Gen. Math. Notes, Vol. 1, No. 2, December 2010, pp. 170-184
ISSN 2219-7184; Copyright © ICSRS Publication, 2010
www.i-csrs.org
Available free online at http://www.geman.in

A Class of Constrained Time Optimal Control Problems in 2- Banach Spaces

[1]Utpalendu Adak and [2]Hora Krishna Samanta

[1]Jadavpur University, Department of Mathematics
Kolkata-700032, West Bengal, India.
E-mail: utpalendu_adak@yahoo.in,
utpalendu_adak@rediffmail.com.

[2]Netaji Mahavidyalaya, Department of Mathematics,
Arambagh, India.

(Received 25.10.2010, Accepted 26.11.2010)

Abstract

In this paper the authors have used certain fundamental concept of functional analysis to tackle a class of constrained time optimal control problems. A class of constrained time optimal control problems has been solved in 2-Banach space setting. An example is exhibited to show the technique of application of the control theory in generalized 2-normed spaces.

Keywords and Phrases: Time optimal control, 2-Banach Space, generalized 2-norm, seminorm, reflexive space, Hahn-Banach.

2000 MSC: 40A05, 46A45, 46B70.

1 Introduction

Minimum time optimal control problem has been solved by different authors using functional analysis technique in Banach Space setting. Minamide and Nakamura [8,9] considered a related problem where the objective function was a continuous convex functional. Choudhury and Mukherjee [1,11,12] developed a uniform theory of time optimal control problem for system which can be represented in terms of linear, bounded and onto transformation from a Banach space of control function to another Banach space. Recently, the concept of 2-Banach spaces has been developed. Many authors like Acikgoz [7]; Lewandowska, Moslehian and Saadatpour [24,25]; Freese and Cho [10]; Cho, Kim and Misiak [23]; Reddy and Dutta [3,5]; Park [4]; Som [14] have developed a uniform theory in 2-Banach space. Optimization in 2-Banach space setting is an important area of application of functional analysis. So, it may be worthwhile to make an attempt to develop an optimization theory in 2-Banach space. In this paper, we have developed a class of constrained time optimal control problems in 2-Banach space.

The control systems, which can be characterized by the following vector matrix differential equation:

$$\frac{dx}{dt} = A(t)\,X(t) + B(t)\,U(t) \qquad (1),$$

where X(t) is an n vector, representing the instantaneous state of the system, u(t) is an r-vector (r≤n) representing the control input to the system, A(t) is (n × n) matrix and B(t) is an (n × r) matrix has received considerable attention in the literature. The solution of the above equation can be expressed in the following integral forms:

$$X(t) = \varphi(t, t_0)\,X(t_0) + \int_{t_0}^{t} \varphi(t,s)\,B(s)\,U(s)\,ds \qquad (2),$$

where ϕ (t, t_0) is the fundamental matrix of the system (1), and $x(t_0)$, the initial state of the system at time $t = t_0$. The minimum time control problem, is to find the optimal control u(t) belonging to the admissible set, which will drive the systems from a given initial state $x(t_0)$ at $t = t_{0,}$ to the desired state x_1 in minimum time t i.e. $x(t) = x_1$. Now (2) can be written as $X(t) - \varphi(t, t_0)\,X(t_0) = \int_{t_0}^{t} \varphi(t,s)\,B(s)\,U(s)\,ds$.

Put $X(t) - \varphi(t, t_0)\,X(t_0) = \xi$. Expression (2) can be written as $\xi = T_t\, u$, where

$T_t u = \int_{t_0}^{t} \varphi(t,s)\,B(s)\,U(s)\,ds$. Thus without any loss of generality one can consider the problems of finding the optimal u to drive the system from the origin to any point ξ in minimum time t.

The above problem can be considered as a mapping from some space to which u belongs to some other space ξ belongs. In the light of the above we can consider following general problem:

Let B_t be a 2-Banach space depending on the parameter t and D be also a 2-Banach space. Let T_t be a bounded linear transformation depending on the parameter mapping B_t onto D. The problem is to find the optimal control $u \in B_t$ to reach ξ from the origin in minimum time t under the constraint $N_1\{(u,u_1): u,u_1 \in B_t\} \leq 1$ where $N_1(.,.)$ denotes the 2-norm function defined on B_t.

2. Some Preliminaries:

Definition of 2-Normed space 2.1: Let B_t be a vector space of dimension greater than one over F, where F is the real or complex number field. Suppose $N_1(.,.)$ be a non negative real valued function on $B_t \times B_t$ which satisfies the conditions: (i) $N_1(u_i,u_j)=0$ if and only if u_i and u_j are linearly dependent vectors, (ii)$N_1(u_i,u_j)= N_1(u_j,u_i)$ for all $u_i,u_j \in B_t$, (iii) $N_1(\lambda u_i,u_j)=|\lambda|\ N_1(u_i,u_j)$ for all $\lambda \in F$ and for all $u_i,u_j \in B_t$, (iv) $N_1(u_i+u_j,z) \leq N_1(u_i,z)+ N_1(u_j,z)$ for all $u_i,u_j,z \in B_t$. Then $N_1(.,.)$ is called a 2-norm function defined on B_t and $(B_t, N_1(.,.))$ is called a linear 2-normed space.

A sequence $\{u_n\}_{n\geq 1}$ in a linear 2-normed space B_t is called Cauchy sequence if there exist two linear independent elements y and z in B_t such that $\{N_1(u_n, y)\}$ and $\{N_1(u_n, z)\}$ are real Cauchy sequence, i.e., $\lim_{m,n}\{N_1(x_m - x_n, y)\} = 0$ and $\lim_{m,n}\{N_1(x_m - x_n, z)\} = 0$

A sequence $\{u_n\}_{n\geq 1}$ in a linear 2-normed space $(B_t, N_1(.,.))$ is called convergent if there exists $u \in B_t$ such that $\lim\{N_1(x_n - x, y)\}_{n\geq 1} \to 0$ $\forall\ y \in B_t$, i.e., $\lim\{N_1(x_n - x, y)\}_{n\geq 1} = 0$ $\forall\ y \in B_t$.

A 2-normed space $(B_t,N_1(.,.))$ is called a 2-Banach space if every Cauchy sequence is convergent. Also if B_t and D are 2-Banach spaces over the field of real numbers, it can be verified that $B_t \times D$ is also 2-Banach space with respect to the 2-norm $N_3(.,.)$ where $N_3\{(u_i,v_i),(u_j,v_j)\}= \min\{N_1(u_i,u_j),N_2(v_i,v_j)\}$, i.e. $N_3(.,.)=\min\{N_1(.,.), N_2(.,.)\}$; $N_1(.,.)$ and $N_2(.,.)$ are 2-norm functions defined on the spaces B_t and D respectively and $N_3\{(u_i,v_i),(u_j,v_j)\}=0$ iff either u_i, u_j are linearly dependent (L.D.) in B_t or v_i,v_j are linearly dependent in D.

Let N_1', N_2', N_3' are the 2-norm functions defined on the spaces $B_t', D', (B_t \times D)'$ respectively, where $N_3'(.,) = \min\{N_1'(.,.), N_2'(.,)\}$ and B_t' denotes the conjugate of B_t. Let B_t be the conjugate of X_t and D be the conjugate of Y. Then $B_t = X_t'$ and $D=Y'$. Let $\phi:D\to R$ & $f: X\to R$ be two functionals. Then $\phi \in D'$, $f \in X'$; $f_1 \in B_t^*$.

Example 2.1: For $X=R^3$, define:
$N_1(x,y)=\max\{|x_1y_2-x_2y_1|+|x_1y_3-x_3y_1|, |x_1y_2-x_2y_1|+|x_2y_3-x_3y_2|\}$, where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)\in R^3$. Then $N_1(.,.)$ is a 2-norm on R^3. See more details Freese [10], Acikgoz [7].
For examples of some known 2-normed spaces, generalized 2-normed space; see Adak [15]-[22].
Definition 2.2: Let X and Y be real linear spaces. Denote by D a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets $D_x = \{y \in Y:(x,y) \in D\}$ and $D^y = \{x \in X:(x,y) \in D\}$ are linear subspaces of the spaces Y and X respectively. A function $N_5(.,.):D \to [0, \infty)$ will be called a generalized 2-norm on D if it satisfies the conditions: (i) $N_5(x, \alpha y) = |\alpha| N_5(x, y) = N_5(\alpha x, y)$ for any real number α and all $(x ,y) \in D$; (ii)$N_5(x, y + z) \le N_5(x, y) + N_5(x, z)$ for $x \in X$, $y, z \in Y$ with $(x, y), (x, z) \in D$; (iii) $N_5(x + y, z) \le N_5(x, z) + N_5(y, z)$ for $x,y \in X$, $z \in Y$ with (x, z) $(y, z) \in D$. Then D is called a 2-normed set.
In particular, if $D = X \times Y$, the function $N_5(.,.)$ is said to be a generalized 2-norm on $X \times Y$ and the pair $(X \times Y, N_5(.,.))$ is called a generalized 2-normed space.
Unfortunately, there is no connection between normed spaces and 2-normed spaces, but in 1999 in order to introduce some connections between normed spaces and 2-normed spaces, Lewandowska [24] introduced generalized 2-normed spaces, as a subspace of 2-normed spaces.

If X = Y, then the generalized 2-normed space $(X \times X, N_1(.,.))$ is denoted by $(X, N_1(.,.))$. In the case that X = Y, $D = D^{-1}$, where $D^{-1} = \{(y, x) : (x, y) \in D\}$, and $N_5(x, y) = N_5(y, x)$ for all $(x, y) \in D$, we call $N_5(.,.)$ a generalized symmetric 2-norm function defined on $X\times X$ and D a symmetric 2-norm set.

Also let (X, N(.)) be a normed space. Then $N_1(x, y) = N(x). N(y)$ for all $x, y \in X$ is a 2-norm function defined on $X \times X$. So, $(X, N_1(.,.))$ is a generalized 2-normed space.
If we take as N(x)=N(y), our generalized 2-normed space will be a generalized symmetric 2-normed space with the symmetric 2-norm defined by $N_1(x, y) = N(x). N(y)$ for all $x, y \in X$. Let us remark that a symmetric 2-normed space need not be a 2-normed space in the sense of Gahler [13]. For instance given above, $x\neq\theta$, $y=kx$, $k\neq 0$, we obtain $N_1(x,y)=N_1(x, kx)=|k|N_1(x,x)>0$, but inspite of this x and y are linearly dependent. So from this, we say that the 2-normed space is not a 2-normed space in the sense of Definition 2.1. Each 2-normed space is a generalized 2-normed space. But, in case of X = Y, $D = D^{-1}$; the generalized 2-normed space is a 2-normed space.

Throughout the paper, $N_1, N_2, N_3, N_1', N_2', N_3'$ denote the 2-norm functions defined on the spaces $B_t, D, (B_t \times D), B_t', D', (B_t \times D)'$ respectively which are defined earlier in Definition 2.1.

Problem Statement

In this paper we shall consider the problem where the constraints on the control function are given as: $|u_\ell| \cdot |u_m| \le N$,

$$\left\{\int_0^t |u_r(\tau)|^2 d\tau\right\}^{\frac{1}{2}} \cdot \left\{\int_0^t |u_s(\tau)|^2 d\tau\right\}^{\frac{1}{2}} \le M,$$ M and N being positive constraints. The problem is to find the optimal control function u which will drive the origin (initial state) to ξ (desired state) in minimum time t, satisfying the above constraints.

For the sake of completeness, we shall now give certain Definitions, Theorems and Lemmas.

Deffinition: Let $U_{X_t}=\{x_t: N_1(\alpha,x)\le 1, x\in B_t\}, \alpha\in X_t, \alpha\ne\theta$; $U_Y=\{y:N_2(\beta,y)\le 1, y\in D\}$, $\beta\in Y$, $\beta\ne\theta$ be the unit balls in B_t, D respectively.

Deffinition: The set of all points $\xi \in D$, such that $T_t u = \xi$ for some $u \in U_t \subset B_t$ will be called the Reachable set and will be denoted by C(t), where U_t is the unit ball in B_t, for some given time t.

Deffinition: Let X be a 2-Normed linear space. A non-negative real valued function $\rho(.,.)$ on $X\times X$ is called a seminorm if it satisfies the conditions:(i) $\rho(x_i+x_j,z)\le\rho(x_i,z)+\rho(x_j,z)$ $\forall$ x_i, x_j, $z\in X$, (ii) $\rho(\lambda x_i,x_j)=|\lambda|\rho(x_i,x_j)$ for all $\lambda\in F$ and for all $x_i, x_j, \in X$.

Deffinition: Let X be a 2-Normed linear space. A 2-norm $\rho_1(.,.)$ on $X\times X$ is said to be equivalent to a 2-norm $\rho_2(.,.)$ on $X\times X$ if there are positive numbers a and b such that $a\,\rho_2(x_i,x_j)\le\rho_1(x_i,x_j)\le b\,\rho_2(x_i,x_j)$. In following theorems B_t, T_t, D will mean the same as define earlier, until they are specially defined.

Theorem 1: If B_t and D be the conjugate spaces of the 2-Normed linear spaces X_t and Y respectively and T_t is the adjoint of some bounded linear transformation S, mapping Y one to one and on to a closed subspace of X_t, then C(t) is closed.

Proof: By [18] (Corollary 2.1) the unit ball in X_t^* is weak* compact. Also, both X_t^* and D are equipped with their weak* topologies. Again, as T_t is adjoint to S, T_t will also be onto and remains continuous with respect to weak* topologies of

X_t^* and D. Consequently, the unit ball of X_t^* will be mapped onto a weak* compact subset of D. Hence C(t) is weak* closed and therefore weakly closed and hence norm closed in D.

Note: Let X be a 2-normed linear space and X^* be its conjugate. Hahn-Banach theorem [16,17] assures that that $N_1\{(x_i, x_j) : x_i, x_j \in X\} \neq 0$. Then there exists a real bounded 2-linear functional $F \in X^*$, defined on the whole space, such that $F(x_i,x_j) = N_1\{(x_i, x_j) : x_i, x_j \in X\}$ and

$$\sup_{x,\, y \text{ are not L.D.}} \frac{\left|F(x_i, x_j)\right|}{N_1\{(x_i, x_j) : x_i, x_j \in X\} \neq 0} = 1 .$$ Such an F will be called an extremal of x.

Note [16,18]: The Reachable set is also convex body, symmetric with respect to the origin of D.

Theorem 2: Let B_t be the conjugate space of the 2-normed linear space X_t and D is the conjugate of some 2-normed linear space Y. Let $\xi \in \delta C(t)$, where $\delta C(t)$ denotes the boundary of C(t) for some given time t. Then there exists at least one $u_\xi(t) \in U_t \subset B_t$ which will transfer the system from origin to $\xi \in \delta C(t)$ in minimum time t, where T_t is an in Theorem 1.

Proof: As Y is reflexive [17], D=Y* is evidently a reflexive space. Now, $S: Y \to X_{t^*}$ implies $S^* : X^*_{t^*} \to Y^*$ that is, $S^* : B_{t^*} \to D$. since Y is reflexive $S^{**} : Y \to B^*_{t^*}$. Therefore $S^{**} = S^*$. But $S^* = \bar{T}_t{}^*$ (by hypothesis). Hence $S^{**} = T^*_{t^*}$. Consequently $S^{**} = S = T^*_{t^*}$. Again $S : Y \to X_{t^*}$ i.e. $S : D^* \to X_{t^*}$. If $\phi \in D^*$ then $S\phi \in X_{t^*}$ and so $\bar{S}\varphi \in X^*_{t^*}$ where $\bar{S}\varphi$ denotes the extremal of $S\phi$ i.e. $\overline{T^*_{t^*}}\varphi \in X^*_{t^*} = B_{t^*}$ with $N_1\{(T^*_t\varphi, f) : T^*_t\varphi, f \in B^{**}_t\} = 1$. Now if t* is the minimum time to reach ξ, then $\xi \in \partial C(t^*)$. Let $\phi\xi \in D^*$ be the supporting hyper plane to $\partial C(t^*)$ at ξ let u_ϕ be optimal control to reach ξ in minimum time t*, then $u_\varphi = \overline{T^*_{t^*}}\varphi, N_1\{(u_\varphi, u_1) : u_\varphi, u_1 \in U_t\} = 1$. Thus $u_\varphi \in B_{t^*}$. See [16,18] for determining $\phi\xi$ and t* for a given ξ. Let N'_1, N'_2, N'_3 are the 2-norms of the spaces $X^*_t, Y^*, (X_t \times Y)^*$ respectively, where X* denotes the conjugate space of X.

Theorem 3: On a finite dimensional 2-normed linear space X, any 2-norm $\rho_1(.,.)$ is equivalent to any other 2-norm $\rho_2(.,.)$.

Remark 1: If D is finite dimensional, then S always exist. We state the following lemmas which can be easily proved.

Lemma 1: Let X be a 2-normed linear space. If $\rho_1(x)$ and $\rho_2(x)$ are the seminorm and 2-norms respectively in X, then, $\text{Max}\{\rho_1(x), \rho_2(x)\}$ is a 2-norm in X, where x $\in$ X.

Corollary: Evidently $\text{Max}\{\rho_1(x), \rho_2(x)\}$ is a 2-norm, where each of $\{\rho_1(x), \rho_2(x)\}$ is a 2-norm.

Lemma 2:

$$\rho_2(u_\ell, u_m) = \text{Max}\left\{\operatorname*{ess\,sup}_{0 \le \tau \le t} \frac{|u_\ell(\tau)|}{N}, \frac{1}{M}\left(\int_0^t |u_\ell(\tau)| \, d\tau\right)^{1/2}\right\} \cdot \text{Max}\left\{\operatorname*{ess\,sup}_{0 \le \tau \le t} \frac{|u_m(\tau)|}{N}, \frac{1}{M}\left(\int_0^t |u_m(\tau)| \, d\tau\right)^{1/2}\right\}$$

is equivalent to $\rho_1(u_i, u_j) = \operatorname*{ess\,sup}_{0\le\tau\le t} |u_i(\tau)| \cdot \operatorname*{ess\,sup}_{0\le\tau\le t} |u_j(\tau)|$ which is a 2-norm on $L_\infty(0,t)$.

Proof: We have

$$\frac{1}{M}\left(\int_0^t |u(\tau)|^2 \, d\tau\right)^{1/2} \le \frac{1}{M}\operatorname*{ess\,sup}_{0\le\tau\le t} |u(\tau)| \cdot \sqrt{t} = \frac{N}{M}\operatorname*{ess\,sup}_{0\le\tau\le t} |u(\tau)| \cdot \sqrt{t}$$

We shall consider two cases, case (i) and case (ii), and two subcases of case (ii).

Case (i): If $t \le \dfrac{M^2}{N^2}$**,**

$$\frac{1}{M}\left(\int_0^t |u_r(\tau)|^2 \, d\tau\right)^{1/2} \cdot \frac{1}{M}\left(\int_0^t |u_s(\tau)|^2 \, d\tau\right)^{1/2} \le \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{|u_\ell(\tau)|}{N} \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{|u_m(\tau)|}{N}$$

$$\therefore \rho_2(u_\ell, u_m) = \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{|u_\ell(\tau)|}{N} \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{|u_m(\tau)|}{N} = \rho_1(u_i, u_j) \text{ for } t \le \frac{M^2}{N^2} \qquad (3).$$

Hence $\rho_2(u_\ell, u_m)$ is equivalent to $\rho_1(u_i, u_j)$ for $t \le \dfrac{M^2}{N^2}$.

Case (ii): $t > \dfrac{M^2}{N^2}$. There will be two subcases:

(a)

$$\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_\ell(\tau)|}{N}\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_m(\tau)|}{N}=\frac{1}{M}\left(\int_0^t|u_r(\tau)|^2\,d\tau\right)^{1/2}\cdot\frac{1}{M}\left(\int_0^t|u_s(\tau)|^2\,d\tau\right)^{1/2}$$

on a set of finite measure.

(b)

$$\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_\ell(\tau)|}{N}\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_m(\tau)|}{N}<\frac{1}{M}\left(\int_0^t|u_r(\tau)|^2\,d\tau\right)^{1/2}\cdot\frac{1}{M}\left(\int_0^t|u_s(\tau)|^2\,d\tau\right)^{1/2}$$

almost everywhere.

We make use of the following notations:

$$\rho_3(u_p,u_q)=\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_p(\tau)|}{N}\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_q(\tau)|}{N},\ \rho_4(u_r,u_s)=\frac{1}{M}\left(\int_0^t|u_r(\tau)|^2\,d\tau\right)^{1/2}\cdot\frac{1}{M}\left(\int_0^t|u_s(\tau)|^2\,d\tau\right)^{1/2}.$$

Obviously $\rho_3(u_p,u_q)$ and $\rho_4(u_r,u_s)$ are 2-norms and they are equivalent to $\rho_1(u_i,u_j)$ and $\rho_2(u_\ell,u_m)$ respectively.

In case (ii) (a): $\rho_2(u_\ell,u_m)=\rho_3(u_p,u_q)=\rho_4(u_r,u_s)$,

$\therefore \rho_2(u_\ell,u_m)\le\rho_3(u_p,u_q)\le\rho_2(u_\ell,u_m)$ (4).

In case (ii) (b): $\rho_2(u_\ell,u_m)=\rho_4(u_r,u_s)\le\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_p(\tau)|}{N}\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_q(\tau)|}{N}\cdot\sqrt{t}$

or

$$\frac{M}{N\sqrt{t}}\rho_2(u_\ell,u_m)\le\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_p(\tau)|}{N}\operatorname*{ess\,sup}_{0\le\tau\le t}\frac{|u_q(\tau)|}{N}=\rho_3(u_p,u_q)<\rho_4(u_r,u_s)=\rho_1(u_i,u_j)\qquad(5).$$

Combining (3), (4) and (5) we obtain

$$\operatorname{Max}\left\{1,\frac{M}{N\sqrt{t}}\right\}\rho_2(u_\ell,u_m)\le\rho_3(u_p,u_q)\le\rho_2(u_\ell,u_m)\qquad(6).$$

But $\rho_3(u_p,u_q)$ is obviously equivalent to $\rho_1(u_i,u_j)$. Hence from (6) $\rho_2(u_\ell,u_m)$ is equivalent to $\rho_1(u_i,u_j)$. Hence the proof.

Definition: We define $L_{\infty,N,M}$ to be the space of all essentially bounded functions u, equipped with the 2-norm $\rho_2(u_\ell,u_m)$.

Definition: We define $L_{\infty,N}$ to be the space of all essentially bounded functions u, equipped with the 2-norm $\rho_3(u_p, u_q) = \operatorname{ess\,sup}_{0\le\tau\le t} \frac{|u_p(\tau)|}{N} \operatorname{ess\,sup}_{0\le\tau\le t} \frac{|u_q(\tau)|}{N}$.

Definition: The space L_M consist of all square integrable functions u, equipped with the 2-norm $\rho_4(u_r, u_s) = \frac{1}{M}(\int_0^t |u_r(\tau)|^2 \, d\tau)^{1/2} \cdot \frac{1}{M}(\int_0^t |u_s(\tau)|^2 \, d\tau)^{1/2}$.

Note: Evidently $\rho_3(u_p, u_q)$ and $\rho_4(u_r, u_s)$ are equivalent to $\rho_1(u_i, u_j)$ and $\rho_2(u_\ell, u_m)$ respectively and hence the space $L_{\infty,N}$ and $L_{\infty,M}$ are complete with respect to their respective 2-norms $\rho_3(u_p, u_q)$ and $\rho_4(u_r, u_s)$.

Consider a system described by (1) where u(t) is a scalar control. Assume that at t = 0 the state of the system is given be x(0). It is required to find u(t) which will bring the system from the initial state x(0) to the origin of the state space in the least time under the constraint

$$|u_\ell(\tau)| \cdot |u_m(\tau)| \le N, (\int_0^t |u_r(\tau)|^2 \, d\tau)^{\frac{1}{2}} \cdot (\int_0^t |u_s(\tau)|^2 \, d\tau)^{\frac{1}{2}} \le M.$$

The above constraints can be expressed in the following alternative form:

$$J(u_\ell, v_m) = \operatorname{Max}\left\{ \operatorname{ess\,sup}_{0\le\tau\le t} \frac{|u_\ell(\tau)|}{N}, \frac{1}{M}(\int_0^t |u_\ell(\tau)| \, d\tau)^{1/2} \right\} \cdot \operatorname{Max}\left\{ \operatorname{ess\,sup}_{0\le\tau\le t} \frac{|u_m(\tau)|}{N}, \frac{1}{M}(\int_0^t |u_m(\tau)| \, d\tau)^{1/2} \right\}$$

From Lemma 1, it follows that $J(u_\ell, v_m)$ is a 2-norm in $L_{\infty,N,M}$.

Now $L_{\infty,N,M}$ can be considered as the conjugate of the space $L_{1,N,M}$ i.e. $L^*_{1,N,M} = L_{\infty,N,M}$ where $*$ denotes the conjugate of the corresponding spaces. Here $T_t : L_{\infty,N,M} \to R^n$ where R^n denotes the n-dimensional Euclidean space. In the finite dimensional case it can be easily shown that $T_t^* = S$ is one to one and onto a closed subspace of $L_{1,N,M}$, where $S: R^n \to L_{1,N,M}$. By Theorem 1 one can easily verify that the corresponding Reachable set is closed. Also By Theorem 2, it follows that there exists an optimal control u_φ.

The Form Of The Optimal Control

The problem is to find u which will maximize $\langle u, T_t^* \varphi \rangle$, under the constraint

$$\left|u_\ell(\tau)\right| \cdot \left|u_m(\tau)\right| \le N, \frac{1}{M}\left(\int_0^t \left|u_r(\tau)\right|^2 d\tau\right)^{\frac{1}{2}} \cdot \frac{1}{M}\left(\int_0^t \left|u_s(\tau)\right|^2 d\tau\right)^{\frac{1}{2}} \le 1 \qquad \text{(A)}$$

Case (I): If $t \le \frac{M^2}{N^2}$, then

$$\rho_2(u_\ell, u_m) = \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{\left|u_\ell(\tau)\right|}{N} \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{\left|u_m(\tau)\right|}{N} = 1.$$

$$\therefore \operatorname*{ess\,sup}_{0\le\tau\le t} \left|u_\ell(\tau)\right| \cdot \operatorname*{ess\,sup}_{0\le\tau\le t} \left|u_m(\tau)\right| = N$$

Now, the optimal u must satisfy the condition $\langle u, T_t^* \varphi\rangle = N_1'\{(T_t^*\varphi, f_1) : T_t^*\varphi, f_1 \in B_t^*\}$ and $\rho_3(u_p, u_q) = 1.$ So the problem is to find a u, which will maximize

$\langle u, T_t^*\varphi\rangle = \int_0^t u(\tau)(T_t^*\varphi)(\tau)\, d\tau$ subject to $\operatorname*{ess\,sup}_{0\le\tau\le t} \left|u(\tau)\right| = N.$ Evidently the optimal u(t) will be given by $u_\varphi(\tau) = N \operatorname{sign}[T_t^*\varphi(\tau)], 0 \le \tau \le t$ and $\langle u, T_t^*\varphi\rangle = N\int_0^t \left|(T_t^*\varphi)(\tau)\right| d\tau$

It can easily verified that $N_1'\{(T_t^*\varphi, f_1) : T_t^*\varphi, f_1 \in B_t^*\} = N\int_0^t \left|(T_t^*\varphi)(\tau)\right| d\tau.$

Case (II) (a)

$$\rho_2(u_\ell, u_m) = \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{\left|u_\ell(\tau)\right|}{N} \operatorname*{ess\,sup}_{0\le\tau\le t} \frac{\left|u_m(\tau)\right|}{N} = \frac{1}{M}\left(\int_0^t \left|u_r(\tau)\right|^2 d\tau\right)^{1/2} \cdot \frac{1}{M}\left(\int_0^t \left|u_s(\tau)\right|^2 d\tau\right)^{1/2} \le 1.$$

Hence $\operatorname*{ess\,sup}_{0\le\tau\le t} \left|u_\ell(\tau)\right| \operatorname*{ess\,sup}_{0\le\tau\le t} \left|u_m(\tau)\right| = N$ and $\int_0^t \left|u_r(\tau)\right|^2 d\tau . \int_0^t \left|u_s(\tau)\right|^2 d\tau = M^2.$

Consequently, one has to find that u(τ) which will maximize

$$\langle u, T_t^*\varphi\rangle = \int_0^t u(\tau)(T_t^*\varphi)(\tau)d\tau.$$

Let E = {t: $\left|u_\ell(\tau)\right| \cdot \left|u_m(\tau)\right| = N$ } and E_c = {t: $\left|u_\ell(\tau)\right| \cdot \left|u_m(\tau)\right| < N$ }

$$\therefore \int_0^t u(\tau)(T_t^*\varphi)(\tau)d\tau = \int_E u(\tau)(T_t^*\varphi)(\tau)d\tau + \int_{E_C} u(\tau)(T_t^*\varphi)(\tau)d\tau.$$

Now $\int_E u(\tau)(T_t^*\varphi)(\tau)d\tau$ will be maximized if $u(\tau) = N\,\mathrm{sign}\,[T_t^*\varphi(\tau)],\ \tau \in E$.

Again $\int_0^t |u(\tau)|^2 d\tau = M^2$ i.e $\int_E |u(\tau)|^2 d\tau + \int_{E_C} |u(\tau)|^2 d\tau = M^2$

or, $\int_{E_C} |u(\tau)|^2 d\tau = M^2 - N^2 m(E)$, where m(E) denotes the measure of the set E.

So, $\int_{E_C} u(\tau)(T_t^*\varphi)(\tau)d\tau$ will be maximized under the constraint (A), if we take

$u(\tau) = \alpha(T_t^*\varphi)(\tau)$ where α is a positive constant. Substituting $u = \alpha(T_t^*\varphi)(\tau)$ in (A), we have $\alpha^2 \int_{E_C} \left|(T_t^*\varphi)(\tau)\right|^2 d\tau = M^2 - N^2 m(E)$, where $\alpha = \dfrac{\sqrt{M^2 - N^2 m(E)}}{\sqrt{\int_{E_C} \left|(T_t^*\varphi)(\tau)\right|^2 d\tau}}$

$\mathrm{Max}\,\langle u, T_t^*\varphi\rangle = N\int_E \left|(T_t^*\varphi)(\tau)\right| d\tau + \sqrt{M^2 - N^2 m(E)} \cdot \sqrt{\int_{E_C} \left|(T_t^*\varphi)(\tau)\right|^2 d\tau}$. It can easily verified that

$$N_1'\{(T_t^*\varphi, f_1) : T_t^*\varphi, f_1 \in B_t^*\} = N\int_E \left|(T_t^*\varphi)(\tau)\right| d\tau + \sqrt{M^2 - N^2 m(E)} \cdot \sqrt{\int_{E_C} \left|(T_t^*\varphi)(\tau)\right|^2 d\tau},$$

from the above it follows that

$$u(\tau) = \begin{cases} N\,\mathrm{sign}\,[\alpha(T_t^*\varphi)(\tau)], \tau \in E = \{t : \left|\alpha(T_t^*\varphi)(\tau)\right| > N\} \\ \alpha(T_t^*\varphi)(\tau)], \tau \in E_C = \{t : \left|\alpha(T_t^*\varphi)(\tau)\right| \le N\} \end{cases}.$$

Case (II) (b): $\rho_2(u_\ell, u_m) = \frac{1}{M}\left(\int_0^t |u_r(\tau)|^2 d\tau\right)^{1/2} \cdot \frac{1}{M}\left(\int_0^t |u_s(\tau)|^2 d\tau\right)^{1/2} = 1$

Or,

$$\left(\int_0^t |u_r(\tau)|^2 d\tau\right)^{1/2} \cdot \left(\int_0^t |u_s(\tau)|^2 d\tau\right)^{1/2} = M^2 \qquad \text{(B).}$$

Now, the problem becomes, find u which will maximize $\int_0^t u(\tau)(T_t^*\varphi)(\tau)d\tau$ under the constraint (B).

Obviously $u_\varphi = \alpha(T_t^*\varphi)$, such that $\alpha^2\int_0^t \left|(T_t^*\varphi)(\tau)\right|^2 d\tau = M^2$ i.e. $\alpha = \dfrac{M}{\sqrt{\int_0^t \left|(T_t^*\varphi)(\tau)\right|^2 d\tau}}$.

$$u_\varphi(\tau) = \alpha(T_t^*\varphi)(\tau) = \frac{M(T_t^*\varphi)(\tau)}{\sqrt{\int_0^t \left|(T_t^*\varphi)(\tau)\right|^2 d\tau}} \text{ and } \int_0^t u(\tau)(T_t^*\varphi)(\tau)d\tau = \alpha\int_0^t \left|(T_t^*\varphi)(\tau)\right|^2 d\tau$$

$$= M\left\{\int_0^t \left|(T_t^*\varphi)(\tau)\right|^2 d\tau\right\}^{1/2} = N_1'\{(T_t^*\varphi, f_1) : T_t^*\varphi, f_1 \in B_t^*\}\cdot$$

Example: Let us consider the n-th order constant linear system $\dfrac{dx(t)}{dt} = A\,X(t) + B\,U(t)$, where X(t), U(t), A, B have their usual meanings. The problem which we shall consider here is to find the admissible control vector U(t) such that the trajectories described by the system under U(t) remain within an ε-neighbourhood of the target state x^d . $N_1\{(x(t_1)-x^d, u) : x(t_1)-x^d, u \in X\} \le \varepsilon$ where

$X = \underset{t_0\le t\le t_1}{\text{ess sup}}\ \underset{1\le i\le r}{\max}\left|x_i(t)\right| \cdot \underset{t_0\le t\le t_1}{\text{ess sup}}\ \underset{1\le j\le r}{\max}\left|x_j(t)\right|$, while $t_0 \le t \le t_1$ minimizing the fuel functional

$$J(u_i,u_j) = \left[\left\{\underset{t_0\le t\le t_1}{\text{ess sup}}\ \underset{1\le i\le r}{\max}\left|U_i(t)\right|^2\right\} + \left\{\int_{t_0}^{t_1}\left|\eta - TU_i\right|(t)\,dt\right\}^2\right]^{1/2} \cdot \left[\left\{\underset{t_0\le t\le t_1}{\text{ess sup}}\ \underset{1\le j\le r}{\max}\left|U_j(t)\right|^2\right\} + \left\{\int_{t_0}^{t_1}\left|\eta - TU_j\right|(t)\,dt\right\}^2\right]^{1/2}$$

$\tau=[t_0,t_1]$, t_0 & t_1 being initial and final times respectively. Let us now specify the 2-Banach spaces and linear operators as follows:

$$X = B^{(r)}_{\infty,\infty}\times B^{(r)}_{\infty,\infty} = L_\infty(\ell_\infty(r),\tau)\times L_\infty(\ell_\infty(r),\tau),\ Y = \ell_\infty(\eta)\times\ell_\infty(\eta),\ Z = B^{(r)}_{1,1}\times B^{(r)}_{1,1} = L_1(\ell_1(r),\tau)\times L_1(\ell_1(r),\tau),$$

Then by definition (2.2), X Y, Z are generalized 2-normed spaces.

$S : X \to Y,\ Su = \int_{t_0}^{t_1} e^{A(t_1 - s)} BU(s)\, ds, T : X \to Z, Tu = \text{-}u, \text{Taking } \xi = X^d - e^{-A(t_1 - t_0)} X(t_0) \text{ and } \eta = TU_0 = -U_0.$

The auxiliary problem becomes finding U, such that

$N_2\{(\xi - Tu(.,.), w) : \xi - Tu(.,.), w \in Y\} \le \varepsilon,\ \ J(u_i, u_j)$ is minimized. For further details, see [15].

Some examples are given in Adak ([15], [18], [19]) to show the technique of application of the control theory in generalized 2-normed spaces.

Note 1: Any complete 2-normed space is said to be 2-Banach space. Every 2-normed space of dimension 2 is a 2-Banach space when the underlying field is complete. For details see Adak [18, 21] & White [2]. A linear 2-normed space of dimension 3 is not a 2-Banach space. For details see White [2].
Note 2: Every 2-normed space is a locally convex topological vector space. But convers is not true. In fact for a fixed $b \in X$, $P_b(x)=N_1(x,b)$ $\forall x \in X$, is a seminorm and the family $P=\{P_b: b \in X\}$ generates a locally convex topology on X. Such a topology is called the natural topology induced by 2-norm $N_1(.,.)$.

Conclusion: In the previous papers [18, 20, 21], we introduced generalized 2–normed spaces and 2-normed spaces. There are appropriate connections between: (i) normed spaces and generalized 2–normed spaces, (ii) 2-normed spaces and generalized 2–normed spaces, (iii) 2-normed spaces and 2-Banach spaces, (iv) 2-normed spaces and locally convex topological vector spaces, (v) generalized 2-normed spaces and generalized symmetric 2-normed spaces.
In this paper we introduced semi-norm and equivalent norm. There are appropriate connections among semi-norm, 2-norm and equivalent norm.

Acknowledgement: The authors are most grateful to Prof. R. N. Mukherjee of the Department of Mathematics, University of Burdwan, Burdwan, West Bengal, India, for his generous encouragement and suggestions at various steps while preparing this paper.

References

[1] A.K.Chaudhury and R.N. Mukherjee, On the global controllability of a certain class of minimum time control problems, *Ind. J. Pure Appl. Math.*, 13(3) (1982, 163-171).
[2] A.White, 2-Banach Spaces, *Math. Nachr*,. 42 (1969), 43-60.
[3] B.S. Reddy and H. Dutta, Random $n-$Inner Product Space, *Int. Math. Forum*, 5(49) (2010), 2415 – 2423.

[4] C. Park, Generalized Quasi-Banach spaces and Quasi-Normed spaces, *J. Chungcheong Math. Soc.*, 19(2) (2006), 197-206.
[5] H. Dutta, Some statistically convergent difference sequence sequence spaces defined over real 2-Normed linear spaces, *Appl. Sci.*, 12(2010), 37-47.
[6] J.A. Burns, Existence theorems and necessary conditions for a general formulation of the minimum effort problem, *JOTA*, 15(4)(1975), 413-440.
[7] M. Acikgoz: 2-ε Proximinality in Generalized 2-normed Spaces, *Int. Math. Forum*,5(16)(2010),781-786.
[8] N. Minamide and K. Nakamura: A minimum cost control problem in Banach space, *J. Math. Anal. Appl.*, 36(1971), 73-85.
[9] N. Minamide and K. Nakamura: Linear bounded phase coordinate control problems under certain regularity and normality conditions, *SIAM J. Control*, 10(1) (1972), 82-92.
[10] R. Freese and Y. Cho, Geometry of Linear 2-normed spaces, Nova Science Publishers, 2001.
[11] R.N. Mukherjee, Global controllability of a class of minimum time control problems for bounded phase coordinate control problems, *Far East J. Appl. Math.*, 9(3) (2002), 193-206.
[12] R.N. Mukherjee, Existence theorems and necessary conditions for general formulation of linear bounded phase co-ordinate control problems, *Far East J. Math. Sciences*, 27(2) (2007), 381-394.
[13] S. Gahler, Lineare 2-normierte Riume, *Math. Nachr.*, 28 (1965), 1-43.
[14] T.Som, Some fixed point results in 2-Banach space, *Int. J. Math. Sci.*, 4(2) (2005), 323-328.
[15] U. Adak and H.K. Samanta, Time optimal control for linear bounded phase coordinate control problems in 2-Banach Spaces, *Int. Math. Forum*, 5(46) (2010), 2279-2292.
[16] U. Adak and H.K. Samanta, A class of optimal control problems in 2-Banach space, *Journal of Assam Academy of Mathematics*, 1(2010), 65-77.
[17] U. Adak and H.K. Samanta, Global controllability of a class of optimal control problems in 2-Banach spaces, *Journal of Assam Academy of Mathematics*, 2(2010), 1-12.
[18] U. Adak and H.K. Samanta, A Certain Class of Minimum Time Optimal Control Problems in 2-Banach Spaces, *J. Phy. Sci.*, 14(2010), 95-106.
[19] U.Adak and H.K. Samanta, A minimum effort control problem in 2-Normed spaces, *Journal of Assam Academy of Mathematics*, 3(2010), 46-70.
[20] U. Adak and H.K. Samanta, Global controllability for bounded phase coordinate control problems in 2-Banach spaces, *Int. J. Pure Appl. Sci. Technol.*, 1(2) (2010), 13-25.
[21] U.Adak and H.K. Samanta, A minimum cost control problem in 2-Banach space, I*nt. J. Technol. Appl. Sci.*, Vol. 2, 2011.

[22] U. Adak and H.K. Samanta, Existence theorems and necessary condition for a class of time optimal control problems in 2-Banach Spaces, 5(2), *Bull. Pure Appl. Math.*(to appear).
[23] Y. Cho, P. Lin, S.S. Kim and A. Misiak, Theory of 2-Inner Product Spaces, Nova Science Publishers, 2001.
[24] Z. Lewandowska, Linear operators on generalized 2-normed Spaces, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 42(4) (1999), 353-368.
[25] Z. Lewandowska, M.S. Moslehian and A. Saadatpour, Hahn-Banach Theorem in generalized 2-Normed Spaces, *Comm. Math. Anal.*, 1(2) (2006), 109-113.