(t)$ is a positive definite matrix-function. It remains for us to check whether (4) is fulfilled for $k \neq p$ and (5). After substituting α_{jp}^{+} from (4) into $\sum_{j=1}^{q-1} \overline{\alpha}_{jk}^{+}\alpha_{jp}^{+} + \alpha_{qk}^{+}\overline{\alpha}_{qp}^{+} = \delta_{kp}$, $1 \le k \le q$, we obtain $(\sum_{j=1}^{q-1} \overline{\alpha}_{jk}^{+}\varphi_j^{+} + \alpha_{qk}^{+})\overline{\alpha}_{qp}^{+} \in R^{-} = \delta_{kp}$, $1 \le k \le q$. The multiplication by $\overline{F}_p^{+}$ gives $\sum_{j=1}^{q-1} \overline{\alpha}_{jk}^{+}\varphi_j^{+} + \alpha_{qk}^{+} = y_{qk}^{-} \in R^{-}$, $1 \le k \le q$. Diagonalizing these linear equations with respect to φ_j^{+} without taking $k = i$ into account we find that (4) is fulfilled for $k = i$. Finally, $\sum_{j=1}^{q-1} \alpha_{jk}^{+}\overline{\varphi}_j^{+} + \varphi_q\overline{\alpha}_{qk}^{+} = \sum_{j=1}^{q-1}(\overline{\alpha}_{jk}^{+} + \varphi_j^{+}\overline{\alpha}_{qk}^{+})\overline{\varphi}_j^{+} + \overline{\alpha}_{qk}^{+} \in R^{-}$. Thus (5) is also fulfilled, which completes the proof.

The necessity follows from the fact that if $\sum_{j=1}^{q} |\alpha_{qj}^{+}(0)|^2 = 0$ and $U = U^{+}U^{-}$ with invertible $U^{\pm}$, then $U^{+} = U(U^{-})^{-1}$ and the elements of the last row of U^{+} belong both to R^{+} and R_0^{-} and therefore are equal to zero, which contradicts the invertibility of U^{+}.

Since the found positive matrix-function $S(t)$ is effectively factored, the factorization of $U(t)$ can be obtained effectively too.

Corollary 1. *When condition* (2) *is fulfilled, the factorization of a unitary matrix-function of form* (1) *can be found as follows:*

$$U(t) = (Y^{-}(t))^{*}Y^{-}(t)S(t)U(t),$$

where $Y^{-}(t) = \|y_{ij}^{-}(t)\|$, $y_{ij}^{-}(t) = \delta_{ij}$ *for* $1 \le i \le q-1$, $1 \le i \le q$, $y_{qj}^{-}(t) = -\overline{\varphi}_j^{+}$ *with* φ_j^{+} *defined by* (6), $1 \le j \le q-1$, $y_{qq}(t) = 1$, *and* $(S(t))^{-1} = (Y^{-}(t))^{*}Y^{-}(t)$.

The proof is obvious, since $S(t)U(t) \in M_q(R^{-})$, $Y^{-}(t)^{*} \in M_q(R^{*})$, and $(Y^{-}(t)^{*})^{-1} \in M_q(R^{*})$.

More can be said for the case with $q = 2$. For a unitary matrix-function $U(t)$ of form (1) all functions $\alpha_{ij}^{*}(t)$, $1 \le j \le q$, may vanish simultaneously only for $t = 0$. Let $\alpha^+_{ij}(t) = t^{n_{ij}} a^+_{ij}(t)$ with polynomials $a^+_{ij}(t)$, $a^+_{ij}(0) \neq 0$ (for $\alpha^+_{ij}(t) = 0$, $n_{ij} = +\infty$) and let

$$n_1 = \min_{1\le j\le 2} n_{1j}, \quad n_2 = -\min_{1\le j\le 2} n_{2j} = -n_1. \tag{8}$$

Then $U(t)$ can be represented as $U(t) = D(t)U_1(t)$, where $U_1(t)$ is a unitary matrix of form (1) for which condition (2) holds, while $D(t)$ is a diagonal matrix with $d_{ii}(t) = t^{n_i}$, $i = 1, 2$. Let $U_1(t) = (Y_1^-(t))^* Y_1^-(t) X_1^-(t)$ be the factorization of $U_1(t)$ with lower triangular $Y_1^-(t)$. Now we can formulate

Corollary 2. *Partial indices of a second-order unitary matrix-function of form* (1) *are equal to* n_1, $-n_1$, *where* n_1 *is defined by* (8).

A factorization of $U(t)$ *can be found as follows:*

$$U(t) = \big(D(t) y_1^-(t)^* D(t)^{-1}\big) D(t) Y_1^-(t) X_1^-(t).$$

The proof easily follows from the observation that $D(t)Y_1^-(t)^* D(t)^{-1} \in M_2(R^+)$ together with its inverse.

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