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# A REGULARITY CRITERION FOR SEMIGROUP RINGS

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Abstract. An analogue of the Kunz–Frobenius criterion for the regularity of a local ring in a positive characteristic is established for general commutative semigroup rings.

Let $S$ be a commutative semigroup (we always assume that $S$ contains a neutral element), and $K$ a field. For every $m \in \mathbb{Z}_+$ the assignment $x \mapsto x^m$, $x \in S$, induces a $K$-endomorphism $\pi_m$ of the semigroup ring $R = K[S]$. Therefore we can consider $R$ as an $R$-algebra via $\pi_m$, and especially as an $R$-module. Let $R^{[m]}$ denote $R$ with its $R$-module structure induced by $\pi_m$. If $S$ is finitely generated, then $R^{[m]}$ is obviously a finitely generated $R$-module.

In this note we want to give a regularity criterion for $S$ in terms of the homological properties of $R^{[m]}$ that is analogous to Kunz's [1] characterization of regular local rings of a characteristic $p > 0$ in terms of the Frobenius functor. Our criterion, which generalizes the result of Gubeladze [2, 10.2], requires only a mild condition on $S$ and we provide a 'pure commutative algebraic' proof. (In [2] the result was stated for seminormal simplicial affine semigroup rings and derived from the main result of [2] that $K_1$-regularity implies the regularity for such rings.)

**Theorem 1.** *Let $S$ be a finitely generated semigroup, $K$ a field, $R = K[S]$, and $m \in \mathbb{Z}_+$, $m > 0$. Suppose that $S$ has no invertible element $\neq 1$ and is generated by irreducible elements. Then the following conditions are equivalent:*

(a) *$R^{[m]}$ has a finite projective dimension;*
(b) *$R^{[m]}$ is a free module;*
(c) *$S$ is free, in other words, $S \cong \mathbb{Z}_+^n$ for some $n \in \mathbb{Z}_+$.*

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*Proof.* It is obvious that (c) implies (b) and (b) implies (a). Now assume that (a) is satisfied. We first reduce the problem to a question of local algebra.

Set $T = S \setminus \{1\}$. The ideal $\mathfrak{m} = TR$ is a maximal ideal of $R$. Indeed, $R/\mathfrak{m} \cong K$. Furthermore the only prime ideal of $\mathfrak{q}$ of $R$ such that $\mathfrak{m} = \pi_m^{-1}(\mathfrak{q})$ is $\mathfrak{m}$. Therefore $\pi_m \otimes R_\mathfrak{m}$ is an endomorphism of $R_\mathfrak{m}$ that makes $R_\mathfrak{m}$ a finitely generated $R_\mathfrak{m}$-module of finite projective dimension. In particular, $R_\mathfrak{m}$ has the same depth considered as an $R_\mathfrak{m}$-module via $\pi_m \otimes R_\mathfrak{m}$ as it has in its natural $R_\mathfrak{m}$-module structure (for example, see [3, 1.2.26]. The Auslander–Buchsbaum formula [3, 1.3.3] thus implies that $R_\mathfrak{m}$ is a finite free module over itself via $\pi_m \otimes R_\mathfrak{m}$. The lemma below shows that $R_\mathfrak{m}$ is a regular local ring.

Let $x_1, \dots, x_n$ be the irreducible elements of $S$. We claim that their images in $R_\mathfrak{m}$ form a minimal system of generators of the maximal ideal $\mathfrak{m}R_\mathfrak{m}$. Indeed, consider a presentation

$$R^r \xrightarrow{\varphi} R^n \xrightarrow{\psi} \mathfrak{m} \to 0,$$

where the $i$-th element $e_i$ of the natural basis of $R^n$ is mapped to $x_i$. We must show that all the entries of the matrix $\varphi$ are in $\mathfrak{m}$. Suppose on the contrary that there is a relation

$$a_1x_1 + \cdots + a_nx_n = 0$$

with, for example, $a_1 \notin \mathfrak{m}$. Then $a_1 = \alpha_1 + \alpha_2 s_2 + \cdots + \alpha_u s_u$ with $\alpha_i \in K$, $\alpha_i \neq 0$, and $s_2, \dots, s_u \in T$. Writing $a_2, \dots, a_m$ similarly, we see that there are only two possibilities, (i) $x_1 = s_i x_1$ for some $i$, or (ii) $x_1 = v x_j$ for some $v \in S$ and $j > 0$. Both cases are impossible because $x_1$ is irreducible.

However, $R_\mathfrak{m}$ is a regular local ring. Especially it is a factorial ring, in which the (images of the) $x_i$ are pairwise non-associated prime elements. Therefore all the elements $x_1^{e_1} \cdots x_n^{e_n}$, $e_1, \dots, e_n \in \mathbb{Z}_+$ are pairwise different, and it follows that $S \cong \mathbb{Z}_+^n$. □

*Remark* 2. (a) If we omit the hypothesis that $S$ be generated by irreducible elements, then the proof above shows just the following: the subsemigroup generated by $x_1, \dots, x_n \in S$ such that $x_1, \dots, x_n$ form a minimal system of generators of the ideal $\mathfrak{m}R_\mathfrak{m}$ is free of rank $n$.

(b) One can weaken the hypothesis of the theorem by requiring only that the group $S_0$ of invertible elements of $S$ be a free abelian group. Then $T = S \setminus S_0$ generates a prime ideal $\mathfrak{p}$ in $R$, and part (c) of the theorem must be replaced by the condition that $S \cong \mathbb{Z}_+^n \times \mathbb{Z}^q$ for some $n, q \in \mathbb{Z}_+$.

The following lemma is just an abstract version of Herzog's argument [4] characterizing the modules of finite projective dimension in terms of the Frobenius functor.

**Lemma 3.** *Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$. If there exists an endomorphism $\pi$ of $R$ with $\pi(\mathfrak{m}) \subset \mathfrak{m}^2$ and such that $R$ is a flat $R$-module via $\pi$, then $R$ is a regular local ring.*

*Proof.* According to the criterion of Auslander–Buchsbaum–Serre [3, 2.2.7] we must show that $k = R/\mathfrak{m}$ has finite projective dimenion as an $R$-module. Write $R'$ for $R$ considered as an $R$-module via $\pi$, and let $\mathcal{P}$ be the functor that takes an $R$-module $M$ to $M \otimes R'$ considered as an $R$-module via the identification $R = R'$. We choose a minimal free resolution $\mathcal{F}$ of $k$,

$$\mathcal{F}\colon \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \to \cdots \to F_1 \to k \to 0.$$

One has $\mathcal{P}(R) = R$, $\mathcal{P}(F_i) = F_i$, and $\mathcal{P}(\mathcal{F})$ is the complex that we obtain from $\mathcal{F}$ by replacing all entries in its matrices by their images under $\pi$. By hypothesis, $\mathcal{P}(\mathcal{F})$ is again exact, and the exactness is preserved by an $e$-fold iteration of this process. Especially, $\mathcal{P}^e(\mathcal{F})$ is a free resolution of $\mathcal{P}^e(k)$ for all $e > 0$.

Let $x_1, \dots, x_t \in \mathfrak{m}$ be a maximal $R$-sequence. Then $\bar{R} = R/(x_1, \dots, x_t)$ has projective dimension $t$, and so $\operatorname{Tor}_i^R(\bar{R}, \mathcal{P}^e(k)) = 0$ for all $i > t$ and $e > 0$. On the other hand, one can compute $\operatorname{Tor}_i^R(\bar{R}, \mathcal{P}^e(k))$ by tensoring $\mathcal{P}^e(\mathcal{F})$ with $\bar{R}$. Let $B_i$ be the kernel of $\varphi_i$. Then for sufficiently large $i$ and all $e > 0$ we have an exact sequence

$$0 \to \bar{R} \otimes \mathcal{P}^e(B_{i+1}) \to \bar{R} \otimes F_{i+1} \to \bar{R} \otimes \mathcal{P}^e(B_i) \to 0.$$

Since we have chosen a maximal $R$-sequence, $\operatorname{depth} \bar{R} \otimes F_{i+1} = 0$ if $F_{i+1} \neq 0$. On the other hand, for $e$ sufficiently large, $\mathcal{P}^e(B_{i+1}) \subset \mathfrak{m}^{2^e} \bar{R} \otimes F_{i+1}$ and $\mathcal{P}^e(B_i) \subset \mathfrak{m}^{2^e} \bar{R} \otimes F_i$ have a positive depth or are zero according to [4, Lemma 3.2]. This is a contradiction. □

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