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ON THE DIRICHLET PROBLEM IN A CHARACTERISTIC RECTANGLE FOR FOURTH ORDER LINEAR SINGULAR HYPERBOLIC EQUATIONS

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Abstract. In the rectangle $D = (0, a) \times (0, b)$ with the boundary Γ the Dirichlet problem

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = p(x, y)u + q(x, y),$$
$$u(x, y) = 0 \quad \text{for} \quad (x, y) \in \Gamma$$

is considered, where p and $q : D \to \mathbb{R}$ are locally summable functions and may have nonintegrable singularities on Γ. The effective conditions guaranteeing the unique solvability of this problem and the stability of its solution with respect to small perturbations of the coefficients of the equation under consideration are established.

§ 1. Formulation of the Problem and Main Results

In the open rectangle $D = (0, a) \times (0, b)$ consider the linear hyperbolic equation

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = p_0(x, y)u + q(x, y), \tag{1.1}$$

where p and q are real functions, Lebesgue summable on $[\delta, a - \delta] \times [\delta, b - \delta]$ for any small $\delta > 0$. We do not exclude the case, where p and q are not summable on D and have singularities on the boundary of D. In this sense equation (1.1) is singular.

Let Γ be the boundary of D. In the present paper for equation (1.1) we study the homogeneous Dirichlet problem

$$u(x, y) = 0 \quad \text{for} \quad (x, y) \in \Gamma. \tag{1.2}$$

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In the regular case, i.e., when p and q are summable on D, problem (1.1), (1.2) was studied in [1].

Before formulating the main results, we introduce several notations.

$\mathbb{R}$ is the set of real numbers. $\overline{D} = [0,a] \times [0,b]$.

For any $z \in \mathbb{R}$ set $[z]_+ = \frac{|z|+z}{2}$.

$C(\overline{D})$ is the space of continuous functions $z : \overline{D} \to \mathbb{R}$.

$L_{loc}(D)$ is the space of functions $z : D \to \mathbb{R}$ which are Lebesgue summable on $[\delta, a-\delta] \times [\delta, b-\delta]$ for any arbitrarily small $\delta > 0$.

$\widetilde{C}^{1,2}_{loc}(D)$ is the space of functions $z : D \to \mathbb{R}$, absolutely continuous on $[\delta, a-\delta] \times [\delta, b-\delta]$ for any arbitrarily small $\delta > 0$ together with $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x \partial y}$ and satisfying the condition $\int_0^a \int_0^b \left[\frac{\partial^2 z(x,y)}{\partial x \partial y}\right]^2 dx\, dy < +\infty$.

A function $u \in \widetilde{C}^{1,2}_{loc}(D)$ will be called a solution of equation (1.1) if it satisfies (1.1) almost everywhere in D.

A solution of problem (1.1), (1.2) will be sought in the class $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$.

Along with (1.1), we have to consider the equation

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \overline{p}(x,y)u + \overline{q}(x,y), \tag{1.3}$$

where $\overline{p}$ and $\overline{q} \in L_{loc}(D)$.

Definition 1.1. A solution of problem (1.1), (1.2) will be called stable with respect to small perturbations of the coefficients of equation (1.1) if there exist positive numbers δ and r such that for any $\overline{p}$ and $\overline{q} \in L_{loc}(D)$ satisfying the conditions

$$\eta_1(\overline{p} - p) \overset{def}{=} \int_0^a \int_0^b xy(a-x)(b-y)|\overline{p}(x,y) - p(x,y)|\, dx\, dy \le \delta, \tag{1.4}$$

$$\eta_2(\overline{q} - q) \overset{def}{=} \int_0^a \int_0^b [xy(a-x)(b-y)]^{\frac{1}{2}} |\overline{q}(x,y) - q(x,y)|\, dx\, dy < +\infty, \tag{1.5}$$

problem (1.3), (1.2) has a unique solution $\overline{u}$ in $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$ and

$$\left[\int_0^a \int_0^b \Big(\frac{\partial^2(\overline{u}(x,y) - u(x,y))}{\partial x \partial y}\Big)^2 dx\, dy\right]^{\frac{1}{2}} < r[\eta_1(\overline{p} - p) + \eta_2(\overline{q} - q)]. \tag{1.6}$$

*For the definition of absolutely continuous functions in a rectangle see [2, §570] or [3].

Remark 1.1. If inequality (1.6) holds, then, by Lemma 2.1 proved below, in the rectangle D the difference $\overline{u} - u$ admits the estimate

$$|\overline{u}(x,y) - u(x,y)| \leq 2r\Big[xy\Big(1 - \frac{x}{a}\Big)\Big(1 - \frac{y}{b}\Big)\Big]^{\frac{1}{2}}\big[\eta_1(\overline{p} - p) + \eta_2(\overline{q} - q)\big].$$

Definition 1.2. We say that a function $p \in L_{loc}(D)$ belongs to $U(D)$ if there exists a number $\alpha \in [0,1)$ such that for any function $u \in \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$ satisfying the boundary condition (1.2), the estimate

$$\int_0^a \int_0^b [p(x,y)]_+ u^2(x,y)\, dx\, dy \leq \alpha \int_0^a \int_0^b \Big(\frac{\partial^2 u(x,y)}{\partial x \partial y}\Big)^2 dx\, dy \tag{1.7}$$

is valid.

Theorem 1.1. *Let*

$$\begin{gathered} \int_0^a \int_0^b [xy(a-x)(b-y)]^{\frac{3}{2}} |p(x,y)|\, dx\, dy < +\infty, \\ \int_0^a \int_0^b [xy(a-x)(b-y)]^{\frac{1}{2}} |q(x,y)|\, dx\, dy < +\infty, \end{gathered} \tag{1.8}$$

and $p \in \boldsymbol{U}(D)$. *Then problem* (1.1), (1.2) *has a unique solution in* $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$, *stable with respect to small perturbations of the coefficients of equation* (1.1).

Theorem 1.2. *Let conditions* (1.8) *be fulfilled and the inequality*

$$p(x,y) \leq \frac{\lambda_0(x,y) + \lambda_1}{xy(a-x)(b-y)} + \frac{\lambda_2}{x^2y^2(a-x)^2(b-y)^2} \tag{1.9}$$

hold almost everywhere in D, *where* λ_0 *is a nonnegative summable function, and* λ_1 *and* λ_2 *are nonnegative numbers such that*

$$\frac{4}{ab} \int_0^a \int_0^b \lambda_0(x,y)\, dx\, dy + \frac{1}{4}\lambda_1 + \frac{16}{a^2b^2}\lambda_2 < 1. \tag{1.10}$$

Then the statement of Theorem 1.1 *is valid.*

As an example, consider the differential equation

$$\begin{aligned} \frac{\partial^4 u}{\partial x^2 \partial y^2} = \Big[\frac{l_1}{xy(a-x)(b-y)} + \frac{l_2}{x^2y^2(a-x)^2(b-y)^2}\Big] u \, + \\ + l_3 x^{\mu_1} y^{\mu_2} (a-x)^{\nu_1} (b-y)^{\nu_2}, \end{aligned} \tag{1.11}$$

where l_i $(i = 1, 2, 3)$, μ_j and ν_j $(j = 1, 2)$ are some real constants. By Theorem 1.2, if $\mu_j > -\frac{3}{2}$, $\nu_j > -\frac{3}{2}$ $(j = 1, 2)$ and

$$\frac{1}{4}[l_1]_+ + \frac{16}{a^2b^2}[l_2]_+ < 1, \tag{1.12}$$

then problem (1.11), (1.2) has a unique solution in $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$ and this solution is stable with respect to small perturbations of the coefficients of equation (1.11).

Note that condition (1.12) is sharp, since for $l_1 = 4$, $l_2 = l_3 = 0$ problem (1.11), (1.2) has an infinite set of solutions. More precisely, for any $c \in \mathbb{R}$, the function $u(x, y) = cxy(x-a)(y-b)$ is a solution of problem (1.11), (1.2).

§ 2. Auxiliary Statements

In this section, along with the notations introduced in §1, we shall make use of the following notations also.

$$\boldsymbol{A}_0^{1,2} = \{u \in \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D}) : u(x, y) = 0 \quad \text{for} \quad (x, y) \in \Gamma\}.$$

$\widetilde{C}^1(\overline{D})$ is the space of functions $z : \overline{D} \to \mathbb{R}$, absolutely continuous together with $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x \partial y}$.

We introduce

Definition 2.1. Let $\alpha > 0$. We say that a function $p \in L_{loc}(D)$ belongs to $\boldsymbol{U}_\alpha(D)$ if inequality (1.7) holds for any $u \in \boldsymbol{A}^{1,2}$.

2.1. Properties of the functions from $\boldsymbol{A}_0^{1,2}$.

Lemma 2.1. *If $u \in \boldsymbol{A}^{1,2}$, then*

$$u^2(x, y) \le \frac{4}{ab} xy(a-x)(b-y)\rho^2 \quad \textit{for} \quad (x, y) \in D, \tag{2.1}$$

$$\int_0^a \int_0^b \frac{u^2(x, y)}{xy(a-x)(b-y)}\, dx\, dy \le \frac{1}{4}\rho^2, \tag{2.2}$$

$$\int_0^a \int_0^b \Big[\frac{u(x, y)}{xy(a-x)(b-y)}\Big]^2 dx\, dy \le \frac{16}{a^2b^2}\rho^2, \tag{2.3}$$

where

$$\rho = \Big[\int_0^a \int_0^b \Big(\frac{\partial^2 u(x, y)}{\partial x \partial y}\Big)^2 dx\, dy\Big]^{\frac{1}{2}}. \tag{2.4}$$

Proof. First, let us prove estimate (2.1). The condition $u \in \boldsymbol{A}^{1,2}$ yields the representations

$$u(x,y) = \int\limits_{ia}^{x} \int\limits_{jb}^{y} \frac{\partial^2 u(s,t)}{\partial s \partial t}\, ds\, dt \quad (i,j=0,1) \quad \text{for} \quad (x,y) \in D.$$

Hence, by the Schwartz inequality and notation (2.4) it follows that

$$u^2(x,y) \le \rho^2 |x-ia||y-jb| \quad (i,j=0,1) \quad \text{for} \quad (x,y) \in D.$$

Therefore $u^2(x,y) \le \rho^2 \min\{x, a-x\} \min\{y, b-y\}$ for $(x,y) \in D$. But $\min\{x, a-x\} \le \frac{2}{a} x(a-x)$ and $\min\{y, b-y\} \le \frac{2}{b} y(b-y)$ for $(x,y) \in D$. Consequently, estimate (2.1) is valid.

Now pass to proving estimate (2.2). By Hardy–Littlewood theorem (see [4], Theorem 262), we have

$$\int\limits_0^b \frac{u^2(x,y)}{y(b-y)}\, dy \le \frac{1}{2} \int\limits_0^b \Big(\frac{\partial u(x,y)}{\partial y}\Big)^2 dy,$$

$$\int\limits_0^a \frac{1}{x(a-x)} \Big(\frac{\partial u(x,y)}{\partial y}\Big)^2 dx \le \frac{1}{2} \int\limits_0^a \Big(\frac{\partial^2 u(x,y)}{\partial x \partial y}\Big)^2 dx$$

almost everywhere in $(0,a)$ and $(0,b)$, respectively. Therefore

$$\int\limits_0^a \Big[\int\limits_0^b \frac{u^2(x,y)}{x(a-x)y(b-y)}\, dy\Big] dx \le \int\limits_0^b \Big[\int\limits_0^a \frac{1}{x(a-x)} \Big(\frac{\partial u(x,y)}{\partial y}\Big)^2 dx\Big] dy \le$$

$$\le \frac{1}{4} \int\limits_0^a \int\limits_0^b \Big(\frac{\partial^2 u(x,y)}{\partial x \partial y}\Big)^2 dx\, dy = \frac{1}{4}\rho^2.$$

Consequently, estimate (2.2) is valid.

As for estimate (2.3), it follows from V. I. Levin's inequality (see [5] or [4, D.79]). Indeed,

$$\int\limits_0^a \Big[\int\limits_0^b \Big[\frac{u(x,y)}{x(a-x)y(b-y)}\Big]^2 dy\Big] dx \le$$

$$\le \frac{4}{b^2} \int\limits_0^b \Big[\int\limits_0^a \frac{1}{x^2(a-x)^2} \Big(\frac{\partial u(x,y)}{\partial y}\Big) dy\Big] dx \le$$

$$\leq \frac{16}{a^2b^2}\int\limits_0^a\int\limits_0^b\Big(\frac{\partial^2 u(x,y)}{\partial x\partial y}\Big)^2\,dx\,dy=\frac{16}{a^2b^2}\rho^2. \quad \square$$

Lemma 2.2. *Let $u\in \boldsymbol{A}^{1,2}$. Then there exist sequences $(x_{ik})_{k=1}^{+\infty}$ and $(y_{ik})_{k=1}^{+\infty}$ such that*

$$0<x_{1k}<x_{2k}<a, \quad 0<y_{1k}<y_{2k}<b \quad (k=1,2,\dots), \tag{2.5}$$

$$\lim_{k\to+\infty}x_{1k}=0, \quad \lim_{k\to+\infty}x_{2k}=a, \quad \lim_{k\to+\infty}y_{1k}=0, \quad \lim_{k\to+\infty}y_{2k}=b \tag{2.6}$$

and

$$\lim_{k\to+\infty}\int\limits_{x_{1k}}^{x_{2k}}\int\limits_{y_{1k}}^{y_{2k}}\frac{\partial^4 u(x,y)}{\partial x^2\partial y^2}u(x,y)\,dx\,dy=\rho^2, \tag{2.7}$$

where ρ is the number given by (2.4)*.*

Proof. Let

$$w(x,y)=\frac{\partial^2 u(x,y)}{\partial x\partial y}. \tag{2.8}$$

For any natural k set

$$\alpha_{0k}=\frac{a}{2k+4}, \quad \alpha_k=\frac{a}{k+2}, \quad a_{0k}=\frac{(k+1)a}{k+2}, \quad a_k=\frac{(2k+3)a}{2k+4},$$
$$\beta_{0k}=\frac{b}{2k+4}, \quad \beta_k=\frac{b}{k+2}, \quad b_{0k}=\frac{(k+1)b}{k+2}, \quad b_k=\frac{(2k+3)b}{2k+4},$$

$$w_{1k}(x)=\int\limits_{\beta_{0k}}^{b_k}w^2(x,y)\,dy, \quad w_{2k}(y)=\int\limits_{\alpha_{0k}}^{a_k}w^2(x,y)\,dx, \tag{2.9}$$

$$\int\limits_{\alpha_{0k}}^{\alpha_k}w_{1k}(x)\,dx+\int\limits_{a_{0k}}^{a_k}w_{1k}(x)\,dx+\int\limits_{\beta_{0k}}^{\beta_k}w_{2k}(y)\,dy+\int\limits_{b_{0k}}^{b_k}w_{2k}(y)\,dy=\varepsilon_k. \tag{2.10}$$

Then $\lim\limits_{k\to+\infty}\varepsilon_k=0$.

In view of the continuity of w, w_{1k} and w_{2k}, for any natural k there exist

$$x_{1k}\in[\alpha_{0k},\alpha_k], \quad x_{2k}\in[a_{0k},a_k], \quad y_{1k}\in[\beta_{0k},\beta_k], \quad y_{2k}\in[b_{0k},b_k]$$

such that

$$w_{1k}(x_{1k})=\min\{w_{1k}(x):\alpha_{0k}\leq x\leq\alpha_k\},$$
$$w_{1k}(x_{2k})=\min\{w_{1k}(x):a_{0k}\leq x\leq a_k\},$$

$$\frac{a}{2k+4}[w^2(x_{1k},y_{1k})+w^2(x_{2k},y_{1k})]+w_{2k}(y_{1k})=$$
$$=\min\Big\{\frac{a}{2k+4}[w^2(x_{1k},y)+w^2(x_{2k},y)]+w_{2k}(y):\beta_{0k}\le y\le\beta_k\Big\},$$
$$\frac{a}{2k+4}[w^2(x_{1k},y_{2k})+w^2(x_{2k},y_{2k})]+w_{2k}(y_{2k})=$$
$$=\min\Big\{\frac{a}{2k+4}[w^2(x_{1k},y)+w^2(x_{2k},y)]+w_{2k}(y):b_{0k}\le y\le b_k\Big\}.$$

Then it is obvious that the sequences $(x_{ik})_{k=1}^{+\infty}$ and $(y_{ik})_{k=1}^{+\infty}$ $(i=1,2)$ satisfy conditions (2.5) and (2.6). On the other hand, from (2.9) and (2.10) we have

$$\varepsilon_k\ge\frac{a}{2k+4}[w_{1k}(x_{1k})+w_{1k}(x_{2k})]+\int\limits_{\beta_{0k}}^{\beta_k}w_{2k}(y)\,dy+\int\limits_{b_{0k}}^{b_k}w_{2k}(y)\,dy\ge$$
$$\ge\int\limits_{\beta_{0k}}^{\beta_k}\Big[\frac{a}{2k+4}w^2(x_{1k},y)+\frac{a}{2k+4}w^2(x_{2k},y)+w_{2k}(y)\Big]\,dy+$$
$$+\int\limits_{b_{0k}}^{b_k}\Big[\frac{a}{2k+4}w^2(x_{1k},y)+\frac{a}{2k+4}w^2(x_{2k},y)+w_{2k}(y)\Big]\,dy\ge$$
$$\ge\frac{ab}{(2k+4)^2}[w^2(x_{1k},y_{1k})+w^2(x_{2k},y_{1k})+w^2(x_{1k},y_{2k})+w^2(x_{2k},y_{2k})]+$$
$$+\frac{b}{2k+4}[w_{2k}(y_{1k})+w_{2k}(y_{2k})]\qquad(k=1,2,\dots).$$

Therefore

$$w_{1k}(x_{ik})\le(k+2)\varepsilon_{0k},\quad w_{2k}(y_{ik})\le(k+2)\varepsilon_{0k}\ (i=1,2;\ k=1,2,\dots),\tag{2.11}$$
$$|w(x_{ik},y_{jk})|\le(k+2)\varepsilon_{0k}\quad(i,j=1,2;\ k=1,2,\dots),\tag{2.12}$$

where $\varepsilon_{0k}=\max\Big\{\frac{2\varepsilon_k}{a},\frac{2\varepsilon_k}{b},2\Big(\frac{\varepsilon_k}{ab}\Big)^{\frac12}\Big\}$ and

$$\lim_{k\to+\infty}\varepsilon_{0k}=0.\tag{2.13}$$

Moreover, inequality (2.1) implies that

$$|u(x_{ik},y_{jk})|\le\frac{\gamma}{k+2}\quad(i,j=1,2;\ k=1,2,\dots),\tag{2.14}$$

where $\gamma=2\rho(ab)^{\frac12}$.

For every natural k consider the integral

$$I_k = \int\limits_{x_{1k}}^{x_{2k}} \int\limits_{y_{1k}}^{y_{2k}} \frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} u(x,y)\, dx\, dy. \tag{2.15}$$

By equality (2.8) and the formula of integration by parts, we have

$$\begin{aligned}
I_k = &\int\limits_{x_{1k}}^{x_{2k}} \Big(\int\limits_{y_{1k}}^{y_{2k}} \frac{\partial^2 w(x,y)}{\partial x \partial y} u(x,y)\, dy \Big)\, dx = \int\limits_{x_{1k}}^{x_{2k}} \Big[\frac{\partial w(x,y_{2k})}{\partial x} u(x,y_{2k}) - \\
&- \frac{\partial w(x,y_{1k})}{\partial x} u(x,y_{1k}) \Big]\, dx - \int\limits_{y_{1k}}^{y_{2k}} \Big(\int\limits_{x_{1k}}^{x_{2k}} \frac{\partial w(x,y)}{\partial x} \frac{\partial u(x,y)}{\partial y}\, dx \Big)\, dy = \\
&= w(x_{2k},y_{2k})u(x_{2k},y_{2k}) - w(x_{1k},y_{2k})u(x_{1k},y_{2k}) - \\
&- w(x_{2k},y_{1k})u(x_{2k},y_{1k}) + w(x_{1k},y_{1k})u(x_{1k},y_{1k}) - \\
&- \int\limits_{x_{1k}}^{x_{2k}} \Big[w(x,y_{2k}) \frac{\partial u(x,y_{2k})}{\partial x} - w(x,y_{1k}) \frac{\partial u(x,y_{1k})}{\partial x} \Big]\, dx - \\
&- \int\limits_{y_{1k}}^{y_{2k}} \Big[w(x_{2k},y) \frac{\partial u(x_{2k},y)}{\partial y} - w(x_{1k},y) \frac{\partial u(x_{1k},y)}{\partial y} \Big]\, dy + I_{0k},
\end{aligned} \tag{2.16}$$

where

$$I_{0k} = \int\limits_{x_{1k}}^{x_{2k}} \int\limits_{y_{1k}}^{y_{2k}} \Big(\frac{\partial^2 u(x,y)}{\partial x \partial y} \Big)^2\, dx\, dy.$$

Moreover, as it follows from (2.4) and (2.6),

$$\lim_{k \to +\infty} I_{0k} = \rho^2. \tag{2.17}$$

By virtue of the condition $u \in \boldsymbol{A}^{1,2}$ and equality (2.8) we have

$$\frac{\partial u(x,y_{ik})}{\partial x} = \int\limits_{(i-1)b}^{y_{ik}} w(x,y)\, dy, \quad \frac{\partial u(x_{ik},y)}{\partial y} = \int\limits_{(i-1)a}^{x_{ik}} w(x,y)\, dx \quad (i=1,2).$$

If along with this we take into account equalities (2.4) and (2.9) and inequality (2.11), then we get

$$\Big| \int\limits_{x_{1k}}^{x_{2k}} w(x,y_{ik}) \frac{\partial u(x,y_{ik})}{\partial x}\, dx \Big| = \Big| \int\limits_{x_{1k}}^{x_{2k}} w(x,y_{ik}) \Big(\int\limits_{(i-1)b}^{y_{ik}} w(x,y)\, dy \Big)\, dx \Big| \leq$$

$$\leq \Big[\int\limits_{x_{1k}}^{x_{2k}} w^2(x,y_{ik})\,dx\Big]^{\frac{1}{2}} \Big[\int\limits_{x_{1k}}^{x_{2k}} \Big(\int\limits_{(i-1)b}^{y_{ik}} w(x,y)\,dy\Big)^2 dx\Big]^{\frac{1}{2}} \leq [w_{2k}(y_{ik})]^{\frac{1}{2}} \times$$

$$\times \Big[\frac{b}{k+2}\int\limits_0^a\int\limits_0^b w^2(x,y)\,dx\,dy\Big]^{\frac{1}{2}} \leq (b\varepsilon_{0k})^{\frac{1}{2}}\rho \ \ (i=1,2;\ k=1,2,\dots), \tag{2.18}$$

$$\Big|\int\limits_{y_{1k}}^{y_{2k}} w(x_{ik},y)\frac{\partial u(x_{ik},y)}{\partial y}\,dy\Big| \leq (a\varepsilon_{0k})^{\frac{1}{2}}\rho \quad (i=1,2;\ k=1,2,\dots). \tag{2.19}$$

Using conditions (2.12)–(2.14), (2.18), and (2.19), from (2.16) we find

$$|I_k - I_{0k}| \leq 4\gamma\varepsilon_{0k} + 2(b\varepsilon_{0k})^{\frac{1}{2}}\rho + 2(a\varepsilon_{0k})^{\frac{1}{2}}\rho \to 0 \quad \text{for} \quad k\to+\infty.$$

Hence, according to (2.15) and (2.17), there follows equality (2.7). □

2.2. On one property of the set $\boldsymbol{U}_\alpha(D)$.

Lemma 2.3. *Let $\alpha>0$, $\delta>0$, $p\in \boldsymbol{U}_\alpha(D)$ and the function $\overline{p}\in L_{loc}(D)$ satisfy the inequality*

$$\int\limits_0^a\int\limits_0^b xy(a-x)(b-y)[\overline{p}(x,y)-p(x,y)]_+\,dx\,dy \leq \delta. \tag{2.20}$$

Then

$$\overline{p}\in \boldsymbol{U}_\beta(D), \tag{2.21}$$

where $\beta = \alpha + \frac{4}{ab}\delta$.

Proof. Let u be an arbitrary function from $\boldsymbol{A}^{1,2}$, and ρ be the number given by (2.4). Then by Definition 2.1 and Lemma 2.1 inequalities (1.7) and (2.1) are valid. Moreover, if we take into account inequalities (2.20) and

$$[\overline{p}(x,y)]_+ \leq [p(x,y)]_+ + [\overline{p}(x,y)-p(x,y)]_+,$$

then we get

$$\int\limits_0^a\int\limits_0^b [\overline{p}(x,y)]_+ u^2(x,y)\,dx\,dy \leq \int\limits_0^a\int\limits_0^b [p(x,y)]_+ u^2(x,y)\,dx\,dy +$$

$$+\int\limits_0^a\int\limits_0^b [\overline{p}(x,y)-p(x,y)]_+ u^2(x,y)\,dx\,dy \leq \alpha\rho^2 +$$

$$+\frac{4\rho^2}{ab}\int_0^a\int_0^b xy(a-x)(b-y)[\overline{p}(x,y)-p(x,y)]_+ dx\,dy \le \Big(\alpha+\frac{4}{ab}\delta\Big)\rho^2=\beta\rho^2.$$

Consequently, inclusion (2.21) is true. □

2.3. Lemmas on a priori estimates.

Lemma 2.4. *Let $p \in \boldsymbol{U}_\alpha(D)$, where $0 < \alpha < 1$, and the function $q \in L_{loc}(D)$ satisfy the condition*

$$\eta_2(q) \overset{def}{=} \int_0^a\int_0^b [xy(a-x)(b-y)]^{\frac{1}{2}}|q(x,y)|\,dx\,dy < +\infty. \tag{2.22}$$

Moreover, if problem (1.1), (1.2) *has a solution $u \in \widetilde{C}^{1,2}_{loc}(D)\cap C(\overline{D})$, then*

$$\rho \le \frac{2}{(1-\alpha)\sqrt{ab}}\eta_2(q), \tag{2.23}$$

where ρ is a number given by (2.4).

Proof. By virtue of Definition 2.1 and Lemmas 2.1 and 2.2 the function u satisfies conditions (1.7) and (2.1), and there exist sequences $(x_{ik})_{k=1}^{+\infty}$ and $(y_{ik})_{k=1}^{+\infty}$ $(i = 1, 2)$ satisfying conditions (2.5) and (2.6) such that equality (2.7) is true.

Multiply both sides of (1.1) by $u(x, y)$ and integrate them over $[x_{1k}, x_{2k}]\times[y_{1k}, y_{2k}]$ for any natural k. Then with regard to (1.7),(2.1) and (2.22) we find

$$\int_{x_{1k}}^{x_{2k}}\int_{y_{1k}}^{y_{2k}} u(x,y)\frac{\partial^4 u(x,y)}{\partial x^2\partial y^2}\,dx\,dy = \int_{x_{1k}}^{x_{2k}}\int_{y_{1k}}^{y_{2k}} p(x,y)u^2(x,y)\,dx\,dy +$$
$$+\int_{x_{1k}}^{x_{2k}}\int_{y_{1k}}^{y_{2k}} q(x,y)u(x,y)\,dx\,dy \le \int_0^a\int_0^b [p(x,y)]_+u^2(x,y)\,dx\,dy +$$
$$+\int_0^a\int_0^b |q(x,y)||u(x,y)|\,dx\,dy \le \alpha\rho^2+\frac{2}{\sqrt{ab}}\eta_2(q)\rho \quad (k=1,2,\dots).$$

If we pass in this inequality to the limit as $k \to +\infty$, then by (2.7) we get

$$\rho^2 \le \alpha\rho^2 + \frac{2}{\sqrt{ab}}\eta_2(q)\rho.$$

Consequently, estimate (2.23) is true. □

Lemma 2.5. *Let* $0 < \alpha < 1,\ \ \delta > 0,$

$$\beta = \alpha + \frac{4}{ab}\delta < 1, \tag{2.24}$$

and the functions $p \in \boldsymbol{U}_\alpha(D)$, $\overline{p}, q$ *and* $\overline{q} \in L_{loc}(D)$ *satisfy conditions* (1.4) *and* (1.5). *Moreover, let problem* (1.1), (1.2) *have a solution* $u \in \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$, *and problem* (1.3), (1.2) *have a solution* $\overline{u} \in \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$. *Then inequality* (1.6) *is valid, where*

$$r = \max\Big\{\frac{4\rho}{(1-\beta)ab}, \frac{2}{(1-\beta)\sqrt{ab}}\Big\} \tag{2.25}$$

and ρ *is the number given by* (2.4).

Proof. Note that by Lemma 2.3 inclusion (2.21) is true. Set

$$v(x,y) = \overline{u}(x,y) - u(x,y).$$

Then from (1.1)–(1.3) we have

$$\frac{\partial^4 v(x,y)}{\partial x^2 \partial y^2} = \overline{p}(x,y)v(x,y) + [\overline{p}(x,y) - p(x,y)]u(x,y) + \overline{q}(x,y) - q(x,y),$$
$$v(x,y) = 0 \quad \text{for} \quad (x,y) \in \Gamma.$$

Hence, by Lemma 2.4 and conditions (2.21) and (2.24), there follows the estimate

$$\rho^* \overset{def}{=} \Big[\int\limits_0^a \int\limits_0^b \Big(\frac{\partial^2 v(x,y)}{\partial x \partial y}\Big)^2 dx\,dy\Big]^{\frac{1}{2}} \le \frac{2}{(1-\beta)\sqrt{ab}}\eta_2(\overline{q} - q) +$$
$$+\frac{2}{(1-\beta)\sqrt{ab}} \int\limits_0^a \int\limits_0^b [xy(a-x)(b-y)]^{\frac{1}{2}} |\overline{p}(x,y) - p(x,y)||u(x,y)|\,dx\,dy.$$

Now if we apply conditions (1.5) and (2.1), then it becomes clear that

$$\rho^* \le \frac{4\rho}{(1-\beta)ab}\eta_1(\overline{p} - p) + \frac{2}{(1-\beta)\sqrt{ab}}\eta_2(\overline{q} - q).$$

Consequently, estimate (1.6) is true, where the constant r is given by (2.25). $\square$

2.4. Lemmas on the existence and uniqueness of solutions of problem (1.1),(1.2).

Lemma 2.6. *If* $p \in \boldsymbol{U}_\alpha(D)$*, where* $0 < \alpha < 1$*, then problem* $(1.1),(1.2)$ *has at most one solution in* $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$*.*

Proof. Let $u_i \in \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$ $(i = 1,2)$ be arbitrary solutions of problem $(1.1),(1.2)$. Set $u(x,y) = u_2(x,y) - u_1(x,y)$. It is obvious that u is a solution of the homogeneous problem

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = p(x,y)u \tag{2.26}$$

and $u \in \boldsymbol{A}^{1,2}$. Hence by Lemma 2.4 it follows that $\frac{\partial^2 u(x,y)}{\partial x \partial y} \equiv 0$ and

$$u(x,y) \equiv \int\limits_0^x \int\limits_0^y \frac{\partial^2 u(s,t)}{\partial s \partial t} \equiv 0.$$

Consequently, $u_1(x,y) \equiv u_2(x,y)$. $\square$

Lemma 2.7. *If the functions* p *and* q *are summable on* D *and* $p \in \boldsymbol{U}_\alpha(D)$*, where* $0 < \alpha < 1$*, then problem* $(1.1),(1.2)$ *has one and only one solution in* $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$*.*

Proof. By Lemma 2.6, problem $(2.26),(1.2)$ has only the trivial solution in the space $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$. Consequently, this problem has only the trivial solution in $\widetilde{C}^1(\overline{D})$, since

$$\widetilde{C}^1(\overline{D}) \subset \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D}). \tag{2.27}$$

But by Theorem 1.1 from [1] the summability of p and q on D and the unique solvability of problem (2.26),(1.2) in the space $\widetilde{C}^1(\overline{D})$ guarantee the existence and uniqueness of a solution $u \in \widetilde{C}^1(\overline{D})$ of problem (1.1),(1.2). Hence Lemma 2.6 and condition (2.27) imply that u is the unique solution of problem (1.1),(1.2) in $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$. $\square$

§ 3. Proofs of the Main Results

Proof of Theorem 1.1. Note that by Definitions 1.2 and 2.1 there exists $\alpha \in (0,1)$ such that

$$p \in \boldsymbol{U}_\alpha(D). \tag{3.1}$$

For any natural m set

$$D_m = \Big(\frac{a}{4m}, \frac{(4m-1)a}{4m}\Big) \times \Big(\frac{b}{4m}, \frac{(4m-1)}{4m}\Big),$$

$$p_m(x,y)=\begin{cases}p(x,y) & \text{for} \quad (x,y)\in D_m,\\ 0 & \text{for} \quad (x,y)\in D\backslash D_m,\end{cases}$$

$$q_m(x,y)=\begin{cases}q(x,y) & \text{for} \quad (x,y)\in D_m,\\ 0 & \text{for} \quad (x,y)\in D\backslash D_m\end{cases}$$

and consider the differential equation

$$\frac{\partial^4 u}{\partial x^2\partial y^2}=p_m(x,y)u+q(x,y). \tag{3.2$_m$}$$

It is clear that p_m and q_m $(m=1,2,\dots)$ are summable on D and the conditions

$$|p_m(x,y)|\le|p(x,y)|, \quad |q_m(x,y)|\le|q(x,y)| \quad (m=1,2,\dots), \tag{3.3}$$

$$\lim_{m\to\infty}p_m(x,y)=p(x,y), \qquad \lim_{m\to\infty}q_m(x,y)=q(x,y) \tag{3.4}$$

hold almost everywhere in D. Moreover, by (3.1),

$$p_m\in\boldsymbol{U}_\alpha(D) \quad (m=1,2,\dots). \tag{3.5}$$

By Lemmas 2.4 and 2.7, for any natural m problem $(3.2_m),(1.2)$ has the unique solution u_m in $\widetilde{C}^{1,2}_{loc}(D)\cap C(\overline{D})$ and

$$\rho_m\le\frac{2}{(1-\alpha)\sqrt{ab}}\int\limits_0^a\int\limits_0^b[xy(a-x)(b-y)]^{\frac12}|q_m(x,y)|\,dx\,dy,$$

where $\rho_m=\Big[\int_0^a\int_0^b\Big(\frac{\partial^2u_m(x,y)}{\partial x\partial y}\Big)^2\,dx\,dy\Big]^{\frac12}$. Hence, taking into account (1.8) and (3.3), we find

$$\rho_m\le\gamma \quad (m=1,2,\dots), \tag{3.6}$$

where $\gamma=\frac{2}{(1-\alpha)\sqrt{ab}}\int_0^a\int_0^b[xy(a-x)(b-y)]^{\frac12}|q(x,y)|\,dx\,dy$.

By Lemma 2.1 and condition (3.6), we have

$$|u_m(x,y)|\le\gamma_0[xy(a-x)(b-y)]^{\frac12} \ \text{ for } \ (x,y)\in\overline{D} \ \ (m=1,2,\dots), \tag{3.7}$$

$$\begin{aligned}|u_m(\overline{x},\overline{y})-u_m(x,y)|&=\Big|\int\limits_x^{\overline{x}}\int\limits_0^{\overline{y}}\frac{\partial^2u_m(s,t)}{\partial s\partial t}\,ds\,dt+\int\limits_0^{x}\int\limits_y^{\overline{y}}\frac{\partial^2u_m(s,t)}{\partial s\partial t}\,ds\,dt\Big|\le\\ &\le(\overline{y}|\overline{x}-x|)^{\frac12}\rho_m+(x|\overline{y}-y|)^{\frac12}\rho_m\le\gamma_0(|\overline{x}-x|)^{\frac12}+|\overline{y}-y|)^{\frac12})\\ &\text{for} \quad 0\le x\le\overline{x}\le a, \ \ 0\le y\le\overline{y}\le b \ \ (m=1,2,\dots),\end{aligned} \tag{3.8}$$

where $\gamma_0=\max\{2(ab)^{-\frac12},a^{\frac12},b^{\frac12}\}\gamma$.

By the Arzela–Ascoli lemma, conditions (3.7) and (3.8) guarantee the existence of a subsequence $(u_{m_k})_{k=1}^{+\infty}$ of the sequence $(u_m)_{m=1}^{+\infty}$, uniformly convergent on $\overline{D}$. Set

$$\lim_{k\to+\infty} u_{m_k}(x,y) = u(x,y). \tag{3.9}$$

Then from (3.7) we get

$$|u(x,y)| \le \gamma_0 [xy(a-x)(b-y)]^{\frac{1}{2}} \quad \text{for} \quad (x,y)\in\overline{D}. \tag{3.10}$$

For any natural k the function u_{m_k} admits the representation

$$u_{m_k}(x,y) = \int\limits_0^a \int\limits_0^b g_1(x,s) g_2(y,t)[p_{m_k}(s,t)u_{m_k}(s,t) + q_{m_k}(s,t)]\, ds\, dt, \tag{3.11}$$

where

$$g_1(x,s) = \begin{cases} s\big(\frac{x}{a}-1\big) & \text{for} \quad s\le x, \\ x\big(\frac{s}{a}-1\big) & \text{for} \quad s> x, \end{cases} \qquad g_2(y,t) = \begin{cases} t\big(\frac{y}{b}-1\big) & \text{for} \quad t\le y, \\ y\big(\frac{t}{b}-1\big) & \text{for} \quad t> y. \end{cases}$$

Moreover, it is obvious that the functions g_1 and g_2 admit the estimates

$$|g_1(x,s)| \le \Big(1-\frac{s}{a}\Big)s, \quad |g_2(y,t)| \le \Big(1-\frac{t}{b}\Big)t, \tag{3.12}$$

$$\begin{aligned} \Big|\frac{\partial g_1(x,s)}{\partial x}\Big| &\le \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{-1} s\Big(1-\frac{s}{a}\Big), \\ \Big|\frac{\partial g_2(y,t)}{\partial y}\Big| &\le \Big[y\Big(1-\frac{y}{b}\Big)\Big]^{-1} t\Big(1-\frac{t}{b}\Big). \end{aligned} \tag{3.13}$$

If along with this we take into account conditions (1.8), (3.3), and (3.7), then we obtain the inequalities

$$|g_1(x,s)g_2(y,t)[p_{m_k}(s,t)u_{m_k}(s,t)+q_{m_k}(s,t)]| \le q^*(s,t) \ (k=1,2,\dots), \tag{3.14}$$

$$\begin{aligned} \Big|\frac{\partial g_1(x,s)}{\partial x}\frac{\partial g_2(y,t)}{\partial y}[p_{m_k}(s,t)u_{m_k}(s,t)+q_{m_k}(s,t)]\Big| \le \\ \le \Big[xy\Big(1-\frac{x}{a}\Big)\Big(1-\frac{y}{b}\Big)\Big]^{-1} q^*(s,t) \quad (k=1,2,\dots), \end{aligned} \tag{3.15}$$

where $q^*(s,t) = \gamma_0\Big[st\Big(1-\frac{s}{a}\Big)\Big(1-\frac{t}{b}\Big)\Big]^{\frac{3}{2}}|p(s,t)| + st\Big(1-\frac{s}{a}\Big)\Big(1-\frac{t}{b}\Big)|q(s,t)|$ and q^* is summable on D.

Now if we apply the Lebesgue's theorem on the passage to the limit under the integral, then, with regard to (3.4), (3.9), and (3.14), from (3.11) we get

$$u(x,y) = \int\limits_0^a \int\limits_0^b g_1(x,s)g_2(y,t)[p(s,t)u(s,t)+q(s,t)]\, ds\, dt. \tag{3.16}$$

By virtue of (3.15), equalities (3.11) and (3.16) yield

$$\lim_{k\to+\infty} \frac{\partial^2 u_{m_k}(x,y)}{\partial x \partial y} = \frac{\partial^2 u(x,y)}{\partial x \partial y}$$

uniformly on every closed subset of D. Taking into account this fact, from (3.6) we get

$$\int\limits_0^a \int\limits_0^b \Big(\frac{\partial^2 u(x,y)}{\partial x \partial y} \Big)^2 \, dx\, dy \le \gamma. \tag{3.17}$$

By virtue of (1.8), (3.12), (3.13) it follows from (3.16) and (3.17) that $u \in \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$ and u is a solution of equation (1.1). On the other hand, it is clear from (3.10) that u satisfies the boundary condition (1.2).

By Lemma 2.6, problem (1.1),(1.2) has no solution different from u in $\widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$.

To complete the proof, we have to show the stability of the solution u with respect to small perturbation of the coefficients of equation (1.1).

Let δ be an arbitrary positive number satisfying inequality (2.24), and ρ and r be numbers given by equalities (2.4) and (2.25). Consider arbitrary functions $\overline{p}$ and $\overline{q} \in L_{loc}(D)$ satisfying conditions (1.4) and (1.5). Then by conditions (1.8),(3.1) and Lemma 2.3,

$$\int\limits_0^a \int\limits_0^b [xy(a-x)(b-y)]^{\frac{3}{2}} |\overline{p}(x,y)| \, dx\, dy < +\infty,$$

$$\int\limits_0^a \int\limits_0^b [xy(a-x)(b-y)]^{\frac{1}{2}} |\overline{q}(x,y)| \, dx\, dy < +\infty$$

and $\overline{p} \in \boldsymbol{U}_\beta(D)$. But according to the above-said, these conditions guarantee the existence and uniqueness of a solution $\overline{u} \in \widetilde{C}^{1,2}_{loc}(D) \cap C(\overline{D})$ of problem (1.1),(1.3). On the other hand, by Lemma 2.5, the solutions u and $\overline{u}$ satisfy conditions (1.6). $\square$

Proof of Theorem 1.2. Let

$$\alpha = \frac{4}{ab} \int\limits_0^a \int\limits_0^b \lambda_0(x,y)\, dx\, dy + \frac{1}{4}\lambda_1 + \frac{16}{a^2 b^2}\lambda_2.$$

By Theorem 1.1 and inequality (1.10), to prove Theorem 1.2 it is sufficient to establish that the function p satisfies condition (3.1).

Let u be an arbitrary funciton from $\boldsymbol{A}^{1,2}$. Then by Lemma 2.1, inequalities (2.1)–(2.3) are valid, where ρ is the number given by (2.4). If along with this we take into account inequality (1.9), then we get

$$\int_0^a\int_0^b [p(x,y)]_+ u^2(x,y)\,dx\,dy \le \int_0^a\int_0^b \lambda_0(x,y)\frac{u^2(x,y)}{xy(a-x)(b-y)}\,dx\,dy +$$

$$+\lambda_1\int_0^a\int_0^b \frac{u^2(x,y)}{xy(a-x)(b-y)}\,dx\,dy + \lambda_2\int_0^a\int_0^b \Big[\frac{u(x,y)}{xy(a-x)(b-y)}\Big]^2\,dx\,dy \le$$

$$\le \left[\frac{4}{ab}\int_0^a\int_0^b \lambda_0(x,y)\,dx\,dy + \frac{\lambda_1}{4} + \frac{16}{a^2b^2}\lambda_2\right]\rho^2 = \alpha\rho^2.$$

Hence, in view of the arbitrariness of u there follows inclusion (3.1). $\square$

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References

1. T. Kiguradze, On the correctness of the Dirichlet problem in a characteristic rectangle for fourth order linear hyperbolic equations. *Georgian Math. J.* **6**(1999), No. 5, 447–470.

2. C. Carathéodory, Vorlesungen über reelle Funktionen. *Chelsea Publ. Comp., New York,* 1968.

3. K. Deimling, A Carathéodory theory for systems of integral equations. *Ann. Mat. Pura Appl., IV. Ser.,* **86**(1970), 217–260.

4. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities. *Cambridge Univ. Press, New York,* 1934.

5. V. I. Levin, On inequalities. II. On a class of integral inequalities. (Russian) *Mat. Sb.* **4**(1938), 45–48.

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