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SUMMATION OF SINGULAR SERIES CORRESPONDING TO REPRESENTATIONS OF NUMBERS BY SOME QUADRATIC FORMS IN TWELVE VARIABLES

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ABSTRACT. Formulas for calculating the sum of singular series corresponding to the number of representations of integers by some quadratic forms in 12 variables with integral coefficient are derived.

INTRODUCTION

In this paper a formula is derived for the sum of the singular series corresponding to the number of representations of positive integers by positive primitive quadratic forms

$$f = a_1(x_1^2 + x_2^2) + a_2(x_3^2 + x_4^2) + a_3(x_5^2 + x_6^2) + \\ + a_4(x_7^2 + x_8^2) + a_5(x_5^2 + x_{10}^2) + a_6(x_{11}^2 + x_{12}^2) \qquad (1)$$

with integral coefficients $a_1, \dots, a_6$.

In our next paper a way to find explicit exact formulas for the number of representations of positive integers by the quadratic forms of type (1) will be suggested.

1. PRELIMINARIES

1.1. In this paper a, k, n, q, λ denote positive integers; b, m, u, v are odd positive integers; p is a prime number; α, β, γ, ν, l are non-negative integers,; h, j, x, y are integers; i is an imaginary unit; $\sum_{h \bmod q}$and $\sum'_{h \bmod q}$ denote respectively sums in which h runs a complete and a reduced residue system modulo q; $(\frac{h}{u})$ is the generalized Jacobi symbol; $\sigma_5(u)$ is the sum of the fifth powers of positive divisors of u; $e(z) = e^{2\pi i z}$ for arbitrary complex number z; Δ is the determinant of the form (1).

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Let

$$S(h,q)=\sum_{j \bmod q} e\Big(\frac{hj^2}{q}\Big) \quad \text{(Gaussian sum);} \tag{1.1}$$

$$c(h,q)=\sum_{j \bmod q}{}' e\Big(\frac{hj}{q}\Big)=\sum_{j \bmod q}{}' e\Big(-\frac{hj}{q}\Big) \quad \text{(Ramanujan's sum);} \tag{1.2}$$

$$\rho(n;f)=\frac{\pi^6}{5!\Delta^{1/2}}\, n^5 \sum_{q=1}^{\infty} A(q) \quad \text{(singular series of the problem),} \tag{1.3}$$

where

$$A(q)=q^{-12}\sum_{h \bmod q}{}' e\Big(-\frac{hn}{q}\Big)\prod_{k=1}^{6} S^2(a_k h,q). \tag{1.4}$$

1.2. For the convenience of the reader we quote some known results as the following lemmas:

Lemma 1. *If $(h,q)=1$, then $S(kh,kq)=kS(h,q)$.*

Lemma 2 ([1], p. 13, Lemma 6). *If $(h,q)=1$, then*

$$\begin{aligned} S^2(h,q) &= \Big(\frac{-1}{q}\Big)q \quad \textit{for} \quad q\equiv 1 \pmod 2,\\ &= 2i^h q \quad \textit{for} \quad q\equiv 0 \pmod 4,\\ &= 0 \quad \textit{for} \quad q\equiv 2 \pmod 4. \end{aligned}$$

Lemma 3 ([1], p. 177, formula (20)). *Let $q=p^\lambda$ and $p^\nu \,||\, h$. Then*

$$\begin{aligned} c(h,q) &= 0 \quad \textit{for} \quad \nu<\lambda-1,\\ &= -p^{\lambda-1} \quad \textit{for} \quad \nu=\lambda-1,\\ &= p^{\lambda-1}(p-1) \quad \textit{for} \quad \nu>\lambda-1. \end{aligned}$$

Lemma 4. *Let*

$$\chi_p = 1+A(p)+A(p^2)+\cdots . \tag{1.5}$$

Then

$$\sum_{q=1}^{\infty} A(q)=\prod_p \chi_p.$$

2. The summation of the singular series $\rho(n;f)$

Lemma 5. *Let* $n = 2^{\alpha}m$, $a_k = 2^{\gamma_k}b_k$ $(k=1,2,\dots,6)$, $\gamma_6 \ge \gamma_5 \ge \gamma_4 \ge \gamma_3 \ge \gamma_2 \ge \gamma_1 = 0$, $\gamma = \sum_{k=2}^{6}\gamma_k$, $(b_1,b_2,\dots,b_6)=1$, $b=[b_1,b_2,\dots,b_6]$. *Then*

$$\chi_2 = 1 + (-1)^{(b_1-m)/2} \quad \textit{for} \quad 0 \le \alpha \le \gamma_2 - 2;$$

$$= 1 \quad \textit{for} \quad \alpha = \gamma_2 - 1,\ \ \alpha = \gamma_2 < \gamma_3,\ \ \gamma_2 = \gamma_3 \le \alpha = \gamma_4 - 1,$$
$$\gamma_2 + 1 = \gamma_3 \le \alpha = \gamma_4 - 1,\ \ \gamma_2 = \gamma_3 \le \alpha = \gamma_4 < \gamma_5,$$
$$\gamma_2 + 1 = \gamma_3 \le \alpha = \gamma_4 < \gamma_5,\ \ \gamma_4 = \gamma_5 \le \alpha = \gamma_6 - 1,$$
$$\gamma_4 + 1 = \gamma_5 \le \alpha = \gamma_6 - 1,\ \ \gamma_4 = \gamma_5 \le \alpha = \gamma_6,$$
$$\gamma_4 + 1 = \gamma_5 \le \alpha = \gamma_6$$

(*in the last four conditions* $\gamma_3 = \gamma_2$ *or* $\gamma_3 = \gamma_2 + 1$);

$$\chi_2 = 2^{-\alpha}\Big\{2^{\alpha} + (-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}(2^{\alpha-\gamma_2} - 3)\Big\} \quad \textit{for} \quad \gamma_2 + 1 \le \alpha < \gamma_3,$$

$$= 2^{-2(\alpha+1)}\Big\{2^{2(\alpha+1)} + (-1)^{(\sum_{k=1}^{3} b_k - m)/2}\, 2^{\gamma_2+\gamma_3}\Big\}$$
$$\textit{for} \quad \gamma_2 = \gamma_3 \le \alpha \le \gamma_4 - 2,\ \ \gamma_2 + 1 = \gamma_3 \le \alpha \le \gamma_4 - 2,$$

$$= 2^{-2(\alpha+1)}\Big\{2^{2(\alpha+1)} + 2^{\gamma_2}\Big((-1)^{(b_1+b_2)/2}\big(2^{2(\alpha+1)-\gamma_2} - 2^{2\alpha-\gamma_3+3}\big) +$$
$$+ (-1)^{(\sum_{k=1}^{3} b_k - m)/2}\cdot 2^{\gamma_3}\Big)\Big\} \quad \textit{for} \quad \gamma_2 + 2 \le \gamma_3 \le \alpha = \gamma_4 - 2,$$

$$= 2^{-\gamma_3}\Big\{2^{\gamma_3} + (-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2+1}(2^{\gamma_3-\gamma_2-1} - 1)\Big\}$$
$$\textit{for} \quad \gamma_2 + 2 \le \gamma_3 \le \alpha = \gamma_4 - 1, \quad \gamma_2 + 2 \le \gamma_3 \le \alpha = \gamma_4 < \gamma_5,$$

$$= 2^{-3\alpha}\Big\{2^{3\alpha} + (-1)^{(b_1+b_2)/2}\cdot 2^{3\alpha+\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1} - 1) +$$
$$+ (-1)^{(\sum_{k=1}^{4} b_k)/2}\cdot 2^{\sum_{k=2}^{4}\gamma_k}\Big(2^3\big(2^{3(\alpha-\gamma_4-1)} - 1\big)7^{-1} - 1\Big)\Big\}$$
$$\textit{for} \quad \gamma_4 + 1 \le \alpha < \gamma_5, \quad \textit{but} \quad \gamma_3 \ge \gamma_2 + 2,$$

$$= 2^{-3\alpha}\Big\{2^{3\alpha} + (-1)^{(\sum_{k=1}^{4} b_k)/2}\cdot 2^{\sum_{k=2}^{4}\gamma_k}\Big(2^3\big(2^{3(\alpha-\gamma_4-1)} - 1\big)7^{-1} - 1\Big)\Big\}$$
$$\textit{for} \quad \gamma_4 + 1 \le \alpha < \gamma_5, \quad \textit{but} \quad \gamma_3 = \gamma_2 \ \ \textit{or} \ \ \gamma_3 = \gamma_2 + 1,$$

$$= 2^{-4(\alpha+1)}\Big\{2^{4(\alpha+1)} + (-1)^{(\sum_{k=1}^{5} b_k - m)/2}\cdot 2^{\gamma-\gamma_6}\Big\}$$

$$\textit{for}\quad \gamma_4=\gamma_5\le\alpha\le\gamma_6-2,\quad \gamma_4+1=\gamma_5\le\alpha\le\gamma_6-2,$$
$$\textit{but}\ \ \gamma_3=\gamma_2\ \ \textit{or}\ \ \gamma_3=\gamma_2+1,$$

$$\chi_2=2^{-4(\alpha+1)}\Big\{2^{4(\alpha+1)}+(-1)^{(b_1+b_k)/2}\cdot 2^{4\alpha+5+\gamma_2-\gamma_3}(2^{\gamma_3-\gamma_2-1}-1)+$$
$$+(-1)^{(\sum\limits_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6}\Big\}\quad \textit{for}\quad \gamma_4=\gamma_5\le\alpha\le\gamma_6-2,$$
$$\gamma_4+1=\gamma_5\le\alpha\le\gamma_6-2,\quad \textit{but}\ \ \gamma_3\ge\gamma_2+2,$$
$$=2^{-(\gamma_3-\gamma_2-1)}\Big\{2^{\gamma_3-\gamma_2-1}+(-1)^{(b_1+b_2)/2}(2^{\gamma_3-\gamma_2-1}-1)\Big\}$$
$$\textit{for}\quad \gamma_4=\gamma_5\le\alpha=\gamma_6-1,\quad \gamma_4+1=\gamma_5\le\alpha=\gamma_6-1,$$
$$\gamma_4=\gamma_5\le\alpha=\gamma_6,\quad \gamma_4+1=\gamma_5\le\alpha=\gamma_6,$$
$$\textit{but}\ \ \gamma_3\ge\gamma_2+2,$$
$$=2^{-4(\alpha+1)}\Big\{2^{4(\alpha+1)}+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}\Big(2^{4(\alpha+1)-3\gamma_4}+$$
$$-2^{4(\alpha+1)-3(\gamma_5-1)}\Big)7^{-1}+(-1)^{(\sum\limits_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6}\Big\}$$
$$\textit{for}\quad \gamma_4+2\le\gamma_5\le\alpha\le\gamma_6-2,\quad \textit{but}\ \ \gamma_3=\gamma_2\ \ \textit{or}\ \ \gamma_3=\gamma_2+1,$$
$$=2^{-4(\alpha+1)}\Big\{2^{4(\alpha+1)}+(-1)^{(b_1+b_2)/2}\cdot 2^{4\alpha+5+\gamma_2-\gamma_3}(2^{\gamma_3-\gamma_2-1}-1)+$$
$$+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}\Big(2^{\alpha+7}\big(2^{3(\alpha-\gamma_4-1)}-2^{3(\alpha-\gamma_5)}\big)7^{-1}\Big)+$$
$$+(-1)^{(\sum\limits_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6}\Big\}\quad \textit{for}\quad \gamma_4+2\le\gamma_5\le\alpha\le\gamma_6-2,$$
$$\textit{but}\ \ \gamma_3\ge\gamma_2+2,$$
$$=2^{-4(\alpha+1)}\Big\{2^{4(\alpha+1)}+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}\,2^{\alpha+7}\times$$
$$\times\big(2^{3(\alpha-\gamma_4-1)}-2^{3(\alpha-\gamma_5-1)}\big)7^{-1}\Big\}\quad \textit{for}\quad \gamma_4+2\le\gamma_5\le\alpha=\gamma_6-1,$$
$$\gamma_4+2\le\gamma_5\le\alpha=\gamma_6,\quad \textit{but}\ \ \gamma_3=\gamma_2\ \ \textit{or}\ \ \gamma_3=\gamma_2+1,$$
$$=2^{-3(\gamma_5-1)}\Big\{2^{3(\gamma_5-1)}+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2-\gamma_3+3\gamma_5-2}(2^{\gamma_3-\gamma_2-1}-1)+$$
$$+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}\big(2^{3(\gamma_5-\gamma_4-1)}-1\big)7^{-1}\Big\}$$

$$\text{for}\quad \gamma_4+2\le\gamma_5\le\alpha=\gamma_6,$$
$$\gamma_4+2\le\gamma_5\le\alpha=\gamma_6-1,\quad \text{but}\quad \gamma_3\ge\gamma_2+2,$$

$$\chi_2=2^{-5\alpha}\Big\{2^{5\alpha}+(-1)^{(\sum_{k=1}^{6}b_k)/2}\cdot 2^{\gamma}\Big(2^5\big(2^{5(\alpha-\gamma_6-1)}-1\big)31^{-1}-1\Big)\Big\}$$
$$\text{for}\quad \alpha\ge\gamma_6+1,\quad \text{but}\quad \gamma_3=\gamma_2\quad \text{or}\quad \gamma_3=\gamma_2+1$$
$$\text{and}\quad \gamma_5=\gamma_4\quad \text{or}\quad \gamma_5\le\gamma_4+1,$$

$$=2^{-5\alpha}\Big\{2^{5\alpha}+(-1)^{(\sum_{k=1}^{4}b_k)/2}\cdot 2^{5\alpha+\gamma-4\gamma_5-\gamma_6+3}\big(2^{3(\gamma_5-\gamma_4-1)}-1\big)7^{-1}+$$
$$+(-1)^{(\sum_{k=1}^{6}b_k)/2}\cdot 2^{\gamma}\Big(2^5\big(2^{5(\alpha-\gamma_6+1)}-1\big)31^{-1}-1\Big)\Big\}\quad \text{for}\quad \alpha\ge\gamma_6+1,$$
$$\text{but}\quad \gamma_3=\gamma_2\quad \text{or}\quad \gamma_3=\gamma_2+1\quad \text{and}\quad \gamma_5\ge\gamma_4+2,$$

$$=2^{-5\alpha}\Big\{2^{5\alpha}+(-1)^{(b_1+b_2)/2}\cdot 2^{5\alpha+\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1}-1)+$$
$$+(-1)^{(\sum_{k=1}^{6}b_k)/2}\cdot 2^{\gamma}\Big(2^5\big(2^{5(\alpha-\gamma_6-1)}-1\big)31^{-1}-1\Big)\Big\}\quad \text{for}\quad \alpha\ge\gamma_6+1,$$
$$\text{but}\quad \gamma_3\ge\gamma_2+2\quad \text{and}\quad \gamma_5=\gamma_4\quad \text{or}\quad \gamma_5=\gamma_4+1,$$

$$=2^{-5\alpha}\Big\{2^{5\alpha}+(-1)^{(b_1+b_2)/2}\cdot 2^{5\alpha+\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1}-1)+$$
$$+(-1)^{(\sum_{k=1}^{4}b_k)/2}\cdot 2^{5\alpha+\gamma-4\gamma_5-\gamma_6+3}\big(2^{3(\gamma_5-\gamma_4-1)}-1\big)7^{-1}+$$
$$+(-1)^{(\sum_{k=1}^{6}b_k)/2}\cdot 2^{\gamma}\Big(2^5\big(2^{5(\alpha-\gamma_6+1)}-1\big)31^{-1}-1\Big)\Big\}$$
$$\text{for}\quad \alpha\ge\gamma_6+1,\quad \text{but}\quad \gamma_3\ge\gamma_2+2\quad \text{and}\quad \gamma_5\ge\gamma_4+2.$$

Proof. I. If in (1.4) we put $q=2^{\lambda}$ and then instead of h introduce a new letter of summation y defined by the congruence $h\equiv by \pmod{2^{\lambda}}$, then we get

$$A(2^{\lambda})=2^{-12\lambda}\sum_{y \bmod 2^{\lambda}}{}' e(-2^{\alpha-\lambda}mby)\prod_{k=1}^{6}S^2(2^{\gamma_k}b_kby,2^{\lambda}).\tag{2.1}$$

From (2.1), according to Lemmas 1, 2 and 3 it follows that:
(1) for $\lambda=\gamma_k+1$, as $S^2(2^{\gamma_k}b_kby,2^{\gamma_k+1})=0$ $(k=1,2,\ldots,6)$,

$$A(2^{\lambda})=0;\tag{2.2}$$

(2) for $2 \le \lambda \le \gamma_2$

$$A(2^\lambda) = 2^{-12\lambda} \sideset{}{'}\sum_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda} mby)(2i^{b_i by} \cdot 2^\lambda) 2^{10\lambda} =$$

$$= 2^{1-\lambda} \sideset{}{'}\sum_{y \bmod 2^\lambda} e\Big(\frac{b_1 by}{4} - \frac{2^\alpha mby}{2^\lambda}\Big) =$$

$$= 2^{1-\lambda} e\Big(\frac{b_1 b - 2^{\alpha-\lambda+2} mb}{4}\Big) \sum_{y=0}^{2^{\lambda-1}-1} e\Big(\frac{(2^{\lambda-2} b_1 - 2^\alpha m) by}{2^{\lambda-1}}\Big) =$$

$$= \begin{cases} e\Big(\dfrac{b_1 b - 2^{\alpha-\lambda+2} mb}{4}\Big) & \text{if} \quad 2^{\lambda-1} \mid (2^{\lambda-2} b_1 - 2^\alpha m) b, \\ 0 & \text{if} \quad 2^{\lambda-1} \nmid (2^{\lambda-2} b_1 - 2^\alpha m) b, \end{cases}$$

i.e.,

$$A(2^\lambda) = \begin{cases} (-1)^{(b_1 - m)/2} & \text{if} \quad \lambda = \alpha + 2, & (2.3) \\ 0 & \text{if} \quad \lambda \ne \alpha + 2; & (2.3_1) \end{cases}$$

(3) for $\gamma_2 + 2 \le \lambda \le \gamma_3$

$$A(2^\lambda) = 2^{-12\lambda} \sideset{}{'}\sum_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda} mby)(2i^{b_i by} \cdot 2^\lambda) 2^{2\gamma_2} (2i^{b_2 by} \cdot 2^{\lambda-\gamma_2}) 2^{8\lambda} =$$

$$= (-1)^{(b_1+b_2)/2} \cdot 2^{-2\lambda+\gamma_2+2} c(2^\alpha mb, 2^\lambda) =$$

$$= \begin{cases} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2 - (\lambda-1)} & \text{if} \quad \lambda < \alpha + 1, & (2.4) \\ (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2 - \alpha} & \text{if} \quad \lambda = \alpha + 1, & (2.4_1) \\ 0 & \text{if} \quad \lambda > \alpha + 1; & (2.4_2) \end{cases}$$

(4) for $\gamma_3 + 2 \le \lambda \le \gamma_4$

$$A(2^\lambda) = 2^{-12\lambda} \sideset{}{'}\sum_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda} mby)(2i^{b_i by} \cdot 2^\lambda) \times$$

$$\times 2^{\gamma_2} (2i^{b_2 by} \cdot 2^{\lambda-\gamma_2}) 2^{\gamma_3} (2i^{b_3 by} \cdot 2^{\lambda-\gamma_3}) 2^{6\lambda} =$$

$$= 2^{-3\lambda+\gamma_2+\gamma_3+3} e\Big(\frac{(b_1+b_2+b_3)b}{4} - \frac{2^\alpha mb}{2^\lambda}\Big) \times$$

$$\times \sum_{y=0}^{2^{\lambda-1}-1} e\Big(\frac{(2^{\lambda-2}(b_1+b_2+b_3) - 2^\alpha m) by}{2^{\lambda-1}}\Big) =$$

$$= \begin{cases} (-1)^{(b_1+b_2+b_3-m)/2} \cdot 2^{\gamma_2+\gamma_3-2\alpha-2} & \text{if} \quad \lambda = \alpha + 2, & (2.5) \\ 0 & \text{if} \quad \lambda \ne \alpha + 2; & (2.5_1) \end{cases}$$

(5) for $\gamma_4+2\le\lambda\le\gamma_5$, similarly as in (3),

$$A(2^\lambda)=\begin{cases}(-1)^{(\sum_{k=1}^{4}b_k)/2}\cdot 2^{\sum_{k=2}^{4}\gamma_k-3(\lambda-1)} & \text{if}\quad \lambda<\alpha+1, & (2.6)\\ -(-1)^{(\sum_{k=1}^{4}b_k)/2}\cdot 2^{\sum_{k=2}^{4}\gamma_k-3\alpha} & \text{if}\quad \lambda=\alpha+1, & (2.6_1)\\ 0 & \text{if}\quad \lambda>\alpha+1; & (2.6_2)\end{cases}$$

(6) for $\gamma_5+2\le\lambda\le\gamma_6$, similarly as in (4),

$$A(2^\lambda)=\begin{cases}(-1)^{(\sum_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6-4\alpha-4} & \text{if}\quad \lambda=\alpha+2, & (2.7)\\ 0 & \text{if}\quad \lambda\ne\alpha+2; & (2.7_1)\end{cases}$$

(7) for $\lambda\ge\gamma_6+2$, similarly as in (3) and in (5),

$$A(2^\lambda)=\begin{cases}(-1)^{(\sum_{k=1}^{6}b_k)/2}\cdot 2^{\gamma-5(\lambda-1)} & \text{if}\quad \lambda<\alpha+1, & (2.8)\\ -(-1)^{(\sum_{k=1}^{6}b_k)/2}\cdot 2^{\gamma-5\alpha} & \text{if}\quad \lambda=\alpha+1, & (2.8_1)\\ 0 & \text{if}\quad \lambda>\alpha+1; & (2.8_2)\end{cases}$$

II. According to (1.5) and (2.2), we have

$$\chi_2=1+\sum_{\lambda=2}^{\gamma_2}A(2^\lambda)+\sum_{\lambda=\gamma_2+2}^{\gamma_3}A(2^\lambda)+\sum_{\lambda=\gamma_3+2}^{\gamma_4}A(2^\lambda)+ \\ +\sum_{\lambda=\gamma_4+2}^{\gamma_5}A(2^\lambda)+\sum_{\lambda=\gamma_5+2}^{\gamma_6}A(2^\lambda)+\sum_{\lambda=\gamma_6+2}^{\infty}A(2^\lambda). \qquad (2.9)$$

Consider the following cases:

(1) Let $0\le\alpha\le\gamma_2-2$. Then from (2.9), (2.3), (2.3_1), (2.4_2), (2.5_1), (2.6_2), (2.7_1), and (2.8_2) we get

$$\chi_2=1+\sum_{\lambda=2}^{\gamma_2}A(2^\lambda)=1+(-1)^{(b_1-m)/2}.$$

(2) Let $\alpha=\gamma_2-1$, or $\alpha=\gamma_2<\gamma_3$ or $\gamma_2=\gamma_3\le\alpha=\gamma_4-1$ or $\gamma_2+1=\gamma_3\le\alpha=\gamma_4-1$ or $\gamma_2=\gamma_3\le\alpha=\gamma_4<\gamma_5$ or $\gamma_2+1=\gamma_3\le\alpha=\gamma_4<\gamma_5$ or $\gamma_4=\gamma_5\le\alpha=\gamma_6-1$ or $\gamma_4+1=\gamma_5\le\alpha=\gamma_6-1$, or $\gamma_4=\gamma_5\le\alpha=\gamma_6$ or $\gamma_4+1=\gamma_5\le\alpha=\gamma_6$ (in the last four conditions $\gamma_3=\gamma_2$ or $\gamma_3=\gamma_2+1$). Then from (2.9), (2.3_1), (2.4_2), (2.5_1), (2.6_2), (2.7_1), and (2.8_2) we get

$$\chi_2=1.$$

(3) Let $\gamma_2+1\le\alpha<\gamma_3$. Then from (2.9), (2.3_1), (2.4)–(2.4_2), (2.5_1), (2.6_2), (2.7_1), and (2.8_2) we get

$$\begin{aligned}\chi_2&=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}\sum_{\lambda=\gamma_2+2}^{\alpha}2^{-(\lambda-1)}-(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2-\alpha}=\\&=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}(2^{-\gamma_2-1}-2^{-\alpha})2-(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2-\alpha}=\\&=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}(2^{-\gamma_2}-3\cdot 2^{-\alpha}).\end{aligned}$$

(4) Let $\gamma_2=\gamma_3\le\alpha\le\gamma_4-2$ or $\gamma_2+1=\gamma_3\le\alpha\le\gamma_4-2$. Then from (2.9), (2.3_1), (2.4_2), (2.5), (2.5_1), (2.6_2), (2.7_1), and (2.8_2) we get

$$\chi_2=1+(-1)^{(b_1+b_2+b_3-m)/2}\cdot 2^{\gamma_2+\gamma_3-2\alpha-2}.$$

(5) Let $\gamma_2+2\le\gamma_3\le\alpha\le\gamma_4-2$. Then from (2.9), (2.3_1), (2.4), (2.5), (2.5_1), (2.6_2), (2.7_1), and (2.8_2) we get

$$\begin{aligned}\chi_2&=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}\sum_{\lambda=\gamma_2+2}^{\gamma_3}2^{-(\lambda-1)}+\\&+(-1)^{(b_1+b_2+b_3-m)/2}\cdot 2^{\gamma_2+\gamma_3-2\alpha-2}=\\&=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}(2^{-\gamma_2}-2^{-\gamma_3+1})+\\&+(-1)^{(\sum\limits_{k=1}^{3}b_k-m)/2}\cdot 2^{\gamma_2+\gamma_3-2\alpha-2}.\end{aligned}$$

(6) Let $\gamma_2+2\le\gamma_3\le\alpha=\gamma_4-1$ or $\gamma_2+2\le\gamma_3\le\alpha=\gamma_4<\gamma_5$. Then from (2.9), (2.3_1), (2.4), (2.4_2), (2.5_1), (2.6_2), (2.7_1), and (2.8_2) we get

$$\chi_2=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}(2^{-\gamma_2}-2^{-\gamma_3+1}).$$

(7) Let $\gamma_4+1\le\alpha<\gamma_5$, but $\gamma_3\ge\gamma_2+2$. Then from (2.9), (2.3_1), (2.4), (2.5_1), (2.6), (2.6_1), (2.7_1), and (2.8_2) we get

$$\begin{aligned}\chi_2&=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}\sum_{\lambda=\gamma_2+2}^{\gamma_3}2^{-(\lambda-1)}+\\&+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}\sum_{\lambda=\gamma_4+2}^{\alpha}2^{-3(\lambda-1)}-(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k-3\alpha}=\\&=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}(2^{-\gamma_2}-2^{-\gamma_3+1})+\\&+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}(2^{-3\gamma_4-3}-2^{-3\alpha})2^3\cdot 7^{-1}+\end{aligned}$$

$$- (-1)^{(\sum_{k=1}^{4} b_k)/2} \cdot 2^{\sum_{k=2}^{4} \gamma_k - 3\alpha} =$$
$$= 1 + (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1} - 1) +$$
$$+ (-1)^{(\sum_{k=1}^{4} b_k)/2} \cdot 2^{\sum_{k=2}^{4} \gamma_k} \big(2^{-3\alpha+3}(2^{3(\alpha-\gamma_4-1)} - 1)7^{-1} - 2^{-3\alpha}\big).$$

(8) Let $\gamma_4 + 1 \leq \alpha < \gamma_5$, but $\gamma_3 = \gamma_2$ or $\gamma_3 = \gamma_2 + 1$. Then from (2.9), (2.3_1), (2.4_2), (2.5_1), (2.6), (2.6_1), (2.7_1), and (2.8_2) we get

$$\chi_2 = 1 + (-1)^{(\sum_{k=1}^{4} b_k)/2} \cdot 2^{\sum_{k=2}^{4} \gamma_k} \sum_{\lambda=\gamma_4+2}^{\alpha} 2^{-3(\lambda-1)} - (-1)^{(\sum_{k=1}^{4} b_k)/2} \cdot 2^{\sum_{k=2}^{4} \gamma_k - 3\alpha}.$$

(9) Let$\gamma_4 = \gamma_5 \leq \alpha \leq \gamma_6 - 2$ or $\gamma_4 + 1 = \gamma_5 \leq \alpha \leq \gamma_6 - 2$, but $\gamma_3 = \gamma_2$ or $\gamma_3 = \gamma_2 + 1$. Then from (2.9), (2.3_1), (2.4_2), (2.5_1), (2.6_2), (2.7), and (2.8_2) we get

$$\chi_2 = 1 + (-1)^{(\sum_{k=1}^{5} b_k - m)/2} \cdot 2^{\gamma-\gamma_6-4\alpha-4}.$$

(10) Let $\gamma_4 = \gamma_5 \leq \alpha \leq \gamma_6 - 2$ or $\gamma_4 + 1 = \gamma_5 \leq \alpha \leq \gamma_6 - 2$, but $\gamma_3 \geq \gamma_2 + 2$. Then from (2.9), (2.3_1), (2.4), (2.5_1), (2.6_2), (2.7), and (2.8_2) we get

$$\chi_2 = 1 + (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2} \sum_{\lambda=\gamma_2+2}^{\gamma_3} 2^{-(\lambda-1)} +$$
$$+ (-1)^{(\sum_{k=1}^{5} b_k - m)/2} \cdot 2^{\gamma-\gamma_6-4\alpha-4} =$$
$$= 1 + (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1} - 1) +$$
$$+ (-1)^{(\sum_{k=1}^{5} b_k - m)/2} \cdot 2^{\gamma-\gamma_6-4\alpha-4}.$$

(11) Let $\gamma_4 = \gamma_5 \leq \alpha = \gamma_6 - 1$ or $\gamma_4 + 1 = \gamma_5 \leq \alpha = \gamma_6 - 1$or $\gamma_4 = \gamma_5 \leq \alpha = \gamma_6$ or $\gamma_4 + 1 = \gamma_5 \leq \alpha = \gamma_6$, but $\gamma_3 \geq \gamma_2 + 2$. Then from (2.9), (2.3_1), (2.4), (2.5_1), (2.6_2), (2.7_1), and (2.8_2) we get

$$\chi_2 = 1 + (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2} \sum_{\lambda=\gamma_2+2}^{\gamma_3} 2^{-(\lambda-1)} =$$
$$= 1 + (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1} - 1).$$

(12) Let $\gamma_4+2\le\gamma_5\le\alpha\le\gamma_6-2$, but $\gamma_3=\gamma_2$ or $\gamma_3=\gamma_2+1$. Then from (2.9), (2.3_1), (2.4_2), (2.5_1), (2.6), (2.7), and (2.8_2) we get

$$\chi_2=1+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}\sum_{\lambda=\gamma_4+2}^{\gamma_5}2^{-3(\lambda-1)}+$$
$$+(-1)^{(\sum\limits_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6-4\alpha-4}=$$
$$=1+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}(2^{-3\gamma_4-3}-2^{-3\gamma_5})2^3\cdot 7^{-1}+$$
$$+(-1)^{(\sum\limits_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6-4\alpha-4}.$$

(13) Let $\gamma_4+2\le\gamma_5\le\alpha\le\gamma_6-2$, but $\gamma_3\ge\gamma_2+2$. Then from (2.9), (2.3_1), (2.4), (2.5_1), (2.6), (2.7), and (2.8_2) we get

$$\chi_2=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2}\sum_{\lambda=\gamma_2+2}^{\gamma_3}2^{-(\lambda-1)}+$$
$$+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}\sum_{\lambda=\gamma_4+2}^{\gamma_5}2^{-3(\lambda-1)}+$$
$$+(-1)^{(\sum\limits_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6-4\alpha-4}=$$
$$=1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1}-1)+$$
$$+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}(2^{-3\gamma_4}-2^{-3\gamma_5+3})7^{-1}+$$
$$+(-1)^{(\sum\limits_{k=1}^{5}b_k-m)/2}\cdot 2^{\gamma-\gamma_6-4\alpha-4}.$$

(14) Let $\gamma_4+2\le\gamma_5\le\alpha=\gamma_6-1$ or $\gamma_4+2\le\gamma_5\le\alpha=\gamma_6$, but $\gamma_3=\gamma_2$ or $\gamma_3=\gamma_2+1$. Then from (2.9), (2.3_1), (2.4_2), (2.5_1), (2.6), (2.7_1), and (2.8_2) we get

$$\chi_2=1+(-1)^{(\sum\limits_{k=1}^{4}b_k)/2}\cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}(2^{-3\gamma_4}-2^{-3\gamma_5+3})7^{-1}.$$

(15) Let $\gamma_4+2 \le \gamma_5 \le \alpha = \gamma_6 - 1$ or $\gamma_4 + 2 \le \gamma_5 \le \alpha = \gamma_6$, but $\gamma_3 \ge \gamma_2 + 2$. Then from (2.9), (2.3_1), (2.4), (2.5_1), (2.6), (2.7_1), and (2.8_2) we get

$$\chi_2 = 1 + (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1} - 1) + \\ + (-1)^{(\sum\limits_{k=1}^{4} b_k)/2} \cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}(2^{-3\gamma_4} - 2^{-3\gamma_5+3})7^{-1}.$$

(16) Let $\alpha \ge \gamma_6+1$, but $\gamma_3 = \gamma_2$ or $\gamma_3 = \gamma_2+1$ and $\gamma_5 = \gamma_4$ or $\gamma_5 = \gamma_4+1$. Then from (2.9), (2.3_1), (2.4_2), (2.5_1), (2.6_2), (2.7_1), (2.8)–(2.8_2) we get

$$\chi_2 = 1 + (-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma} \sum_{\lambda=\gamma_6+2}^{\alpha} 2^{-5(\lambda-1)} - (-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma-5\alpha} = \\ = 1 + (-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma}(2^{-5\gamma_6-5} - 2^{-5\alpha})2^5 \cdot 31^{-1} + \\ - (-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma-5\alpha} = \\ = 1 + (-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma}\Big(2^{-5(\alpha-1)}\big(2^{5(\alpha-\gamma_6-1)} - 1\big)31^{-1} - 2^{-5\alpha}\Big).$$

(17) Let $\alpha \ge \gamma_6 + 1$, but $\gamma_3 = \gamma_2$ or $\gamma_3 = \gamma_2 + 1$ and $\gamma_5 \ge \gamma_4 + 2$. Then from (2.9), (2.3_1), (2.4_2), (2.5_1), (2.6), (2.7_1), (2.8)–(2.8_2) we get

$$\chi_2 = 1 + (-1)^{(\sum\limits_{k=1}^{4} b_k)/2} \cdot 2^{\sum\limits_{k=2}^{4}\gamma_k}(2^{-3\gamma_4} - 2^{-3\gamma_5+3})7^{-1} + \\ + (-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma}\Big((2^{-5\gamma_6-5} - 2^{-5\alpha})2^5 \cdot 31^{-1} - 2^{-5\alpha}\Big) = \\ = 1 + (-1)^{(\sum\limits_{k=1}^{4} b_k)/2} \cdot 2^{\gamma-4\gamma_5-\gamma_6+3}\big(2^{3(\gamma_5-\gamma_4-1)} - 1\big)7^{-1} + \\ + (-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma}\Big(2^{-5(\alpha-1)}\big(2^{5(\alpha-\gamma_6-1)} - 1\big)31^{-1} - 2^{-5\alpha}\Big).$$

(18) Let $\alpha \ge \gamma_6 + 1$, but $\gamma_3 \ge \gamma_2 + 2$ and $\gamma_5 = \gamma_4$ or $\gamma_5 = \gamma_4 + 1$. Then from (2.9), (2.3_1), (2.4), (2.5_1), (2.6_2), (2.7_1), (2.8)–(2.8_2) we get

$$\chi_2 = 1 + (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1} - 1) +$$

$$+(-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma}\Big(2^{-5(\alpha-1)}\big(2^{5(\alpha-\gamma_6-1)}-1\big)31^{-1}-2^{-5\alpha}\Big).$$

(19) Let $\alpha \geq \gamma_6+1$, but $\gamma_3 \geq \gamma_2+2$ and $\gamma_5 \geq \gamma_4+2$. Then from (2.9), (2.3_1), (2.4), (2.5_1), (2.6), (2.7_1), (2.8)–(2.8_2) we get

$$\chi_2 = 1+(-1)^{(b_1+b_2)/2}\cdot 2^{\gamma_2-\gamma_3+1}(2^{\gamma_3-\gamma_2-1}-1)+$$
$$+(-1)^{(\sum\limits_{k=1}^{4} b_k)/2} \cdot 2^{\gamma-4\gamma_5-\gamma_6+3}\big(2^{3(\gamma_5-\gamma_4-1)}-1\big)7^{-1}+$$
$$+(-1)^{(\sum\limits_{k=1}^{6} b_k)/2} \cdot 2^{\gamma}\Big(2^{-5(\alpha-1)}\big(2^{5(\alpha-\gamma_6-1)}-1\big)31^{-1}-2^{-5\alpha}\Big).$$

From all that was said above the formulas for χ_2 follow. □

Lemma 6. *Let $p>2$, $p^{\beta} \,\|\, n$, $p^{l_k} \,\|\, a_k$ $(k=1,2,\dots,6)$. Let the values of l_k taken in decreasing order be $l_6 \geq l_5 \geq l_4 \geq l_3 \geq l_2 \geq l_1 = 0$, $l=\sum_{k=1}^{6} l_k$; $\eta(l_2)=1$ if $2 \mid l_2$ and $\eta(l_2)=0$ if $2 \nmid l_2$. Then*

$$\chi_p = p^{-1}(p-1)(\beta+1) \quad \textit{for} \quad l_2 \geq \beta+1,\ \ p\equiv 1 \pmod 4,$$
$$= p^{-1}(p+1) \quad \textit{for} \quad l_2 \geq \beta+1,\ \ p\equiv 3 \pmod 4,\ \ 2\mid\beta,$$
$$= 0 \quad \textit{for} \quad l_2 \geq \beta+1,\ \ p\equiv 3 \pmod 4,\ \ 2\nmid\beta;$$
$$\chi_p = p^{-(\beta-l_2+2)}\Big\{p^{\beta-l_2+1}\big((p+1)+(p-1)l_2\big)-(p+1)\Big\}$$
$$\textit{for} \quad l_2 \leq \beta < l_3,\ \ p\equiv 1 \pmod 4,$$
$$= -p^{-(\beta-l_2+2)}(p+1)\Big\{p^{\beta-l_2+1}\eta(l_2)-(-1)^{l_2}\Big\}$$
$$\textit{for} \quad l_2 \leq \beta < l_3,\ \ p\equiv 3 \pmod 4;$$
$$\chi_p = p^{-(2\beta-l_2-l_3+3)}\Big\{p^{2(\beta+1)-l_2-l_3}\big((p+1)+(p-1)l_2\big)+$$
$$+p^2\Big(\big(p^{2(\beta-l_3)}-1\big)(p+1)^{-1}-p^{2(\beta-l_3)}\Big)-1\Big\}$$
$$\textit{for} \quad l_3 \leq \beta < l_4,\ \ p\equiv 1 \pmod 4,$$
$$\chi_p = p^{-(2\beta-l_2-l_3+3)}\Big\{p^{2\beta-l_2-l_3+2}(p+1)\eta(l_2)-(-1)^{l_2}p^2\times$$
$$\times\Big(p^{2(\beta-l_3)}+\big(p^{2(\beta-l_3)}-(-1)^{\beta+l_3}\big)(p-1)(p^2+1)^{-1}\Big)+(-1)^{\beta+l_2+l_3}\Big\}$$
$$\textit{for} \quad l_3 \leq \beta < l_4,\ \ p\equiv 3 \pmod 4;$$

$$\chi_p = p^{-(3\beta-l_2-l_3-l_4+4)}\Big\{p^{3(\beta+1)-l_2-l_3-l_4}\big((p+1)+(p-1)l_2\big) +$$
$$+ p^{3(\beta-l_4+1)}\Big((p^{2(l_4-l_3)}-1)(p+1)^{-1}-p^{2(l_4-l_3)}\Big) +$$
$$+ p^3(p^{3(\beta-l_4)}-1)(p-1)(p^3-1)^{-1}-1\Big\}$$
$$\textit{for}\quad l_4 \le \beta < l_5,\quad p \equiv 1 \pmod 4,$$

$$\chi_p = p^{-(3\beta-l_2-l_3-l_4+4)}\Big\{p^{3\beta-l_2-l_3-l_4+3}(p+1)\eta(l_2) - (-1)^{l_2}p^{3(\beta-l_4+1)} \times$$
$$\times \Big(p^{2(l_4-l_3)} + \big((-1)^{l_3-l_4+1}p^{2(l_4-l_3)}-1\big)(p-1)(p^2+1)^{-1}\Big) +$$
$$+ (-1)^{l_2+l_3+l_4}\Big(p^3(p^{3(\beta-l_4)}-1)(p-1)(p^3-1)^{-1}-1\Big\}$$
$$\textit{for}\quad l_4 \le \beta < l_5,\quad p \equiv 3 \pmod 4;$$

$$\chi_p = p^{-(4\beta-l+l_6+5)}\Big\{p^{4(\beta+1)-l+l_6}\big((p+1)+(p-1)l_2\big) +$$
$$+ p^{4(\beta-l_5+1)}\Big(p^{3(l_5-l_4)}\Big((p^{2(l_4-l_3)}-1)(p+1)^{-1}-p^{3l_5-2l_3-l_4}\Big) +$$
$$+ (p^{3(l_5-l_4)}-1)(p-1)(p^3-1)^{-1}\Big) +$$
$$+ (p^{4(\beta-l_5)}-1)p^4(p-1)(p^4-1)^{-1}-1\Big\}$$
$$\textit{for}\quad l_5 \le \beta < l_6,\quad p \equiv 1 \pmod 4,$$

$$\chi_p = p^{-(4\beta-l+l_6+5)}\Big\{p^{4\beta-l+l_6+4}(p+1)\eta(l_2) - (-1)^{l_2}p^{4(\beta+1)-3l_4-l_5} \times$$
$$\times \Big(p^{2(l_4-l_3)} + \big(p^{2(l_4-l_3)}-(-1)^{l_3+l_4}\big)(p-1)(p^2+1)^{-1}\Big) +$$
$$+ (-1)^{l_2+l_3+l_4}p^{4(\beta+1-l_5)}(p^{3(l_5-l_4)}-1)(p-1)(p^3-1)^{-1} +$$
$$+ (-1)^{l-l_6}\Big(\big((-1)^{l_5+1}p^{4(\beta-l_5)}+(-1)^{\beta}\big)(p-1)p^4(p^4+1)^{-1}+(-1)^{\beta}\Big)\Big\}$$
$$\textit{for}\quad l_5 \le \beta < l_6,\quad p \equiv 3 \pmod 4;$$

$$\chi_p = p^{-(5\beta-l+6)}\Big\{p^{5(\beta+1)-l}\big((p+1)+(p-1)l_2\big) +$$
$$+ p^{5(\beta+1)-4l_5-l_6}\Big(p^{3(l_5-l_4)}\Big((p^{2(l_4-l_3)}-1)(p+1)^{-1}-p^{2(l_4-l_3)}\Big) +$$
$$+ (p^{3(l_5-l_4)}-1)(p-1)(p^3-1)^{-1}\Big) +$$
$$+ p^5(p-1)\Big(p^{5(\beta-l_6)}(p^{4(l_6-l_5)}-1)(p^4+1)^{-1} +$$

$$+\,(p^{5(\beta-l_6)}-1)(p^5-1)^{-1}\Big)-1\Big\} \quad \textit{for} \quad \beta\ge l_6,\ \ p\equiv 1 \pmod 4,$$

$$\chi_p=p^{-(5\beta-l+6)}\Big\{p^{5\beta-l+5}(p+1)\eta(l_2)-(-1)^{l_2}p^{5(\beta+1)-l+l_2+l_3-2l_4}\times$$
$$\times\Big(p^{2(l_4-l_3)}+\big(p^{2(l_4-l_3)}-(-1)^{l_3-l_4}\big)(p-1)(p^2+1)^{-1}\Big)+$$
$$+\,(-1)^{l_2+l_3+l_4}p^{5(\beta+1)-4l_5-l_6}(p^{3(l_5-l_4)}-1)(p-1)(p^3-1)^{-1}+$$
$$-\,(-1)^{l}p^5(p-1)\Big(p^{5(\beta-l_6)}\big((-1)^{l_5-l_6}p^{4(l_6-l_5)}-1\big)(p^4+1)^{-1}+$$
$$-\,(p^{5(\beta-l_6)}-1)(p^5-1)^{-1}\Big)-(-1)^{l}\Big\} \quad \textit{for} \quad \beta\ge l_6,\ \ p\equiv 3 \pmod 4).$$

Proof. I. In (1.4) put $2\nmid q$, $q=(q,a_k)q_k$ $(k=1,2,\dots,6)$. Then from Lemmas 1 and 2 we get

$$A(q)=q^{-12}\sideset{}{'}\sum_{h \bmod q}e\Big(-\frac{hn}{q}\Big)\prod_{k=1}^{6}(q,a_k)^2S^2\Big(\frac{a_k}{(q,a_k)}\,h,q_k\Big)=$$
$$=q^{-6}\sideset{}{'}\sum_{h \bmod q}e\Big(-\frac{hn}{q}\Big)\prod_{k=1}^{6}(q,a_k)\Big(\frac{-1}{q_k}\Big).$$

Putting $q=p^\lambda$ and taking into account that $(a_1,\dots,a_6)=1$, it follows that

$$A(p^\lambda)=p^{-6\lambda}\sideset{}{'}\sum_{h \bmod p^\lambda}e\Big(-\frac{hn}{p^\lambda}\Big)p^{\sum\limits_{k=2}^{6}\min(\lambda,l_k)}\Big(\frac{-1}{p}\Big)^{\sum\limits_{k=2}^{6}\min(\lambda,l_k)}=$$
$$=\Big(\frac{-1}{p}\Big)^{\sum\limits_{k=2}^{6}\min(\lambda,l_k)}p^{\sum\limits_{k=2}^{6}\min(\lambda,l_k)}\cdot p^{-6\lambda}c(n,p^\lambda). \tag{2.10}$$

From (2.10) amd Lemma 3 it follows that
(1) for $\lambda\le l_2$

$$A(p^\lambda)=\Big(\frac{-1}{p}\Big)^{\lambda}(1-p^{-1}) \quad \text{if} \quad \lambda<\beta+1, \tag{2.11}$$
$$=-\Big(\frac{-1}{p}\Big)^{\beta+1}p^{-1} \quad \text{if} \quad \lambda=\beta+1; \tag{2.11$_1$}$$

(2) for $l_2<\lambda\le l_3$

$$A(p^\lambda)=\Big(\frac{-1}{p}\Big)^{l_2}p^{l_2-\lambda}(1-p^{-1}) \quad \text{if} \quad \lambda<\beta+1, \tag{2.12}$$
$$=-\Big(\frac{-1}{p}\Big)^{l_2}p^{l_2-\beta-2} \quad \text{if} \quad \lambda=\beta+1; \tag{2.12$_1$}$$

(3) for $l_3 < \lambda \le l_4$

$$A(p^\lambda) = \Big(\frac{-1}{p}\Big)^{l_2+l_3+\lambda} p^{l_2+l_3-2\lambda}(1-p^{-1}) \quad \text{if} \quad \lambda < \beta+1, \tag{2.13}$$

$$= -\Big(\frac{-1}{p}\Big)^{l_2+l_3+\beta+1} p^{l_2+l_3-2\beta-3} \quad \text{if} \quad \lambda = \beta+1; \tag{2.13$_1$}$$

(4) for $l_4 < \lambda \le l_5$

$$A(p^\lambda) = \Big(\frac{-1}{p}\Big)^{l_2+l_3+l_4} p^{l_2+l_3+l_4-3\lambda}(1-p^{-1}) \ \text{if} \ \lambda < \beta+1, \tag{2.14}$$

$$= -\Big(\frac{-1}{p}\Big)^{l_2+l_3+l_4} p^{l_2+l_3+l_4-3\beta-4} \quad \text{if} \quad \lambda = \beta+1; \tag{2.14$_1$}$$

(5) for $l_5 < \lambda \le l_6$

$$A(p^\lambda) = \Big(\frac{-1}{p}\Big)^{l-l_6+\lambda} p^{l-l_6-4\lambda}(1-p^{-1}) \quad \text{if} \quad \lambda < \beta+1, \tag{2.15}$$

$$= -\Big(\frac{-1}{p}\Big)^{l-l_6+\beta+1} p^{l-l_6-4\beta-5} \quad \text{if} \quad \lambda = \beta+1; \tag{2.15$_1$}$$

(6) for $\lambda \ge l_6$

$$A(p^\lambda) = \Big(\frac{-1}{p}\Big)^{l} p^{l-5\lambda}(1-p^{-1}) \quad \text{if} \quad \lambda < \beta+1, \tag{2.16}$$

$$= -\Big(\frac{-1}{p}\Big)^{l} p^{l-5\beta-6} \quad \text{if} \quad \lambda = \beta+1. \tag{2.16$_1$}$$

In all the above-mentioned cases

$$A(p^\lambda) = 0 \quad \text{if} \quad \lambda > \beta+1. \tag{2.17}$$

II. According to (1.5) we have

$$\chi_p = 1 + \sum_{\lambda=1}^{l_2} A(p^\lambda) + \sum_{\lambda=l_2+1}^{l_3} A(p^\lambda) + \sum_{\lambda=l_3+1}^{l_4} A(p^\lambda) + \\ + \sum_{\lambda=l_4+1}^{l_5} A(p^\lambda) + \sum_{\lambda=l_5+1}^{l_6} A(p^\lambda) + \sum_{\lambda=l_6+1}^{\infty} A(p^\lambda). \tag{2.18}$$

Also it is obvious that

$$1 + (1-p^{-1})\sum_{\lambda=1}^{l_2}(-1)^\lambda = \begin{cases} 1 & \text{if} \quad 2 \mid l_2, \\ p^{-1} & \text{if} \quad 2 \nmid l_2. \end{cases} \tag{2.19}$$

Consider the following cases:

(1) Let $l_2 \geq \beta + 1$. Then from (2.18), (2.11), (2.11_1), and (2.17) we get

$$\chi_p = 1 + (1 - p^{-1}) \sum_{\lambda=1}^{\beta} \Big(\frac{-1}{p}\Big)^{\lambda} - \Big(\frac{-1}{p}\Big)^{\beta+1} p^{-1}.$$

(2) Let $l_2 \leq \beta < l_3$. Then from (2.18), (2.11), (2.12), (2.12_1), and (2.17) we get

$$\chi_p = 1 + (1 - p^{-1}) \sum_{\lambda=1}^{l_2} \Big(\frac{-1}{p}\Big)^{\lambda} +$$
$$+ (1 - p^{-1}) \Big(\frac{-1}{p}\Big)^{l_2} \sum_{\lambda=l_2+1}^{\beta} p^{l_2-\lambda} - \Big(\frac{-1}{p}\Big)^{l_2} p^{l_2-\beta-2}.$$

(3) Let $l_3 \leq \beta < l_4$. Then from (2.18), (2.11), (2.12), (2.13), (2.13_1), and (2.17) we get

$$\chi_p = 1 + (1 - p^{-1}) \sum_{\lambda=1}^{l_2} \Big(\frac{-1}{p}\Big)^{\lambda} + (1 - p^{-1}) \Big(\frac{-1}{p}\Big)^{l_2} \sum_{\lambda=l_2+1}^{l_3} l^{l_2-\lambda} +$$
$$+ (1 - p^{-1}) \Big(\frac{-1}{p}\Big)^{l_2} \sum_{\lambda=l_3+1}^{\beta} \Big(\frac{-1}{p}\Big)^{\lambda+l_3} p^{l_2+l_3-2\lambda} +$$
$$- \Big(\frac{-1}{p}\Big)^{l_2+l_3+\beta+1} p^{l_2+l_3-2\beta-3}.$$

(4) Let $l_4 \leq \beta < L_5$. Then from (2.18), (2.11), (2.12), (2.13), (2.14), (2.14_1), and (2.17) we get

$$\chi_p = 1 + (1 - p^{-1}) \sum_{\lambda=1}^{l_2} \Big(\frac{-1}{p}\Big)^{\lambda} + (1 - p^{-1}) \Big(\frac{-1}{p}\Big)^{l_2} \sum_{\lambda=l_2+1}^{l_3} p^{l_2-\lambda} +$$
$$+ (1 - p^{-1}) \Big(\frac{-1}{p}\Big)^{l_2+l_3} \sum_{\lambda=l_3+1}^{l_4} \Big(\frac{-1}{p}\Big)^{\lambda} p^{l_2+l_3-2\lambda} + (1 - p^{-1}) \times$$
$$\times \Big(\frac{-1}{p}\Big)^{l_2+l_3+l_4} \sum_{\lambda=l_4+1}^{\beta} p^{l_2+l_3+l_4-3\lambda} - \Big(\frac{-1}{p}\Big)^{l_2+l_3+l_4} p^{l_2+l_3+l_4-3\beta-4}.$$

(5) Let $l_5 \leq \beta < l_6$. Then from (2.18), (2.11)–(2.15), (2.15_1), and (2.17) we get

$$\chi_p = 1 + (1 - p^{-1}) \sum_{\lambda=1}^{l_2} \Big(\frac{-1}{p}\Big)^{\lambda} + (1 - p^{-1}) \Big(\frac{-1}{p}\Big)^{l_2} \sum_{\lambda=l_2+1}^{l_3} p^{l_2-\lambda} +$$

$$+ (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l_2+l_3} \sum_{\lambda=l_3+1}^{l_4} \Big(\frac{-1}{p}\Big)^{\lambda} p^{l_2+l_3-2\lambda} +$$

$$+ (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l_2+l_3+l_4} \sum_{\lambda=l_4+1}^{l_5} p^{l_2+l_3+l_4-3\lambda} + (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l-l_6} \times$$

$$\times \sum_{\lambda=l_5+1}^{\beta} \Big(\frac{-1}{p}\Big)^{\lambda} p^{l-l_6-4\lambda} - \Big(\frac{-1}{p}\Big)^{l-l_6+\beta+1} p^{l-l_6-4\beta-5}.$$

(6) Let $\beta \geq l_6$. Then from (2.18), (2.11)–(2.16), (2.16_1), and (2.17) we get

$$\chi_p = 1 + (1-p^{-1}) \sum_{\lambda=1}^{l_2} \Big(\frac{-1}{p}\Big)^{\lambda} + (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l_2} \sum_{\lambda=l_2+1}^{l_3} p^{l_2-\lambda} +$$

$$+ (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l_2+l_3} \sum_{\lambda=l_3+1}^{l_4} \Big(\frac{-1}{p}\Big)^{\lambda} p^{l_2+l_3-2\lambda} +$$

$$+ (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l_2+l_3+l_4} \sum_{\lambda=l_4+1}^{l_5} p^{l_2+l_3+l_4-3\lambda} +$$

$$+ (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l-l_6} \sum_{\lambda=l_5+1}^{l_6} \Big(\frac{-1}{p}\Big)^{\lambda} p^{l-l_6-4\lambda} +$$

$$+ (1-p^{-1})\Big(\frac{-1}{p}\Big)^{l} \sum_{\lambda=l_6+1}^{\beta} p^{l-5\lambda} - \Big(\frac{-1}{p}\Big)^{l} p^{l-5\beta-6}. \tag{2.20}$$

From the above expressions for χ_p one can get the corresponding formulas stated in the lemma in the cases where $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$. □

Theorem. *Let* $n = 2^{\alpha} m = 2^{\alpha} uv$, $u = \prod_{p|n, p\nmid 2\Delta} p^{\beta}$, $v = \prod_{\substack{p|n \\ p|\Delta, p>2}} p^{\beta}$. *Then*

$$\rho(n; f) = \frac{2^{5\alpha+3} v^5}{\Delta^{1/2}} \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p|\Delta \\ p>2}} (1-p^{-6})^{-1} \sigma_5(u).$$

Proof. Let $p > 2$, $p^{\beta} \,||\, n$, $p \nmid \Delta$ (i.e., $l = 0$). Then in (2.20), putting $l = 0$, we get

$$\chi_p = 1 + (1-p^{-1}) \sum_{\lambda=1}^{\beta} p^{-5\lambda} - p^{-5\beta-6} =$$

$$= 1 + \sum_{\lambda=1}^{\beta} p^{-5\lambda} - p^{-6} \sum_{\lambda=1}^{\beta} p^{-5(\lambda-1)} - p^{-5\beta-6} =$$

$$= \sum_{\lambda=0}^{\beta} p^{-5\lambda} - p^{-6} \sum_{\lambda=0}^{\beta} p^{-5\lambda} = (1-p^{-6}) \sum_{d|p^\beta} d^{-5}. \tag{2.21}$$

For $p \nmid \Delta n$, i.e., for $\beta = 0$, from (2.21) we get

$$\chi_p = 1 - p^{-6}. \tag{2.22}$$

Thus from Lemma 4, (2.21), and (2.22) it follows that

$$\sum_{q=1}^{\infty} A(q) = \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p\nmid\Delta \\ p>2}} \chi_p = \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p|n \\ p\nmid 2\Delta}} \chi_p \prod_{\substack{p\nmid\Delta n \\ p>2}} \chi_p =$$

$$= \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p|n \\ p\nmid 2\Delta}} \Big((1-p^{-6}) \sum_{d|p^\beta} d^{-5}\Big) \prod_{\substack{p\nmid\Delta n \\ p>2}} (1-p^{-6}) =$$

$$= \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{p>2} (1-p^{-6}) \prod_{\substack{p|\Delta \\ p>2}} (1-p^{-6})^{-1} \sum_{d|u} d^{-5} =$$

$$= \frac{2^6 \cdot 3 \cdot 5}{\pi^6} \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p|\Delta \\ p>2}} (1-p^{-6})^{-1} \frac{1}{u^5} \sigma_5(u), \tag{2.23}$$

as it is well known that

$$\prod_{p>2} (1-p^{-6}) = (1-2^{-6})^{-1} \zeta^{-1}(6) = \frac{2^6 \cdot 3 \cdot 5}{\pi^6}.$$

Thus the theorem follows from (1.3) and (2.23). □

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