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# BOUNDS FOR THE CHARACTERISTIC FUNCTIONS OF THE SYSTEM OF MONOMIALS IN RANDOM VARIABLES AND OF ITS TRIGONOMETRIC ANALOGUE

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Abstract. Using a multidimensional analogue of Vinogradov's inequality for a trigonometric integral, the upper bounds are constructed for the moduli of the characteristic functions both of the system of monomials in components of a random vector with an absolutely continuous distribution in $\mathbb{R}^s$ and of the system

$$(\cos j_1\pi\xi_1\cdots\cos j_s\pi\xi_s,\ \ 0\le j_1,\dots,j_s\le k,\ \ j_1+\cdots+j_s\ge 1),$$

where $(\xi_1,\dots,\xi_s)$ is uniformly distributed in $[0,1]^s$.

**Introduction.** This note continues the series of studies [1–3] carried out on the initiative of Yu. V. Prokhorov and dealing with estimates of the characteristic functions (c.f.) of degenerate multidimensional distributions of the form

$$\left|\int_{\mathbb{R}^s}\exp\{i(t,\varphi(x))\}p(x)\,dx\right|\le C|t|^{-\alpha}\quad\text{for}\quad |t|>t_0,\tag{1}$$

where $\varphi:\mathbb{R}^s\to\mathbb{R}^k$, $s<k$, $t\in\mathbb{R}^k$; $p(x)$ is the distribution density; $C$, $\alpha$, and $t_0$ are positive constants. In what follows we shall use the notation $I(t;\varphi(x),p(x))$ for the integral from (1) and omit the condition $|t|>t_0$.

When $s=1$, Sadikova has shown in [1] that for $\varphi(x)=\varphi_0(x)=(x,x^2,\dots,x^k)$ one can take $\alpha=(1+1/k)^{-(k-1)}/k!$ if the integrals of $|p'(x)|$ and $|x|^{k-1}p(x)$ are finite. Her method is based on van der Corput's lemma whose generalization allowed Yurinskii to assert that "for decreasing of the c.f. like a negative power of the argument modulus it is, roughly speaking, sufficient that the surface carrying the distribution have no tangencies of an

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infinitely high order with $(k-1)$-dimensional hyperplanes, and that the surface density of the distribution be bounded, satisfy the Lipschitz condition in $L_1$-norm, and have several moments" (see [2]).

The author's paper [3] deals with the case $s=1$, where $\varphi(x)=\varphi_1(x)=(\cos\pi x,\dots,\cos k\pi x)$, $p(x)=\mathbb{1}_{[0,1]}(x)$, where $\mathbb{1}_A(x)$ is the indicator of the set $A$. Using Vinogradov's famous inequality for a trigonometric integral [4], it is shown in [3] that in that case $\alpha=1/2k$ (for $k=1$ it can be deduced from [2] as well). An order with respect to $|t|$ turned out to be exact (because of the exactness of Vinogradov's inequality), i.e., in some directions of $\mathbb{R}^k$ the c.f. behaves like $|t|^{-1/2k}$ for $|t|\to\infty$. Concomitantly, we obtained bounds with $\alpha=1/k$ for the case $\varphi(x)=\varphi_0(x)$ assuming that $\int_{\mathbb{R}^1}\max(1,|x|)|p'(x)|dx<\infty$ for the density $p(x)$; for analogous estimates see [5].

$\varphi_0(x)$ arises in problems connected with behavior of the joint distribution of sample moments. For $\varphi_1(x)$ the inequality from [3] turns out important in proving the fact that the deviation of the distribution function of $\omega^2$-test statistic from the limiting one has order $n^{-1}$ (see [6], cf. [7]).

**1. Multidimensional Analogue of Vinogradov's Inequality. The Results.** Let $x=(x_1,\dots,x_s)\in\mathbb{R}^s$ and the the multiindex $\mathbf{j}=(j_1,\dots,j_s)\neq\mathbf{0}=(0,\dots,0)\in\mathbb{R}^s$ vary in $J_{k,s}=\{0,1,\dots,k\}^s\setminus\{\mathbf{0}\}$. Consider the system of monomials in $s$ real variables $\varphi_0^{(s)}(x)=\big(x_1^{j_1}\cdots x_s^{j_s},\,\mathbf{j}\in J_{k,s}\big)$ and also the corresponding system of cosines products

$$\varphi_1^{(s)}(x)=\big(\cos j_1\pi x_1\cdots\cos j_s\pi x_s,\;\;\mathbf{j}\in J_{k,s}\big).$$

Denote $\tau=\tau(t)=\max\big\{|t_{\mathbf{j}}|:\mathbf{j}\in J_{k,s}\big\}$, $\quad t=(t_{\mathbf{j}},\mathbf{j}\in J_{k,s})\in\mathbb{R}^{(k+1)^s-1}$.

For $\varphi_j^{(s}(x)$, $j=0,1$, inequalities of form (1), which can be used to study distributions of a system of mixed sample moments and other multivariate statistics, can be derived by the following multidimensional analogue of Vinogradov's inequality [8, p. 39] written for our convenience in the form

$$\big|I(t;\varphi_0^{(s)}(x),\mathbb{1}_{[0,1]^s}(x)\big|\le 32^s(2\pi)^{1/k}\tau^{-1/k}\ln^{s-1}(2+\tau/2\pi); \qquad (2)$$

note that by [8, p. 41] we have for $\gamma>1$ that

$$\left|\int_{[0,1]^s}\exp\{i\gamma x_1^k\cdots x_s^k\}dx\right|\ge[2\pi k^s(s-1)!]^{-1}(2\pi)^{1/k}\gamma^{-1/k}\ln^{s-1}\frac{\gamma}{2\pi}.$$

By virtue of the fact that the inequality $\tau(\widetilde{t})\ge\prod_{j=1}^s\big[\min(1,|y_j|)\big]^k\tau$ holds for $y=(y_1,\dots,y_s)\in\mathbb{R}^s$ and for $\widetilde{t}=(y_1^{j_1}\cdots y_s^{j_s}t_{\mathbf{j}},\,\mathbf{j}\in J_{k,s})$ and the function on the right-hand side of (2) decreases with respect to $\tau$, we have

$$|I_y(t)|\le\prod_{j=1}^s|\operatorname{sgn}y_j|\max(1,|y_j|)32^s(2\pi)^{1/k}\tau^{-1/k}\ln^{s-1}(2+\tau/2\pi) \qquad (3)$$

for

$$I_y(t) = \int_0^{y_1} \cdots \int_0^{y_s} \exp\{i(t, \varphi_0^{(s)}(x))\} dx.$$

Let $D = [a_1, b_1] \times \cdots \times [a_s, b_s]$ with positive $h_j = b_j - a_j$, $j = 1, \ldots, s$; denote $D(j_1, \ldots, j_r) = [a_{j_1}, b_{j_1}] \times \cdots \times [a_{j_r}, b_{j_r}]$ and $D_c(j_1 \ldots j_r) = \prod\{[a_j, b_j] : j = 1, \ldots, s,\ j \neq j_1, \ldots, j_r\}$ for $1 \leq j_1 < \cdots < j_r \leq s$, $1 < r < s$. The notation $x(j_1, \ldots, j_s)$ and $x_c(j_1 \ldots j_r)$ for $x \in \mathbb{R}^s$ is evident. Denote further by $V$, $V_c(j_1 \ldots j_r)$ the sets of vertices of the parallelepipeds $D$, $D_c(j_1 \ldots j_r)$, respectively, and $\Pi(y) = \prod_{j=1}^{r} \max(1, |y_j|)$ for $y = (y_1, \ldots, y_r) \in \mathbb{R}^r$, $1 \leq r \leq s$.

**Proposition 1.** *If a random vector $\xi$ has the density*

$$u_D(x) = (h_1 \cdots h_s)^{-1} \mathbb{1}_D(x),$$

*then for the c.f. of $\varphi_0^{(s)}(\xi)$ the inequality*

$$\left| I(t; \varphi_0^{(s)}(x), u_D(x)) \right| \leq \Pi(h_1^{-1}, \ldots, h_s^{-1}) 32^s (2\pi)^{1/k} \tau^{-1/k} \ln^{s-1}(2 + \tau/2\pi)$$

*holds.*

To proceed to the general case with density $p(x)$ concentrated on $D$ one should use

**Proposition 2.** *If for $x \in D$ a function $p(x) \geq 0$ is continuous in $D$ and has, in $D$, continuous partial derivatives $p_i = p_{x_i}$, $p_{ij} = p_{x_i x_j}, \ldots, p_{1 \cdots s} = p_{x_1 \ldots x_s}$, then the estimate*

$$\left| I(t; \varphi_0^{(s)}(x), \mathbb{1}_D(x) p(x)) \right| \leq C 32^s (2\pi)^{1/k} \tau^{-1/k} \ln^{s-1}(2 + \tau/2\pi)$$

*holds, where $C = C_0 + \cdots + C_s$ with*

$$C_r = \sum_{1 \leq j_1 < \cdots < j_r \leq s} \ \sum_{x_c(j_1 \ldots j_r) \in V_c(j_1 \ldots j_r)} \Pi(x_c(j_1 \ldots j_r)) \times$$
$$\times \int_{D(j_1 \ldots j_r)} \Pi(x(j_1 \ldots j_r)) p_{j_1 \ldots j_r}(x)\, dx(j_1 \ldots j_r), \quad 1 \leq r < s,$$
$$C_0 = \sum_{x \in V} \Pi(x) p(x), \qquad C_s = \int_D \Pi(x) p_{1 \ldots s}(x)\, dx.$$

If the above functions are unbounded or discontinuous in $D$ or $D$ is unbounded, then the estimate will demand obvious changes (conditions of the existence of some limits and integrals).

Since $|t| \leq [(k+1)^s - 1]^{1/2} \tau$ for $t \in \mathbb{R}^{(k+1)^s - 1}$, Propositions 1–2 can be expressed in terms of $|t|$.

Let now $T_r(x) = \sum_{j=0}^{r} a_{rj} x^j$, $x \in \mathbb{R}^1$, be a Chebyshev polynomial of the first kind and $A_k$ be the matrix with rows $(a_{r0}, \ldots, a_{rr}, 0, \ldots, 0)$, $r = 0, \ldots, k$.

For a matrix $B=(b_{pq})_{p,q=0,\dots,k}$ denote $\rho(B)=\max\{\sum\limits_q |b_{pq}| : p=0,\dots,k\}$ and $\rho_k=\rho((A_k^{-1})')$ ($'$ means transposition). Let $\lambda_k=\lambda_{\min}(A_kA_k')$ be the least eigenvalue of the matrix $A_kA_k'$.

**Proposition 3.** *The following inequalities hold:*

(a) $|I(t;\varphi_1^{(s)}(x),\mathbb{1}_{[0,1]^s}(x))|\le 2^{\frac{9ks+1}{k(s+1)}}\pi^{-\frac{2ks-1}{k(s+1)}}s^{\frac{1}{s+1}}\times$
$$\times\rho_k^{\frac{s}{k(s+1)}}\tau^{-\frac{1}{k(s+1)}}\ln^{\frac{s-1}{s+1}}(2+\tau/2\pi\rho_k^s);$$

(b) $|I(t;\varphi_1^{(s)}(x),\mathbb{1}_{[0,1]^s}(x))|\le 2^{\frac{9ks+1}{k(s+1)}}\pi^{-\frac{2ks-1}{k(s+1)}}s^{\frac{1}{s+1}}[(k+1)^s-1]^{\frac{1}{2k(s+1)}}\times$
$$\times\lambda_k^{-\frac{1}{2k(s+1)}}|t|^{-\frac{1}{k(s+1)}}\ln^{\frac{s-1}{s+1}}\left(2+\frac{|t|\lambda_k^{s/2}}{2\pi[(k+1)^s-1]^{1/2}}\right).$$

**2. Proofs.** Proposition 1 is evident.

*Proof of Proposition* 2. If the functions $u:\mathbb{R}^s\to\mathbb{C}$ and $v:\mathbb{R}^s\to\mathbb{C}$ are continuous in $D$ along with the partial derivatives $u_{x_i}$, $v_{x_i}$, $u_{x_ix_j}$, $v_{x_ix_j},\dots,u_{x_1\dots x_s}$, $v_{x_1\dots x_s}$, then

$$\int_D uv_{x_1\dots x_s}dx=\Delta_a^b(uv)-\sum_{1\le i\le s}\Delta_{a_c(i)}^{b_c(i)}\int_{D(i)}vu_{x_i}dx(i)+$$
$$+\sum_{1\le i<j\le s}\Delta_{a_c(ij)}^{b_c(ij)}\int_{D(ij)}vu_{x_ix_j}dx(ij)-\dots+(-1)^s\int_D vu_{x_1\dots x_s}dx.$$

Substituting here $u=p(x)$ and $v=I_x(t)$, passing to moduli, and applying (3), we complete the proof. $\square$

*Proof of Proposition* 3. By the change of variables $x_j=\arccos\pi y_j$, $j=1,\dots,s$, we obtain

$$I(t;\varphi_1^{(s)}(x),\mathbb{1}_{(0,1)^s}(x))=I(t;\widetilde{\varphi}_0^{(s)}(y),q(y)),\qquad(4)$$

where

$$q(y)=\pi^{-s}\prod_{j=1}^{s}(1-y_j^2)^{-1/2}\mathbb{1}_{(-1,1)^s}(y),$$
$$\widetilde{\varphi}_0^{(s)}(y)=(T_{j_1}(y_1)\cdots T_{j_s}(y_s),\ \mathbf{j}\in J_{k,s}).$$

Apply Proposition 2 for a cube $[-(1-\varepsilon),1-\varepsilon]^s$ with $0<\varepsilon<1$. Since $(1-(1-\varepsilon)^2)^{1/2}<\varepsilon^{-1/2}$ and $\int\limits_0^{1-\varepsilon}x(1-x^2)^{-3/2}dx<\varepsilon^{-1/2}$, we have

$$C_r<(2/\pi)^s\binom{s}{r}\varepsilon^{-s/2},\ \ 0\le r\le s,\quad\text{i.e.,}\quad C<(4/\pi)^s\varepsilon^{-s/2}.$$

The remaining part of $|I(t;\varphi_0^{(s)}(y),q(y)|$ is less than $1-(1-z)^s<sz$ with $z=\frac{2}{\pi}(\frac{\pi}{2}-\arcsin(1-\varepsilon))\le\frac{4}{\pi}\sqrt{\varepsilon}$. Finally, we have

$$|I(t;\varphi_0^{(y)}(s),q(y))|\le\Bigl(\frac{4}{\pi}\Bigr)^s 32^s(2\pi)^{1/k}\tau^{-1/k}\ln^{s-1}(2+\tau/2\pi)\varepsilon^{-s/2}+\frac{4s}{\pi}\sqrt{\varepsilon},$$

whence, minimizing with respect to $\varepsilon$, we get

$$|I(t;\varphi_0^{(s)}(y),q(y))|\le 2^{\frac{9ks+1}{k(s+1)}}\pi^{-\frac{2ks-1}{k(s+1)}}s^{\frac{1}{s+1}}\tau^{-\frac{1}{k(s+1)}}\ln^{\frac{s-1}{s+1}}(2+\tau/2\pi). \quad (5)$$

Further,

$$(t,\widetilde{\varphi}_0^{(s)}(y))=\sum_{\mathbf{j}\in J_{k,s}}t_{\mathbf{j}}T_{j_1}(y_1)\cdots T_{j_s}(y_s)=(t^0)'A_k^{(s)}(1,\varphi_0^{(s)}(y))$$

with $A_k^{(s)}=\underbrace{A_k\otimes\cdots\otimes A_k}_{s\text{ times}}$, $A_k^{(1)}=A_k$, where $\otimes$ denotes the Kronecker product, $t^0=(0,t)$ and $\varphi_0^{(s)}(y)$ and $t$ are understood as row vectors with lexicographically arranged components.

From (4) we now obtain

$$I(t;\varphi_1^{(s)}(x),\mathbb{1}_{(0,1)^s}(x))=I(A_k^{(s)'}t^0;(1,\varphi_0^{(s)}(y)),q(y)). \quad (6)$$

Since $\rho(B_1\otimes B_2)=\rho(B_1)\rho(B_2)$ for matrices $B_1$ and $B_2$, according to [9, Theorem 6.5.1], $\tau((A_k^{(s)})'t^0)\ge\tau/\rho((A_k^{(s)'})^{-1})\ge\tau/\rho_k^s$, which by virtue of (5) and (6) leads to (a).

To prove (b), note that since $A_k^{(s)}(A_k^{(s)})'=(A_kA_k')^{(s)}$ (see [10]) and for two positive definite matrices $\lambda_{\min}(B_1\otimes B_2)=\lambda_{\min}(B_1)\lambda_{\min}(B_2)$ (see, e.g., Supplement to [11]), we obtain $|A_k^{(s)'}t^0|=((t^0)'A_k^{(s)}(A_k^{(s)})'(t^0))^{1/2}\ge\lambda_k^{s/2}|t|$ and $\tau(A_k^{(s)'}t^0)\ge[(k+1)^s-1]^{-1/2}\lambda_k^{s/2}|t|$. □

Other applications of (2) to obtain bounds for c.f. can be found in [12]. Inequalities similar to the general ones from [2] can be found in [13].

Based on the theory of singularities and asymptotic expansions of oscillating integrals [14] one can study the exactness properties of bounds (a) and (b) with respect to $|t|$ and $\tau$ for the case $s>1$, too.

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