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ON PERIODIC SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

I. KIGURADZE AND B. PŮŽA

Abstract. Sufficient conditions are established for the existence and uniqueness of an ω-periodic solution of the functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t),$$

where f is a continuous operator acting from the space of n-dimensional ω-periodic continuous vector functions into the space of n-dimensional ω-periodic and summable on $[0,\omega]$ vector functions.

1. Statement of the Problem and Basic Notation

Let n be a natural number, $\omega > 0$, and $f : C_\omega(R^n) \to L_\omega(R^n)$ be a continuous operator. Consider the vector functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t). \tag{1.1}$$

A vector function $x : R \to R^n$ is called *an ω-periodic solution* of equation (1.1) if it is absolutely continuous, satisfies (1.1) almost everywhere on R and

$$x(t+\omega) = x(t) \quad \text{for} \quad t \in R.$$

In the second section of this paper, using the principle of a priori boundedness we establish new sufficient conditions for the existence and uniqueness of an ω-periodic solution of equation (1.1). In the third section we give corollaries of the main results for the vector differential equation with deviating arguments

$$\frac{dx(t)}{dt} = f_0\big(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))\big), \tag{1.2}$$

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where $f_0 : R \times R^{(m+1)n} \to R^n$ satisfies the local Carathéodory conditions and is ω-periodic in the first argument, i.e., satisfies the equality

$$f_0(t+\omega, x_0, x_1, \dots, x_m) = f_0(t, x_0, x_1, \dots, x_m) \tag{1.3}$$

for almost all $t \in R$ and for all $x_k \in R^n$ $(k = 0, 1, \dots, m)$. As for $\tau_k : R \to R$ $(k = 1, \dots, m)$, they are measurable and such that

$$(\tau_k(t+\omega) - \tau_k(t))/\omega \quad (k = 1, \dots, m) \quad \text{are integer numbers.} \tag{1.4}$$

The above-mentioned propositions strengthen the earlier results on periodic solutions of systems of ordinary differential equations and functional differential equations of types (1.1) and (1.2) (see [1–23] and the references cited therein).

Throughout this paper, use will be made of the following notation:

R^n is the space of all n-dimensional column vectors $x = (x_i)_{i=1}^n$ with the elements $x_i \in R$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|.$$

$R^{n\times n}$ is the space of all $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with the elements $x_{ik} \in R$ $(i, k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|.$$

$$R_+^n = \big\{(x_i)_{i=1}^n \in R^n : \;\; x_i \ge 0 \;\; (i = 1, \dots, n)\big\}.$$
$$R_+^{n\times n} = \big\{(x_{ik})_{i,k=1}^n \in R^{n\times n} : \;\; x_{ik} \ge 0 \;\; (i, k = 1, \dots, n)\big\}.$$

If $x, y \in R^n$ and $X, Y \in R^{n\times n}$, then

$$x \le y \iff y - x \in R_+^n, \quad X \le Y \iff Y - X \in R_+^{n\times n}.$$

$x \cdot y$ is the scalar product of the vectors x and $y \in R^n$.

If $x = (x_i)_{i=1}^n \in R^n$ and $X = (x_{ik})_{i,k=1}^n \in R^{n\times n}$, then

$$|x| = (|x_i|)_{i=1}^n, \quad |X| = (|x_{ik}|)_{i,k=1}^n,$$
$$\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^n.$$

$\det(X)$ is the determinant of the matrix X.

X^{-1} is the matrix inverse to X.

$r(X)$ is the spectral radius of the matrix X. E is the unit matrix.

$C([0,\omega]; R^n)$ is the space of all continuous vector functions $x : [0,\omega] \to R^n$ with the norm

$$\|x\|_C = \max\big\{\|x(t)\| : \;\; 0 \le t \le \omega\big\}.$$

$C_\omega(R^n)$ with $\omega > 0$ is the space of all continuous ω-periodic vector functions $x : R \to R^n$ with the norm

$$\|x\|_{C_\omega} = \max\big\{\|x(t)\| : \quad 0 \le t \le \omega\big\};$$

if $x = (x_i)_{i=1}^n \in C_\omega(R^n)$, then

$$|x|_{C_\omega} = (\|x_i\|_{C_\omega})_{i=1}^n.$$

$L([0,\omega]; R^n)$ is the space of all vector functions $x : R \to R^n$ with summable on $[0,\omega]$ elements and with the norm

$$\|x\|_L = \int_0^\omega \|x(t)\|\, dt.$$

$L_\omega(R^n)$ is the space of all ω-periodic vector functions $x : R \to R^n$ with summable on $[0,\omega]$ elements and with the norm

$$\|x\|_{L_\omega} = \int_0^\omega \|x(t)\|\, dt.$$

$$L_\omega(R_+) = \big\{x \in L_\omega(R) : \quad x(t) \ge 0 \ \text{ for } \ t \in R\big\};$$
$$L_\omega(R_-) = \big\{x \in L_\omega(R) : \quad x(t) \le 0 \ \text{ for } \ t \in R\big\}.$$

$L_\omega(R^{n\times n})$ is the space of all matrix functions $X : R \to R^{n\times n}$ with elements from $L_\omega(R)$.

If $Z : R \to R^{n\times n}$ is an ω-periodic continuous matrix function with columns $z_1, \dots, z_n$, and $g : C_\omega(R^n) \to L_\omega(R^n)$ is a linear operator, then by $g(Z)$ we understand the matrix function with columns $g(z_1), \dots, g(z_n)$.

2. Periodic Solutions of Equation (1.1)

Throughout this section, $f : C_\omega(R^n) \to L_\omega(R^n)$ is assumed to be a continuous operator such that

$$f^*(\cdot, \rho) \in L_\omega(R_+) \quad \text{for} \quad \rho \in]0, +\infty[\,,$$

where

$$f^*(t, \rho) = \sup\big\{\|f(x)(t)\| : \quad x \in C_\omega(R^n), \ \ \|x\|_{C_\omega} \le \rho\big\}.$$

We introduce

Definition 2.1. Let β be a positive number. We say that an operator $p : C_\omega(R^n)\times C_\omega(R^n) \to L_\omega(R^n)$ belongs to the class $V^n_\omega(\beta)$ if it is continuous and satisfies the following three conditions:

(i) $p(x,\cdot) : C_\omega(R^n) \to L_\omega(R^n)$ is a linear operator for any arbitrarily fixed $x \in C_\omega(R^n)$;

(ii) there exists a nondecreasing in the second argument function $\alpha : R \times R_+ \to R_+$ such that $\alpha(\cdot,\rho) \in L_\omega(R)$ for $\rho \in]0,+\infty[$, and for any $x, y \in C_\omega(R)$ and for almost all $t \in R$ the inequality

$$\|p(x,y)(t)\| \le \alpha(t, \|x\|_{C_\omega})\|y\|_{C_\omega}$$

holds;

(iii) for any $x \in C_\omega(R^n)$ and $q \in L_\omega(R^n)$, an arbitrary ω-periodic solution y of the differential equation

$$\frac{dy(t)}{dt} = p(x,y)(t) + q(t) \tag{2.1}$$

admits the estimate

$$\|y\|_{C_\omega} \le \beta\|q\|_{L_\omega}. \tag{2.2}$$

Definition 2.2. We say that an operator $p : C_\omega(R^n) \times C_\omega(R^n) \to L_\omega(R^n)$ belongs to the set V^n_ω if there exists $\beta > 0$ such that $p \in V^n_\omega(\beta)$.

Theorem 2.1. *Let there exist a positive number ρ_0 and an operator $p \in V^n_\omega$ such that for any $\lambda \in]0,1[$ an arbitrary ω-periodic solution of the differential equation*

$$\frac{dx(t)}{dt} = (1-\lambda)p(x,x)(t) + \lambda f(x)(t) \tag{2.3}$$

admits the estimate

$$\|x\|_{C_\omega} \le \rho_0. \tag{2.4}$$

Then equation (1.1) *has at least one ω-periodic solution.*

Proof. For arbitrary $x \in C([0,\omega];R^n)$, we denote by $v_\omega(x)$ the vector function defined by the equality

$$v_\omega(x)(t) = x(t-j\omega) + \frac{t-j\omega}{\omega}\,[x(0)-x(\omega)] \quad \text{for} \quad j\omega \le t < (j+1)\omega \tag{2.5}$$
$$(j = 0, 1, -1, 2, -2, \dots),$$

and for any x and $y \in C([0,\omega];R^n)$ we set

$$\widetilde{p}(x,y)(t) = p(v_\omega(x), v_\omega(y))(t), \quad l(x,y) = y(\omega) - y(0), \tag{2.6}$$
$$\widetilde{f}(x)(t) = f(v_\omega(x))(t).$$

Obviously, $v_\omega : C([0,\omega]; R^n) \to C_\omega(R^n)$ is a linear bounded operator, while $\widetilde{f} : C([0,\omega]; R^n) \to L([0,\omega]; R^n)$ and $\widetilde{p} : C([0,\omega]; R^n) \times C([0,\omega]; R^n) \to L([0,\omega]; R^n)$ are continuous operators. Moreover, the restrictions on $[0,\omega]$ of ω-periodic solutions of equations (1.1) and (2.3) are respectively solutions of the differential equations

$$\frac{dx(t)}{dt} = \widetilde{f}(x)(t) \tag{2.7}$$

and

$$\frac{dx(t)}{dt} = (1-\lambda)\widetilde{p}(x,x)(t) + \lambda \widetilde{f}(x)(t) \tag{2.8}$$

with the periodic boundary condition

$$x(\omega) = x(0), \tag{2.9}$$

and vice versa the periodic extension of an arbitrary solution of problem (2.7), (2.9) (problem (2.8), (2.9)) is an ω-periodic solution of equation (1.1) (equation (2.3)). Consequently, for any $\lambda \in]0,1[$, an arbitrary solution of problem (2.8), (2.9) admits estimate (2.4).

On the other hand, it follows from the condition $p \in V_\omega^n$ and equalities (2.6) that the pair of operators $(\widetilde{p}, l)$ is compatible in the sense of Definition 1 from [14].

Thus we have shown that for problem (2.7), (2.9), all the conditions of Theorem 1 from [14] are fulfilled, which guarantees the solvability of this problem. However, according to the above-said, the existence of an ω-periodic solution of equation (1.1) follows from the solvability of problem (2.7), (2.9). $\square$

Corollary 2.1. *Let there exist $\beta > 0$ and $p \in V_\omega^n(\beta)$ such that for any $x \in C_\omega(R^n)$ almost everywhere on R the inequality*

$$\|f(x)(t) - p(x,x)(t)\| \le \gamma(t, \|x\|_{C_\omega}) \tag{2.10}$$

holds, where $\gamma(\cdot,\rho) \in L_\omega(R_+)$ for $0 < \rho < +\infty$, and

$$\limsup_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \gamma(t,\rho)\, dt < \frac{1}{\beta}\,. \tag{2.11}$$

Then equation (1.1) *has at least one ω-periodic solution.*

Proof. By (2.11) there exists $\rho_0 > 0$ such that

$$\beta \int_0^\omega \gamma(t,\rho)\, dt < 1 \quad \text{for} \quad \rho \ge \rho_0. \tag{2.12}$$

Let x be an ω-periodic solution of (2.3) for some $\lambda \in]0,1[$. Assume

$$\delta(t) = f(x)(t) - p(x,x)(t).$$

Then

$$\frac{dx(t)}{dt} = p(x,x)(t) + \lambda\delta(t)$$

and, as follows from (2.10), the vector function δ satisfies almost everywhere on R the inequality

$$\|\delta(t)\| \le \gamma(t, \|x\|_{C_\omega}),$$

whence, owing to $p \in V_\omega^n(\beta)$, we have

$$\|x\|_{C_\omega} \le \beta \int_0^\omega \|\delta(t)\|\,dt \le \beta \int_0^\omega \gamma(t, \|x\|_{C_\omega})\,dt.$$

From this inequality, by virtue of (2.12), follows estimate (2.4).

If now we take into account Theorem 2.1 the validity of the corollary will become obvious. □

Corollary 2.2. *Let for any $x \in C_\omega(R^n)$, inequality* (2.10) *be fulfilled almost everywhere on R, where $\gamma(\cdot,\rho) \in L_\omega(R_+)$ for $0 < \rho < +\infty$, and $p : C_\omega(R^n) \times C_\omega(R^n) \to L_\omega(R^n)$ is a continuous operator such that $p(x,\cdot) : C_\omega(R^n) \to L_\omega(R^n)$ is linear and $\int_0^\omega p(x,E)(s)\,ds$ is a nonsingular matrix for an arbitrarily fixed $x \in C_\omega(R^n)$. Let, moreover, there exist matrices A and $B \in R_+^{n\times n}$ such that*

$$r(A + BA^2) < 1, \tag{2.13}$$

$$\int_0^\omega |p(x,y)(s)|\,ds \le A|y|_{C_\omega}, \quad \left|\Big[\int_0^\omega p(x,E)(s)\,ds\Big]^{-1}\right| \le B \tag{2.14}$$

for any x and $y \in C_\omega(R^n)$ and the function γ satisfies condition (2.11), *where $\beta = \|(E - A - BA^2)^{-1}(E + BA)\|$. Then equation* (1.1) *has at least one ω-periodic solution.*

Proof. By virtue of Corollary 2.1, to prove Corollary 2.2 it suffices to establish that for any $x \in C_\omega(R^n)$ and $q \in L_\omega(R^n)$ an arbitrary ω-periodic solution y of equation (2.1) admits estimate (2.2).

By (2.5), from (2.1) we have

$$y(t) = y(0) + p^1(x,y)(t) + q_0(t), \tag{2.15}$$

where

$$p^1(x,y)(t)=\int_0^t p(x,v_\omega(y))(s)\,ds,\quad q_0(t)=\int_0^t q(s)\,ds. \tag{2.16}$$

Therefore

$$y(t)=y(0)+p^1(x,E)(t)y(0)+p^1(x,p^1(x,y))(t)+p^1(x,q_0)(t),$$

whence because of the ω-periodicity of y and the nonsingularity of the matrix

$$p^1(x,E)(\omega)=\int_0^\omega p(x,E)(s)\,ds$$

we obtain

$$y(0)=-\Big[\int_0^\omega p(x,E)(s)\,ds\Big]^{-1}\big[p^1(x,p^1(x,y))(\omega)+p^1(x,q_0)(\omega)\big].$$

According to (2.14) and (2.16), the latter representation results in

$$\begin{aligned}|y(0)|&\le B\big(A|v_\omega(p^1(x,y))|_{C_\omega}+A|v_\omega(q_0)|_{C_\omega}\big)\le\\ &\le B\big(A|p^1(x,y)|_{C_\omega}+A|q_0|_{C_\omega}\big)\le BA^2|y|_{C_\omega}+BA|q|_{L_\omega}.\end{aligned}$$

Taking this estimate into account, from (2.15) we find that

$$|y|_{C_\omega}\le BA^2|y|_{C_\omega}+BA|q|_{L_\omega}+A|y|_{C_\omega}+|q|_{L_\omega}$$

and

$$(E-A-BA^2)|y|_{C_\omega}\le(E+BA)|q|_{L_\omega}.$$

Hence by (2.13) we have

$$|y|_{C_\omega}\le(E-A-BA^2)^{-1}(E+BA).$$

Thus estimate (2.2) is valid. $\square$

Corollary 2.2 deals with the case where

$$\sup\left\{\frac{1}{1+\|x\|_{C_\omega}}\int_0^\omega\|f(x)(t)\|\,dt:\ \ x\in C_\omega(R^n)\right\}<+\infty,$$

whereas Corollary 2.1 covers the class of equations of type (1.1) for which the last condition is violated. As an example, consider the integro-differential equation

$$\frac{dx(t)}{dt}=p_1(t,x(t))\int_0^\omega p_2(s,x(s))x(s)\,d\varphi(s)+p_0(t,x(t)), \tag{2.17}$$

where the functions $p_i : R\times R \to R$ $(i = 0, 1)$ satisfy the local Carathéodory conditions and are ω-periodic in the first argument, $p_2 : [0, \omega] \times R \to R$ is continuous and $\varphi : [0, \omega] \to R$ is nondecreasing.

Corollary 2.3. *Let on* $[0, \omega] \times R$ *the inequalities*

$$\sigma_i p_i(t, x) \ge \delta_i(t) \quad (i = 1, 2) \tag{2.18}$$

be fulfilled, where $\sigma_i \in \{-1, 1\}$ $(i = 1, 2)$*,* $\delta_1 : [0, \omega] \to R_+$ *is a summable function and* $\delta_2 : [0, \omega] \to R_+$ *is a continuous function such that*

$$\delta = \Big(\int_0^\omega \delta_1(s)\, ds\Big)\Big(\int_0^\omega \delta_2(s)\, d\varphi(s)\Big) > 0. \tag{2.19}$$

Let further

$$\limsup_{\rho\to+\infty} \Big(\frac{1}{\rho} \int_0^\omega \gamma(t, \rho)\, dt\Big) < \frac{\delta}{1 + 3\delta}\,,$$

where

$$\gamma(t, \rho) = \max\big\{|p_0(t, x)| : \ \ |x| \le \rho\big\}.$$

Then equation (2.17) *has at least one* ω*-periodic solution.*

Proof. Suppose

$$p(x, y)(t) = p_1(t, x(t)) \int_0^\omega p_2(s, x(s))y(s)\, d\varphi(s).$$

By virtue of Corollary 2.1, to prove Corollary 2.3 it suffices to establish that $p \in V^1_\omega(\frac{1+3\delta}{\delta})$.

Let $x \in C_\omega(R)$, $q \in L_\omega(R)$ and let y be an arbitrary ω-periodic solution of the equation

$$\frac{dy(t)}{dt} = p_1(t, x(t)) \int_0^\omega p_2(s, x(s))y(s)\, d\varphi(s) + q(t).$$

Suppose

$$\int_0^\omega p_2(s, x(s))y(s)\, d\varphi(s) = c. \tag{2.20}$$

Then

$$y(t) = y(0) + c \int_0^t p_1(s, x(s))\, ds + \int_0^t q(s)\, ds. \tag{2.21}$$

Therefore

$$c\int_0^\omega p_1(s,x(s))\,ds+\int_0^\omega q(s)\,ds=0$$

and consequently

$$c=-\Big(\int_0^\omega p_1(s,x(s))\,ds\Big)^{-1}\Big(\int_0^\omega q(s)\,ds\Big). \tag{2.22}$$

If we substitute (2.21) in (2.20) and calculate $y(0)$, then we obtain

$$\begin{aligned}&y(0)=\\
&=c\Big(\int_0^\omega p_2(s,x(s))\,d\varphi(s)\Big)^{-1}\Big[1-\int_0^\omega p_2(s,x(s))\Big(\int_0^s p_1(\xi,x(\xi))\,d\xi\Big)\,d\varphi(s)\Big]-\\
&\quad-\Big(\int_0^\omega p_2(s,x(s))\,d\varphi(s)\Big)^{-1}\int_0^\omega p_2(s,x(s))\Big(\int_0^s q(\xi)\,d\xi\Big)\,d\varphi(s).\end{aligned} \tag{2.23}$$

Introduce the function

$$\eta(t)=\Big(\int_0^\omega p_1(s,x(s))\,ds\Big)^{-1}\int_0^t p_1(s,x(s))\,ds. \tag{2.24}$$

Then from (2.21) and (2.22) we get

$$y(t)=y(0)+\int_0^t(1-\eta(t))q(s)\,ds-\eta(t)\int_t^\omega q(s)\,ds.$$

On the other hand, taking into account (2.18) and (2.19), from (2.22)–(2.24) we find

$$|y(0)|\le\Big(\frac{1}{\delta}+2\Big)\|q\|_{L_\omega},\quad 0\le\eta(t)\le1\quad\text{for}\quad 0\le t\le\omega.$$

Therefore

$$\|y\|_{L_\omega}\le|y(0)|+\|q\|_{L_\omega}\le\frac{1+3\delta}{\delta}\,\|q\|_{L_\omega},$$

which because of the arbitrariness of $x\in C_\omega(R)$ and $q\in L_\omega(R)$ results in $p\in V_\omega^1(\frac{1+3\delta}{\delta})$. $\square$

Theorem 2.2. *Let for any $x \in C_\omega(R^n)$ almost everywhere on R the inequality*

$$f(x)(t) \cdot \operatorname{sgn}(\sigma x(t)) \leq p_0(t)\|x(t)\| + \gamma(t, \|x\|_{C_\omega}) \tag{2.25}$$

be fulfilled, where $\sigma \in \{-1, 1\}$, $p_0 \in L_\omega(R)$, $\gamma(\cdot, \rho) \in L_\omega(R_+)$ for $0 < \rho < +\infty$,

$$\int_0^\omega p_0(s)\, ds < 0 \tag{2.26}$$

and

$$\limsup_{\rho \to +\infty} \frac{1}{\rho}\bigg(\int_t^{t+\omega} \exp\Big(\sigma \int_s^t p_0(\xi)\, d\xi\Big) \gamma(s, \rho)\, ds\bigg) <$$

$$< \bigg|\exp\Big(-\sigma \int_0^\omega p_0(\xi)\, d\xi\Big) - 1\bigg| \ \text{uniformly with respect to} \ t \in [0, \omega]. \tag{2.27}$$

Then equation (1.1) *has at least one ω-periodic solution.*

To prove this theorem, it is necessary to establish an a priori estimate of nonnegative ω-periodic solutions of the differential inequality

$$\sigma u'(t) \leq p_0(t) u(t) + \gamma(t, \|u\|_{C_\omega}), \tag{2.28}$$

where $\sigma \in \{-1, 1\}$, $p_0 \in L_\omega(R)$, and $\gamma(\cdot, \rho) \in L_\omega(R_+)$ for $0 < \rho < +\infty$.

Note that by an ω-periodic solution of inequality (2.28) we mean an absolutely continuous ω-periodic function $u : R \to R$ satisfying this inequality almost everywhere on R.

Lemma 2.1. *Let inequality* (2.26) *be fulfilled, and let there exist a nonnegative constant ρ_0 such that*

$$\int_t^{t+\omega} \exp\Big(\sigma \int_s^t p_0(\xi)\, d\xi\Big) \gamma(s, \rho)\, ds <$$

$$< \bigg|\exp\Big(- \sigma \int_0^\omega p_0(s)\, ds\Big) - 1\bigg| \rho \quad \text{for} \quad 0 \leq t \leq \omega, \quad \rho > \rho_0. \tag{2.29}$$

Then an arbitrary nonnegative ω-periodic solution u of (2.28) *admits the estimate*

$$\|u\|_{C_\omega} \leq \rho_0. \tag{2.30}$$

Proof. Let u be an arbitrary ω-periodic solution of the differential inequality (2.28). Suppose

$$q(t) = u'(t) - \sigma p_0(t)u(t).$$

Then by Theorem 6.4 from [11] we find

$$u(t) = \left[\exp\Big(-\sigma\int_0^\omega p_0(s)\,ds\Big) - 1\right]^{-1}\int_t^{t+\omega}\exp\Big(\sigma\int_s^t p_0(\xi)\,d\xi\Big)q(s)\,ds. \tag{2.31}$$

On the other hand, owing to (2.28), the inequality

$$\sigma q(t) \le \gamma(t, \|x\|_{C_\omega})$$

holds almost everywhere on R. If along with this inequality we take into consideration inequality (2.26), then from (2.31) we obtain

$$u(t) \le \left|\exp\Big(-\sigma\int_0^\omega p_0(s)\,ds\Big) - 1\right|^{-1} \times$$
$$\times \int_t^{t+\omega}\exp\Big(\sigma\int_s^t p_0(\xi)\,d\xi\Big)\gamma(s, \|x\|_{C_\omega})\,ds \quad \text{for} \quad 0 \le t \le \omega. \tag{2.32}$$

Suppose now that estimate (2.30) is not valid. Then there exists a point $t_0 \in [0, \omega]$ such that

$$\|u\|_{C_\omega} = u(t_0) > \rho_0.$$

Taking into account this fact and condition (2.29), from (2.32) we obtain the contradiction

$$\|u\|_{C_\omega} < \|u\|_{C_\omega},$$

which proves the lemma. $\square$

Proof of Theorem 2.2. First of all it should be noted that by condition (2.27), there exists a positive number ρ_0 such that inequality (2.29) is fulfilled.

For any x and $y \in C_\omega(R^n)$ suppose

$$p(x, y)(t) = \sigma p_0(t)y(t).$$

Then by Theorem 6.4 from [11], inequality (2.26) guarantees the condition

$$p \in V_\omega^n.$$

According to this fact and Theorem 2.1, to prove Theorem 2.2 it suffices to establish that for any $\lambda \in]0, 1[$ an arbitrary ω-periodic solution of the differential equation

$$\frac{dx(t)}{dt} = (1-\lambda)\sigma p_0(t)x(t) + \lambda f(x)(t) \tag{2.33}$$

admits estimate (2.4).

Indeed, let x be such a solution. Suppose

$$u(t) = \|x(t)\|.$$

Then by (2.25) from (2.33) we find

$$\begin{aligned}\sigma u'(t) &= x'(t)\cdot \operatorname{sgn}(\sigma x(t)) = \\ &= (1-\lambda)p_0(t)\|x(t)\| + \lambda f(x)(t)\cdot \operatorname{sgn}(\sigma x(t)) \le \\ &\le p_0(t)\|x(t)\| + \gamma(t, \|x\|_{C_\omega}) = p_0(t)u(t) + \gamma(t, \|u\|_{C_\omega}).\end{aligned}$$

Consequently u is a nonnegative ω-periodic solution of the differential inequality (2.28). This function by Lemma 2.1 admits estimate (2.30). Therefore x admits estimate (2.4). $\square$

Theorem 2.3. *Let for any x and $y \in C_\omega(R^n)$, almost everywhere on R the condition*

$$\begin{aligned}&\big[f(x)(t) - f(y)(t)\big]\cdot \operatorname{sgn}\big[\sigma(x(t)-y(t))\big] \le \\ &\le p_0(t)\|x(t)-y(t)\| + \gamma_0(t)\|x-y\|_{C_\omega}\end{aligned} \tag{2.34}$$

be fulfilled, where $\sigma \in \{-1, 1\}$, the function $p_0 \in L_\omega(R)$ satisfies inequality (2.26) *and the function $\gamma_0 \in L_\omega(R_+)$ satisfies the inequality*

$$\begin{aligned}&\int_t^{t+\omega} \exp\bigg(\sigma\int_s^t p_0(\xi)\,d\xi\bigg)\gamma_0(s)\,ds < \\ &< \bigg|\exp\Big(-\sigma\int_0^\omega p_0(\xi)\,d\xi\Big) - 1\bigg| \quad \text{for} \quad 0 \le t \le \omega.\end{aligned} \tag{2.35}$$

Then equation (1.1) *has one and only one ω-periodic solution.*

Proof. From (2.34) and (2.35) we arrive at conditions (2.25) and (2.27), where

$$\gamma(t, \rho) = \gamma_0(t)\rho + \|f(t, 0, \dots, 0)\|.$$

Therefore by Theorem 2.2, equation (1.1) has at least one ω-periodic solution x.

To complete the proof of the theorem, it remains to show that an arbitrary ω-periodic solution y of equation (1.1) coincides with x. Suppose

$$u(t) = \|x(t) - y(t)\|.$$

Then by (2.34), u is a nonnegative ω-periodic solution of the differential inequality (2.28), where

$$\gamma(t, \rho) = \gamma_0(t)\rho.$$

On the other hand, owing to (2.35), the function γ satisfies inequality (2.29), where $\rho_0 = 0$, whence by Lemma 2.1 it follows that $u(t) \equiv 0$. Consequently $x(t) \equiv y(t)$. $\square$

Example 2.1. Consider the integro-differential equation

$$\frac{dx(t)}{dt} = \sigma p_0(t)x(t) + \sigma_0\Big[|p_0(t)| \int_{-\infty}^{+\infty} p_1(s)|x(s)|\,ds + p_2(t)\Big], \tag{2.36}$$

where σ and $\sigma_0 \in \{-1,1\}$, $p_1 : R \to R_+$ is a summable function, while $p_0 \in L_\omega(R_-)$ and $p_2 \in L_\omega(R_+)$ are functions different from zero on a set of positive measure. Because of the restrictions imposed on p_0 and p_2,

$$\int_t^{t+\omega} |g(t,s)|\,|p_0(s)|\,ds = 1 \tag{2.37}$$

and

$$\int_t^{t+\omega} |g(t,s)|p_2(s)\,ds \geq \delta, \tag{2.38}$$

where

$$g(t,s) = \Big[\exp\Big(-\sigma\int_0^\omega p_0(\xi)\,d\xi\Big) - 1\Big]^{-1} \exp\Big(\sigma\int_s^t p_0(\xi)\,d\xi\Big),$$

and δ is a positive constant. On the other hand, the operator

$$f(x)(t) = \sigma p_0(t)x(t) + \sigma_0\Big[|p_0(t)| \int_{-\infty}^{+\infty} p_1(s)|x(s)|\,ds + p_2(t)\Big]$$

satisfies condition (2.34), where

$$\gamma_0(t) = |p_0(t)| \int_{-\infty}^{+\infty} p_1(s)\,ds.$$

This and equality (2.37) imply that if

$$\int_{-\infty}^{+\infty} p_1(s)\,ds < 1, \tag{2.39}$$

then inequality (2.35) is fulfilled. In that case, by Theorem 2.3 equation (2.36) has a unique ω-periodic solution.

Show that if

$$\int_{-\infty}^{+\infty} p_1(s)\,ds \geq 1, \tag{2.40}$$

then equation (2.36) has no ω-periodic solution. Assume to the contrary that it has a solution x. Then

$$x(t) = \sigma_0 \int_t^{t+\omega} g(t,s)\left[|p_0(s)| \int_{-\infty}^{+\infty} p_1(\xi)|x(\xi)|\,d\xi + p_2(s)\right] ds.$$

Hence, taking into account (2.37), (2.38) and (2.40), we find

$$\mu \geq \mu \int_{-\infty}^{+\infty} p_1(\xi)\,d\xi + \delta \geq \mu + \delta,$$

where $\mu = \min\{|x(t)| : \ t \in R\}$. The obtained contradiction shows that equation (2.36) has no ω-periodic solution when condition (2.40) is fulfilled.

The example under consideration shows that in Theorem 2.2 (Theorem 2.3) it is impossible to replace the strict inequality (2.27) (the strict inequality (2.35)) by the nonstrict one.

Theorem 2.4. *Let there exist* $\rho_0 \in]0,+\infty[$, $\delta \in]0,1[$ *and* $\sigma \in \{-1,1\}$ *such that for any* $x \in C_\omega(R^n)$ *satisfying*

$$\|x\|_{C_\omega} > \rho_0, \tag{2.41}$$

almost everywhere on the set

$$\big\{t \in R : \ \ \|x(t)\| > (1-\delta)\|x\|_{C_\omega}\big\} \tag{2.42}$$

the inequality

$$f(x)(t) \cdot \operatorname{sgn}(\sigma x(t)) \leq 0 \tag{2.43}$$

is fulfilled. Then equation (1.1) *has at least one* ω*-periodic solution.*

Proof. By Theorem 2.1, to prove Theorem 2.4 it suffices to establish that for any $\lambda \in]0,1[$, an arbitrary ω-periodic solution of the differential equation

$$\frac{dx(t)}{dt} = -\sigma(1-\lambda)x(t) + \lambda f(x)(t) \tag{2.44}$$

admits estimate (2.4).

Assume to the contrary that for some $\lambda \in]0,1[$ equation (2.44) has an ω-periodic solution x satisfying (2.41). Then there exist $t_0 \in]0,+\infty[\,$, $t_* \in]0,t_0[$ and $t^* \in]t_0,+\infty[$ such that

$$\|x(t_0)\| = \|x\|_{C_\omega}, \quad \|x(t)\| > (1-\delta)\|x\|_{C_\omega} \quad \text{for} \quad t_* \le t \le t^*.$$

Hence it is clear that $[t_*,t^*]$ is included in (2.42). Therefore inequality (2.43) is fulfilled almost everywhere on $[t_*,t^*]$. Consequently

$$\sigma\,\frac{d\|x(t)\|}{dt} = -(1-\lambda)\|x(t)\| + \lambda f(x)(t)\cdot \operatorname{sgn}(\sigma x(t)) < 0$$
$$\text{for almost all} \quad t \in [t_*,t^*].$$

Hence for $\sigma = 1$ ($\sigma = -1$) it follows that

$$\|x(t_*)\| > \|x(t_0)\| \quad (\,\|x(t^*)\| > \|x(t_0)\|\,).$$

But this is impossible because $\|x(t_0)\| = \|x\|_{C_\omega}$. The obtained contradiction proves the theorem. $\square$

As an example, consider the nonlinear differential system

$$\frac{dx_i(t)}{dt} = -\sigma l_0(t)|x_i(t)|^\lambda \operatorname{sgn} x_i(t) + \sigma f_{1i}(t,x_1(t),\dots,x_n(t)) + \\ + f_{2i}\big(t,l_1(x_1)(t),\dots,l_n(x_n)(t)\big) \quad (i=1,\dots,n), \tag{2.45}$$

where $\sigma \in \{-1,1\}$, $\lambda \in]0,+\infty[\,$, $l_0 \in L_\omega(R_+)$, f_{1i} and $f_{2i} : R\times R^n \to R$ $(i=1,\dots,n)$ are ω-periodic in the first argument functions satisfying the local Carathéodory conditions, and $l_i : C_\omega(R) \to C_\omega(R)$ $(i=1,\dots,n)$ are linear bounded operators with norms $\|l_1\|,\dots,\|l_n\|$.

Set

$$\mu(\lambda) = \begin{cases} \lambda - 1 & \text{for} \quad \lambda > 1 \\ 0 & \text{for} \quad \lambda \le 1 \end{cases}.$$

Theorem 2.4 implies

Corollary 2.4. *Let on $R\times R^n$ the inequalities*

$$\sum_{i=1}^n f_{1i}(t,x_1,\dots,x_n)\operatorname{sgn} x_i \le l_0(t)\left[\eta_1\Big(\sum_{i=1}^n |x_i|\Big)^\lambda + \eta_0\right], \tag{2.46}$$

$$\sum_{i=1}^n |f_{2i}(t,x_1,\dots,x_n)| \le l_0(t)\left[\eta_2\Big(\sum_{i=1}^n |x_i|\Big)^\lambda + \eta_0\right] \tag{2.47}$$

be fulfilled, where η_i $(i=0,1,2)$ are positive constants such that

$$\eta_1 + \|l_i\|^\lambda \eta_2 < n^{-\mu(\lambda)} \quad (i=1,\dots,n). \tag{2.48}$$

Then system (2.45) *has at least one ω-periodic solution.*

Proof. If we assume $x(t)=(x_i(t))_{i=1}^n$,

$$f_i(x)(t)=-\sigma l_0(t)|x_i(t)|^\lambda \operatorname{sgn} x_i(t)+\sigma f_{1i}(t,x_1(t),\dots,x_n(t))+ \\ +f_{2i}\big(t,l_1(x_1)(t),\dots,l_n(x_n)(t)\big)\quad (i=1,\dots,n)$$

and

$$f(x)(t)=(f_i(x)(t))_{i=1}^n,$$

then system (2.45) will take form (1.1). On the other hand, taking into account (2.46) and (2.47), the operator f satisfies the condition

$$f(x)(t)\cdot\operatorname{sgn}(\sigma x(t))\le \\ \le -l_0(t)\sum_{i=1}^n|x_i(t)|^\lambda+l_0(t)\big[(\eta_1+\|l_{i_0}\|^\lambda\eta_2)\|x\|_{C_\omega}^\lambda+2\eta_0\big],$$

where $\|l_{i_0}\|=\max\{\|l_1\|,\dots,\|l_n\|\}$. Hence in view of the inequality

$$\|x\|^\lambda\le n^{\mu(\lambda)}\sum_{i=1}^n|x_i|^\lambda$$

we obtain

$$f(x)(t)\cdot\operatorname{sgn}(\sigma x(t))\le \\ \le -l_0(t)\big[n^{-\mu(\lambda)}\|x(t)\|^\lambda-(\eta_1+\|l_{i_0}\|^\lambda\eta_2)\|x\|_{C_\omega}^\lambda-2\eta_0\big]. \tag{2.49}$$

By virtue of (2.48), there exists $\delta\in]0,1[$ such that

$$\varepsilon=(1-\delta)^\lambda n^{-\mu(\lambda)}-\eta_1-\|l_{i_0}\|^\lambda\eta_2>0.$$

Set

$$\rho_0=(2\eta_0/\varepsilon)^{\frac{1}{\lambda}}.$$

Let $x\in C_\omega(R^n)$ be an arbitrary vector function satisfying (2.41). Then by (2.49) inequality (2.43) holds almost everywhere on set (2.42). Therefore all the conditions of Theorem 2.4 are fulfilled, which guarantees the existence of at least one ω-periodic solution of (2.45). □

3. Periodic Solutions of Equation (1.2)

Throughout this section $f_0: R\times R^{(m+1)n}\to R^n$ is assumed to be a vector function satisfying the local Carathéodory conditions and also condition (1.3), while $\tau_k: R\to R$ $(k=1,\dots,m)$ are assumed to be measurable functions satisfying condition (1.4).

For any $x\in C_\omega(R^n)$ we assume that

$$f(x)(t)=f_0\big(t,x(t),x(\tau_1(t)),\dots,x(\tau_m(t))\big).$$

Then the operator $f: C_\omega(R^n)\to L_\omega(R^n)$ is continuous. Therefore from Corollary 2.2 and Theorems 2.2–2.4 we obtain the following propositions.

Corollary 3.1. *Let the inequality*

$$\Big\|f_0(t,x_0,x_1,\dots,x_m)-\sum_{k=0}^{m}\mathcal{P}_k(t,x_0,x_1,\dots,x_m)x_k\Big\|\leq$$
$$\leq\gamma\big(t,\|x_0\|,\|x_1\|,\dots,\|x_m\|\big)$$

be fulfilled on $R\times R^{(m+1)n}$, *where* $\mathcal{P}_k : R\times R^{(m+1)n}\to R^{n\times n}$ ($k=0,1,\dots,m$) *are* ω*-periodic in the first argument matrix functions satisfying the local Carathéodory conditions, and* $\gamma : R\times R_+^{m+1}\to R_+$ *is nondecreasing in the last* n *arguments and* ω*-periodic in the first argument. Let, moreover, there exist matrices* A *and* $B\in R_+^{n\times n}$ *such that*

$$r(A+BA^2)<1,$$
$$\limsup_{\rho\to+\infty}\frac{1}{\rho}\int_0^{\omega}\gamma(s,\rho,\dots,\rho)\,ds<\big\|(E-A-BA^2)^{-1}(E+BA)\big\|$$

and for any $x\in C_\omega(R^n)$ *the matrix*

$$\sum_{k=0}^{m}\int_0^{\omega}\mathcal{P}_k\big(s,x(s),x(\tau_1(s)),\dots,x(\tau_m(s))\big)\,ds$$

is nondegenerate,

$$\sum_{k=0}^{m}\int_0^{\omega}\big|\mathcal{P}_k\big(s,x(s),x(\tau_1(s)),\dots,x(\tau_m(s))\big)\big|\,ds\leq A,$$

and

$$\bigg|\Big[\sum_{k=0}^{m}\int_0^{\omega}\mathcal{P}_k\big(s,x(s),x(\tau_1(s)),\dots,x(\tau_m(s))\big)\,ds\Big]^{-1}\bigg|\leq B.$$

Then equation (1.2) *has at least one* ω*-periodic solution.*

Corollary 3.2. *Let the inequality*

$$f_0(t,x_0,x_1,\dots,x_m)\cdot\operatorname{sgn}(\sigma x_0)\leq p_0(t)\|x_0\|+\gamma\big(t,\|x_0\|,\dots,\|x_m\|\big)$$

be fulfilled on $R\times R^{(m+1)n}$, *where* $\sigma\in\{-1,1\}$, $p_0\in L_\omega(R)$, $\int_0^{\omega}p_0(s)\,ds<0$,

$\gamma(\cdot,\rho,\dots,\rho) \in L_\omega(R_+)$ *for* $0<\rho<+\infty$, *and*

$$\limsup_{\rho\to+\infty}\bigg(\frac{1}{\rho}\int\limits_t^{t+\omega}\exp\Big(\sigma\int\limits_s^t p_0(\xi)\,d\xi\Big)\gamma(s,\rho,\dots,\rho)\,ds\bigg)<$$

$$<\bigg|\exp\Big(-\sigma\int\limits_0^\omega p_0(\xi)\,d\xi\Big)-1\bigg| \quad \textit{uniformly with respect to} \quad t\in[0,\omega].$$

Then equation (1.2) *has at least one* ω*-periodic solution.*

Corollary 3.3. *Let the condition*

$$\big[f_0(t,x_0,x_1,\dots,x_m)-f_0(t,y_0,y_1,\dots,y_m)\big]\cdot\operatorname{sgn}\big(\sigma(x_0-y_0)\big)\le$$

$$\le p_0(t)\|x_0-y_0\|+\sum_{k=0}^m\gamma_k(t)\|x_k-y_k\|$$

be fulfilled on $R\times R^{(m+1)n}$, *where* $\sigma\in\{-1,1\}$, $p_0\in L_\omega(R)$, $\int\limits_0^\omega p_0(s)\,ds<0$, $\gamma_k\in L_\omega(R_+)$ $(k=1,\dots,m)$ *and*

$$\sum_{k=1}^m\int\limits_t^{t+\omega}\exp\bigg(\sigma\int\limits_s^t p_0(\xi)\,d\xi\bigg)\gamma_k(s)\,ds<$$

$$<\bigg|\exp\Big(-\sigma\int\limits_0^\omega p_0(\xi)\,d\xi\Big)-1\bigg| \quad \textit{for} \quad 0\le t\le\omega.$$

Then equation (1.2) *has one and only one* ω*-periodic solution.*

Corollary 3.4. *Let there exist* $\rho\in\,]0,+\infty[\,$, $\delta\in\,]0,1[$ *and* $\sigma\in\{-1,1\}$ *such that on the set*

$$\big\{(t,x_0,x_1,\dots,x_m)\in R\times R^{(m+1)n}:\ \ \|x_0\|\ge\rho,$$
$$(1-\delta)\|x_k\|\le\|x_0\|\ \ (k=1,\dots,m)\big\}$$

the inequality

$$f_0(t,x_0,x_1,\dots,x_m)\cdot\operatorname{sgn}(\sigma x_0)\le 0$$

is fulfilled. Then equation (1.2) *has at least one* ω*-periodic solution.*

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Authors' addresses:

I. Kiguradze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia

B. Půža
Department of Mathematics
Masaryk University
Janáčkovo nám. 2a, 66295 Brno
Czech Republic