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Unique Common Fixed Point Theorem for Three Pairs of Weakly Compatible Mappings in Complete G-metric Space

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Abstract

In this paper a unique common fixed point theorem has been proved for three pairs of weakly compatible mappings in complete G – metric space. This theorem is the extension of many other results existing in the literature. An example has been provided to validate the main result of this paper.

Keywords: *Common fixed point, Complete G – metric space, G – Cauchy sequence, Weakly compatible maps.*

1 Introduction

The concept of the commutativity has been generalized in several ways. S. Sessa, [11] has introduced the concept of weakly commuting whereas Gerald Jungck [5] initiated the concept of compatibility. It can be easily verified that

- When the two mappings are commuting then they are compatible but not conversely.
- Compatible mappings are more general than commuting and weakly commuting mappings.
- Compatible maps are weakly compatible but not conversely.

Many authors like [3], [4], [1] and [10] worked on compatible mappings in metric space.

Mustafa in collaboration with Sims [14] introduced a new notation of generalized metric space called G- metric space in 2006. He proved many fixed point results for a self mapping in G- metric space under certain conditions.

The main aim of this paper is to prove unique common fixed point theorem for three pairs of weakly compatible maps satisfying a new contractive condition in a complete G – metric space.

Now, we give preliminaries and basic definitions which are used through-out the paper.

Definition 1.1: *Let X be a non empty set, and let* $G : X \times X \times X \to R^+$ *be a function satisfying the following properties:*

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$
(G_2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *,with* $x \neq y$
(G_3) $G(x, x, y) \le G(x, y, z)$ *for all* $x, y, z \in X$ *, with* $y \neq z$
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ *(Symmetry in all three variables)*
(G_5) $G(x, y, z) \le G(x, a, a) + G(a, y, z)$ *, for all* $x, y, z, a \in X$ *(rectangle inequality)*

Then the function G is called a generalized metric space, or more specially a G-metric on X, and the pair (X, G) is called a G−metric space.

Definition 1.2: *Let* (X, G) *be a* G *- metric space and let* $\{x_n\}$ *be a sequence of points of* X *, a point* $x \in X$ *is said to be the limit of the sequence* $\{x_n\}$*, if* $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$*, and we say that the sequence* $\{x_n\}$ *is* G *- convergent to* x *or* $\{x_n\}$ G *-converges to* x.

Thus, $x_n \to x$ in a G - metric space (X, G) if for any $\in > 0$ there exists $k \in N$ such that $G(x, x_n, x_m) < \in$, for all $m, n \ge k$

Proposition 1.3: *Let* (X,G) *be a* G*- metric space. Then the following are equivalent:*

i) $\{x_n\}$ *is* G *- convergent to* x
ii) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$
iii) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$
iv) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$

Proposition 1.4: *Let* (X,G) *be a* G *- metric space. Then for any x, y, z, a in X it follows that*

i) *If* $G(x, y, z) = 0$ *then* $x = y = z$
ii) $G(x, y, z) \le G(x, x, y) + G(x, x, z)$
iii) $G(x, y, y) \le 2G(y, x, x)$
iv) $G(x, y, z) \le G(x, a, z) + G(a, y, z)$

Definition 1.5: *Let* (X,G) *be a* G *- metric space. A sequence* $\{x_n\}$ *is called a* G*- Cauchy sequence if for any* $\in > 0$ *there exists* $k \in N$ *such that* $G(x_n, x_m, x_l) < \in$ *for all* $m, n, l \ge k$ *, that is* $G(x_n, x_m, x_l) \to 0$ *as* $n, m, l \to +\infty$.

Proposition 1.6: *Let* (X,G) *be a* G*- metric space .Then the following are equivalent:*

i) *The sequence* $\{x_n\}$ *is* G *- Cauchy;*
ii) *For any* $\in > 0$ *there exists* $k \in N$ *such that* $G(x_n, x_m, x_m) < \in$ *for all* $m, n \ge k$

Proposition 1.7: *A* G*- metric space* (X,G) *is called* G*-complete if every* G*- Cauchy sequence is* G*-convergent in* (X,G).

Proposition 1.8: *Let* (X,G) *be a G-metric space. Then* $f : X \to X$ *is G-continuous at* $x \in X$ *, if and only if it is G-sequentially continuous at* x *, that is , whenever* $\{x_n\}$ *is G-convergent to* x *,* $\{f(x_n)\}$ *is G-convergent to* $f(x)$.

Definition 1.9: *Let* ***f*** *and g be two self – maps on a set X. Maps f and g are said to be commuting if* $fgx = gfx$ *, for all* $x \in X$

Definition 1.10: *Let* ***f*** *and g be two self – maps on a set X. If* $fx = gx$ *, for some* $x \in X$ *then x is called coincidence point of f and g.*

Definition 1.11[6] : *Let f and g be two self – maps defined on a set X, then f and g are said to be weakly compatible if they commute at coincidence points. That is if $fu = gu$ for some $u \in X$, then $fgu = gfu$.*

Lemma 1.12 [5]: *Let f and g be weakly compatible self mappings of a set X. If f and g have a unique point of coincidence , that is , $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 1.13: *A function $\phi : [0,\infty) \to [0,\infty)$ is said to be special phi function if it satisfies:*

i) $0 < \phi(t) < t$ *, for all* $t > 0$

ii) *The series* $\sum_{n\geq 1} \phi^n(t)$ *converges for all* $t > 0$

i.e. we may have $\lim_{n\to+\infty} \phi^n(t) = 0$ *for all* $t > 0$ *and*

iii) ϕ *is an upper semi continuous function.*

Definition 1.15: *A real valued function ϕ defined on $X \subseteq R$ is said to be upper semi continuous if $\lim_{n\to\infty}\sup \phi(t_n) \leq \phi(t)$, for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n \to \infty$.*

2 Main Result

Theorem 2.1: *Let (X, G) be a complete G - metric space and $A, B, C, L, M, N : X \to X$ be mappings such that*

I) $N(X) \subseteq A(X)$, $L(X) \subseteq B(X)$, $M(X) \subseteq C(X)$

II) $G(Lx, My, Nz) \leq \phi(\lambda(x, y, z))$ *, where ϕ is a special phi function and*

$$\lambda(x, y, z) = \max.\{G(Ax, By, Cz), G(Lx, Ax, Cz), G(My, By, Ax), G(Nz, Cz, By)\}$$

III) *The pairs (L, A), (M, B) and (N, C) are weakly compatible.*

Then A, B, C, L, M and N have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point of X and define the sequence $\{x_n\}$ in X such that

$$y_n = Lx_n = Bx_{n+1} \quad , \quad y_{n+1} = Mx_{n+1} = Cx_{n+2} \quad , \quad y_{n+2} = Nx_{n+2} = Ax_{n+3}$$

Consider, $G(y_n, y_{n+1}, y_{n+2}) = G(Lx_n, Mx_{n+1}, Nx_{n+2})$

$$\leq \phi(\lambda(x_n, x_{n+1}, x_{n+2}))$$

where

$$\lambda(x_n, x_{n+1}, x_{n+2}) = \max.\left\{ \begin{matrix} G(Ax_n, Bx_{n+1}, Cx_{n+2}), \\ G(Lx_n, Ax_n, Cx_{n+2}), G(Mx_{n+1}, Bx_{n+1}, Ax_n), G(Nx_{n+2}, Cx_{n+2}, Bx_{n+1}) \end{matrix} \right\}$$

$$= \max.\left\{ \begin{matrix} G(Nx_{n-1}, Lx_n, Mx_{n+1}), G(Lx_n, Nx_{n-1}, Mx_{n+1}), \\ G(Mx_{n+1}, Lx_n, Nx_{n-1}), G(Nx_{n+2}, Mx_{n+1}, Lx_n) \end{matrix} \right\}$$

$$= \max.\{G(y_{n-1}, y_n, y_{n+1}), G(y_n, y_{n-1}, y_{n+1}), G(y_{n+1}, y_n, y_{n-1}), G(y_{n+2}, y_{n+1}, y_n)\}$$

i.e. $\lambda(x_n, x_{n+1}, x_{n+2}) = \max.\{G(y_{n-1}, y_n, y_{n+1}), G(y_n, y_{n+1}, y_{n+2})\}$

Since ϕ is a phi function,

Therefore $\lambda(x_n, x_{n+1}, x_{n+2}) = G(y_n, y_{n+1}, y_{n+2})$ is not possible.

Therefore $G(y_n, y_{n+1}, y_{n+2}) \le \phi(G(y_{n-1}, y_n, y_{n+1}))$ -------------- (2.1.1)

Since ϕ is an upper semi continuous, special phi function, so equation (2.1.1) implies that the sequence $\{y_n\}$ is monotonic decreasing and continuous.

Hence there exists a real number say $r \ge 0$, such that $\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+2}) = r$
As $n \to \infty$, equation (2.1.1) implies that $r \le \phi(r)$, which is possible only if $r = 0$, because ϕ is a special phi function.

Therefore $\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+2}) = 0$ -------------- (2.1.2)

Now we show that $\{y_n\}$ is a Cauchy sequence.
We have,

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}) &\le \phi(G(y_{n-1}, y_n, y_{n+1})) \qquad \text{by (2.1.1)} \\ &\le \phi(\phi(G(y_{n-2}, y_{n-1}, y_n))) \\ &= \phi^2(G(y_{n-2}, y_{n-1}, y_n)) \\ &\quad , \\ &\quad , \\ &\quad , \\ &\le \phi^n(G(y_0, y_1, y_2)) \end{aligned}$$

By using (G_3) , (G_4), (G_5) and condition (2.1.1) for any $k \in N$, we write

$$G(y_n, y_{n+k}, y_{n+k}) \le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3})$$
$$+ - - - + G(y_{n+k-2}, y_{n+k-1}, y_{n+k-1}) + G(y_{n+k-1}, y_{n+k}, y_{n+k})$$

$$\le G(y_n, y_{n+1}, y_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+3}) + G(y_{n+2}, y_{n+3}, y_{n+4})$$
$$+ - - - + G(y_{n+k-2}, y_{n+k-1}, y_{n+k}) + G(y_{n+k-1}, y_{n+k}, y_{n+k+1})$$

$$\le \varphi^n (G(y_0, y_1, y_2)) + \varphi^{n+1} (G(y_0, y_1, y_2)) + \varphi^{n+2} (G(y_0, y_1, y_2))$$
$$+ - - - + \varphi^{n+k} (G(y_0, y_1, y_2))$$

$$= \sum_{i=n}^{n+k} \phi^i (G(y_0, y_1, y_2))$$

i.e. $G(y_n, y_{n+k}, y_{n+k}) \le \sum_{i=n}^{\infty} \phi^i (G(y_0, y_1, y_2))$ --------------- (2.1.3)

By definition of function phi, we have $\sum_{i=n}^{\infty} \phi^i (G(y_0, y_1, y_2))$ tends to 0 as $n \to \infty$

Therefore $\lim_{n\to\infty} G(y_n, y_{n+k}, y_{n+k}) = 0$, for all $k \in N$ ----------- (2.1.4)

This means that $\{y_n\}$ is a Cauchy sequence and since X is complete, therefore there exists a point $u \in X$, such that $\lim_{n\to\infty} y_n = u$

Therefore $\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} Bx_{n+1} = u$, $\lim_{n\to\infty} Mx_{n+1} = \lim_{n\to\infty} Cx_{n+2} = u$

and $\lim_{n\to\infty} Nx_{n+2} = \lim_{n\to\infty} Ax_{n+3} = u$

Since $N(X) \subseteq A(X)$, there exists a point $v \in X$ such that $u = Av$
Therefore by (II) we have,

$$G(Lv, u, u) \le G(Lv, Mx_{n+1}, u) + G(Mx_{n+1}, u, u)$$
$$\le G(Lv, Mx_{n+1}, Nx_{n+2}) + G(Nx_{n+2}, u, Mx_{n+1}) + G(Mx_{n+1}, u, u)$$
$$\le \phi(\lambda(v, x_{n+1}, x_{n+2})) + G(Nx_{n+2}, u, Mx_{n+1}) + G(Mx_{n+1}, u, u)) \quad \text{----- (2.1.5)}$$

Where,

$$\lambda(v, x_{n+1}, x_{n+2}) = \max.\begin{Bmatrix} G(Av, Bx_{n+1}, Cx_{n+2}), G(Lv, Av, Cx_{n+2}), \\ G(Mx_{n+1}, Bx_{n+1}, Av), G(Nx_{n+2}, Cx_{n+2}, Bx_{n+1}) \end{Bmatrix}$$

$$= \max.\{G(u, Lx_n, Mx_{n+1}), G(Lv, u, Mx_{n+1}), G(Mx_{n+1}, Lx_n, u), G(Nx_{n+2}, Mx_{n+1}, Lx_n)\}$$

Taking limit as $n \to \infty$ in the above relation, we get

$$\lambda(v, x_{n+1}, x_{n+2}) = \max.\{G(u,u,u), G(Lv,u,u), G(u,u,u), G(u,u,u)\}$$

Therefore $\lambda(v, x_{n+1}, x_{n+2}) = G(Lv,u,u)$

Thus as $n \to \infty$, we get from (2.1.5)

$$G(Lv,u,u) \le \phi(G(Lv,u,u)) + G(u,u,u) + G(u,u,u)$$

i.e. $G(Lv,u,u) \le \phi(G(Lv,u,u))$ -------------- (2.1.6)

If $Lv \ne u$, then $G(Lv,u,u) > 0$, and hence as ϕ is a special phi function

$$\phi(G(Lv,u,u)) < G(Lv,u,u)$$

Therefore from (2.1.6) we have $G(Lv,u,u) < G(Lv,u,u)$, which is a contradiction

$\therefore$ we must have $Lv = u$. So we have $Av = Lv = u$.

i.e. v is a coincidence point of L and A.

Since the pair of maps L and A are weakly compatible,

$\therefore$ $LAv = ALv$ i.e. $Lu = Au$

Again, since $L(X) \subseteq B(X)$, there exists a point $w \in X$ such that $u = Bw$

Therefore by (II) we have,

$$\begin{aligned} G(u,u,Mw) &= G(Lv,Lv,Mw) \qquad (\because G(x,x,y) \le G(x,y,z)) \\ &\le G(Lv,Mw,Nx_{n+2}) \\ &\le \phi(\lambda(v,w,x_{n+2})) \end{aligned}$$ -------------------- (2.1.7)

Where $\lambda(v,w,x_{n+2}) = \max.\left\{ \begin{matrix} G(Av,Bw,Cx_{n+2}), G(Lv,Av,Cx_{n+2}), \\ G(Mw,Bw,Ax_{n+2}), G(Nx_{n+2},Cx_{n+2},Bw) \end{matrix} \right\}$

$$= \max.\{G(u,u,Mx_{n+1}), G(u,u,Mx_{n+1}), G(Mw,u,Nx_{n+1}), G(Nx_{n+2},Mx_{n+1},u)\}$$

Taking limit as $n \to \infty$, we get

$\lambda(v,w,x_{n+2}) = \max.\{G(u,u,u), G(u,u,u), G(Mw,u,u), G(u,u,u)\}$
Therefore $\lambda(v,w,x_{n+2}) = G(Mw,u,u) = G(u,u,Mw)$

Therefore from (2.1.7), we get $G(u,u,Mw) \le \phi(G(u,u,Mw))$ ------------ (2.1.8)

If $Mw \ne u$, then $G(u,u,Mw) > 0$ and hence as ϕ is a special phi function,

$\phi(G(u,u,Mw)) < G(u,u,Mw)$

Therefore by using (2.1.8), we get, $G(u,u,Mw) < G(u,u,Mw)$, which is a contradiction.

Hence we have $Mw = u$. Thus we have $Mw = Bw = u$ i.e. w is a coincidence point of M and B.
Since the pair of maps M and B are weakly compatible,

$\therefore\ MBw = BMw$ i.e. $Mu = Bu$

Now again, since $M(X) \subseteq C(X)$, there exists a point $p \in X$, such that $u = Cp$
Therefore by (II), we have,

$$G(u,u,Np) = G(Lv,Mw,Np) \le \phi(\lambda(v,w,p)) \quad \text{----------------- (2.1.9)}$$

Where

$$\lambda(v,w,p) = \max.\{G(Av,Bw,Cp), G(Lv,Av,Cp), G(Mw,Bw,Av), G(Np,Cp,Bw)\} = \max.\{G(u,u,u), G(u,u,u), G(u,u,u), G(Np,u,u)\}$$

Therefore $\lambda(v,w,p) = G(Np,u,u) = G(u,u,Np)$

Therefore from (2.1.9), we have $G(u,u,Np) \le \phi(G(u,u,Np))$ -------- (2.1.10)

If $Np \ne u$, then $G(u,u,Np) > 0$ and hence as ϕ is a special phi function,

$\phi(G(u,u,Np)) < G(u,u,Np)$

Therefore from (2.1.10) we get, $G(u,u,Np) < G(u,u,Np)$, which is a contradiction.

Hence we must have $Np = u$ .Thus we have $Np = Cp = u$ i.e. p is a coincidence point of N and C. Since the pair of maps N and C are weakly compatible,

$\therefore\ NCp = CNp$ i.e. $Nu = Cu$

Now we show that 'u' is a fixed point of L .

By (II), we have $G(Lu,u,u) = G(Lu,Mw,Np)$

$\leq \phi(\lambda(u,w,p))$ ----------- (2.1.11)

Where

$$\lambda(u,w,p) = \max.\{G(Au,Bw,Cp),G(Lu,Au,Cp),G(Mw,Bw,Au),G(Np,Cp,Bw)\}$$
$$= \max.\{G(Lu,u,u),G(Lu,Lu,u),G(u,u,Lu),G(u,u,u)\}$$
$$= G(Lu,u,u) \quad \text{--------- by (iv) of Proposition 1.4}$$

Therefore from (2.1.11), we have, $G(Lu,u,u) \leq \phi(G(Lu,u,u))$ ------ (2.1.12)

If $Lu \neq u$, then $G(Lu,u,u) > 0$ and hence as ϕ is a special phi function,

$\therefore\ \phi(G(Lu,u,u)) < G(Lu,u,u)$

Therefore (2.1.12) implies that $G(Lu,u,u) < G(Lu,u,u)$, which is a contradiction.

Hence we have $Lu = u$. So we get $Lu = Au = u$.

Now, we show that u is a fixed point of M.

Therefore by (II) we have, $G(u,u,Mu) = G(Lu,Np,Mu)$

$= G(Lu,Mu,Np)$

$\leq \phi(\lambda(u,u,p))$ ------------- (2.1.13)

Where

$$\lambda(u,u,p) = \max.\{G(Au,Bu,Cp),G(Lu,Au,Cp),G(Mu,Bu,Au),G(Np,Cp,Bu)\}$$
$$= \max.\{G(Lu,Mu,u),G(Lu,Lu,u),G(Mu,Mu,fu),G(u,u,Mu)\}$$
$$= \max.\{G(u,Mu,u),G(u,u,u),G(Mu,Mu,u),G(u,u,Mu)\}$$
$$= G(u,u,Mu) \quad \text{--------- by (iv) of Proposition 1.4}$$

So from (2.1.13) we get, $G(u,u,Mu) \leq \phi(G(u,u,Mu))$ ------------ (2.1.14)

If $Mu \neq u$, then $G(u,u,Mu) > 0$ and hence as ϕ is a special phi function,

$\phi(G(u,u,Mu) < G(u,u,Mu)$

Thus from (2.1.14) we get, $G(u,u,Mu) < G(u,u,Mu)$, which is a contradiction.

Therefore $Mu = u$. Hence $Mu = Bu = u$

Now we show that u is a fixed point of N.
Therefore from (II) we have, $G(u,u,Nu) = G(Lu,Mu,Nu)$

$$\leq \phi(\lambda(u,u,u)) \quad \text{-------------} \ (2.1.15)$$

Where

$$\begin{aligned}\lambda(u,u,u) &= \max.\{G(Au,Bu,Cu), G(Lu,Au,Cu), G(Mu,Bu,Au), G(Nu,Cu,Bu)\}\\ &= \max.\{G(u,u,Nu), G(u,Lu,Nu), G(u,Mu,Lu), G(Nu,Nu,Mu)\}\\ &= \max.\{G(u,u,Nu), G(u,u,Nu), G(u,u,u), G(Nu,Nu,u)\}\\ &= G(u,u,Nu) \quad \text{---------} \quad \text{by (iv) of Proposition 1.4}\end{aligned}$$

Thus from (2.1.15) we have, $G(u,u,Nu) \leq \phi(G(u,u,Nu))$ --------- (2.1.16)

If $Nu \neq u$, then $G(u,u,Nu) > 0$ and hence as ϕ is a special phi function,

$\phi(G(u,u,hu)) < G(u,u,hu)$

Thus by using (2.1.16) we get, $G(u,u,Nu) < G(u,u,Nu)$, which is a contradiction.

Hence $Nu = u$. Thus we have $Nu = Cu = u$

Therefore $Lu = Au = Mu = Bu = Nu = Cu = u$ i.e. u is a common fixed point of L, A, M, B, N and C.

Now we show that 'u' is unique common fixed point of L, A, M, B, N and C.

If possible, let us assume that 'm' is another common fixed point of L, A, M, B, N and C.

By using (II) we have, $G(u,u,m) = G(Lu,Mu,Nm)$

$$\leq \phi(\lambda(u,u,m)) \quad \text{---------------} \ (2.1.17)$$

Where

$$\begin{aligned}\lambda(u,u,m) &= \max.\{G(Au,Bu,Cm), G(Lu,Au,Cm), G(Mu,Bu,Au), G(Nm,Cm,Bu)\}\\ &= \max.\{G(u,u,m), G(u,u,m), G(u,u,u), G(m,m,u)\}\\ &= G(u,u,m) \quad \text{---------} \quad \text{by (iv) of Proposition 1.4}\end{aligned}$$

Thus from (2.1.17) we have, $G(u,u,m) \leq \phi(G(u,u,m))$ -------------- (2.1.18)

If $u \neq m$, then $G(u,u,m) > 0$ and hence as ϕ is a special phi function, $\phi(G(u,u,m)) < G(u,u,m)$

Hence from (2.1.18) we get, $G(u,u,m) < G(u,u,m)$, which is a contradiction.

Hence we have $u = m$.
Thus 'u' is the unique common fixed point of L, A, M, B, N and C.

Example 2.2: *Let* $X = [0,\infty)$ *and* G *be a mapping defined on* X *as*

$$G(x,y,z) = |x-y| + |y-z| + |z-x| \text{ , for all } x,y,z \in X .$$

Then G *is a complete* G*- metric on* X *and* (X,G) *is a complete* G*-metric space.*

Let $A,B,C,L,M,N : X \to X$ be defined as $Ax = \frac{x}{3}$, $Tx = \frac{x}{6}$, $Cx = \frac{x}{9}$, $Lx = \frac{x}{24}$, $Mx = \frac{x}{36}$

and $Nx = \frac{x}{12}$ then (i) $N(X) \subseteq A(X)$, $L(X) \subseteq B(X)$, $M(X) \subseteq C(X)$

(ii) The pairs (L,A), (M,B) and (N,C) are weakly compatible.
(iii) Also $G(Lx,My,Nz) \le \phi(\lambda(x,y,z))$

Where

$$\lambda(x,y,z) = \max.\{G(Ax,By,Cz), G(Lx,Ax,Cz), G(My,By,Ax), G(Nz,Cz,By)\}$$

Then '0' is unique common fixed point of L, A, M, B, N and C in X .

Corollary 2.3: *Let* (X,G) *be a complete* G *- metric space and* $A,L,M,N : X \to X$ *be mappings such that*

I) $N(X) \subseteq A(X)$, $L(X) \subseteq A(X)$, $M(X) \subseteq A(X)$

II) $G(Lx,My,Nz) \le \phi(\lambda(x,y,z))$, *where* ϕ *is a special phi function and* $\lambda(x,y,z) = \max.\{G(Ax,Ay,Az), G(Lx,Ax,Az), G(My,Ay,Ax), G(Nz,Az,Ay)\}$

III) The pairs (L,A), (M,A) *and* (N,A) *are weakly compatible.*

Then A, L, M and N have a unique common fixed point in X.

Proof: By taking $A = B = C$ in **Theorem 2.1** we get the proof.

Corollary 2.4: *Let* (X,G) *be a complete* G *- metric space and* $A,L : X \to X$ *be mappings such that*

I) $L(X) \subseteq A(X)$

II) $G(Lx, Ly, Lz) \le \phi(\lambda(x, y, z))$, *where* ϕ *is a special phi function and* $\lambda(x, y, z) = \max.\{G(Ax, Ay, Az), G(Lx, Ax, Az), G(Ly, Ay, Ax), G(Lz, Az, Ay)\}$

III) *The pair* (L, A) *is weakly compatible.*

Then A, L have a unique common fixed point in X.

Proof: By taking $A = B = C$ & $L = M = N$ in **Theorem 2.1** we get the proof.

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