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Lie and Jordan Structure in Simple $\Gamma-$ Regular Ring

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Abstract

In this paper, we study Lie and Jordan Structure in Simple $\Gamma-$ *Regular Ring of characteristic not equal to two. Some Properties of these* $\Gamma-$ *Regular Ring are determined.*

Keywords: $\Gamma-$ *Ring,* $\Gamma-$ *Regular Ring, Ideal, Jordan Ring, Lie Ring, Simple* $\Gamma-$ *Regular Ring*

1 Introduction

The concept of $\Gamma-$ ring was first introduced by Nobusawa [4] in 1964 and generalized by Barnes [1] in 1996. The idea of $\Gamma-$ regular ring was studied by Krishnaswamy [2] in 2009. S.Kyuno [3] worked on the Simple $\Gamma-$ ring with simple conditions and Herstein [8] studied the Lie and Jordan Strucutures in Simple ring. In this paper, we have extended the results of Paul[5] into Lie and Jordan Structure in Simple $\Gamma-$ regular ring. Some characterization of this $\Gamma-$ regular ring have been established.

2 Preliminaries

Definition 2.1 *Let M and Γ be two additive abelian groups. There is a mapping from $M \times \Gamma \times M \to M$ such that*

1. $(x+y)\alpha z = x\alpha z + y\alpha z; x(\alpha+\beta)z = x\alpha z + x\beta z; x\alpha(y+z) = x\alpha y + x\alpha z.$

2. $(x\alpha y)\beta z = x\alpha(y\beta z)$ *where* $x, y, z \in M$ *and* $\alpha, \beta \in \Gamma$.

Then, M is called a $\Gamma-$ ring.

Definition 2.2 *An element a of a ring R is said to be regular if there exists an element $x \in R$ such that $axa = a$. The ring R is regular if and only if each element of R is regular.*

Definition 2.3 *Let R and Γ be two additive abelian groups. An element $a \in R$ is said to be $\Gamma-$ Regular if there exists an element $x \in \Gamma$ such that $axa = a$. A $\Gamma-$ ring is said to be $\Gamma-$ regular ring if and only if each element of R is $\Gamma-$ regular.*

Definition 2.4 *A Lie ring L is to be defined as an abelian group with an operation $[\bullet, \bullet]$ having the properties*

1. *for all* $x \in L$, $[x, x] = 0$.

2. *Bilinearity :* $[x+y, z] = [x, z] + [y, z]$; $[z, x+y] = [z, x] + [z, y]$

3. *Jacobi identity :* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in L$.

Remark 2.5 *Any associative ring can be made into a Lie ring by defining the bracket opertaion by $[x, y] = xy - yx$.*

Definition 2.6 *A subset S of the $\Gamma-$ regular ring R is a left(right) ideal of R if S is an additive sub-group of R and $R\Gamma S = \{c\alpha a / c \in R, \alpha \in \Gamma, a \in S\}$ $(S\Gamma R = \{a\alpha c / c \in R, \alpha \in \Gamma, a \in S\})$ is contained in S. If S is both left and right ideal of R, then we say that S is an ideal of two sided ideal of R.*

If A and B are ideals in $\Gamma-$ regular ring R, then the sum of A and B is also an ideal of R that is $A + B = \{a + b / a \in A, b \in B\}$.

Definition 2.7 *Let R be a $\Gamma-$ regular ring. An element $a \in R$ is called a nil-potent of a $\Gamma-$ regular ring for some $\alpha \in \Gamma$ there exists a least positive integer n such that $(a\alpha)^n a = (a\alpha a\alpha a\alpha \ldots\ldots\ldots\ldots ntimes)a = 0$.*

Definition 2.8 *An ideal A of a $\Gamma-$ regular ring R is called a nil-potent ideal of a $\Gamma-$ regular ring R if $(A\Gamma)^n A = (A\Gamma A\Gamma A\Gamma \ldots\ldots\ldots\ldots ntimes)A = 0$ where n is the least positive integer.*

Definition 2.9 *For any $\Gamma-$ regular ring R, the Lie and Jordan Structure of a $\Gamma-$ regular ring is to be defined as the new product of $[x,y]_\alpha = x\alpha y - y\alpha x$ and $(x,y)_\alpha = x\alpha y + y\alpha x$ for every $x, y \in R$ and $\alpha \in \Gamma$.*

Definition 2.10 *A subset S of R is a Lie sub $\Gamma-$ regular ring R if S is an additive sub-group such that for $a, b \in S, a\alpha b - b\alpha a$ must also be in S for all $\alpha \in \Gamma$. A subset S of R is a Jordan sub $\Gamma-$ regular ring R if S is an additive sub-group such that for $a, b \in S, a\alpha b + b\alpha a$ must also be in S for all $\alpha \in \Gamma$.*

Definition 2.11 *Let S be a Lie sub $\Gamma-$ regular ring of R. The additive sub group $V \subset S$ is said to be Lie ideal of S if whenever $v \in V, \alpha \in \Gamma, a \in S$ then $[V,a]_\alpha = V\alpha a - a\alpha V$ is in V. Again let S be a Jordan sub $\Gamma-$ regular ring of R. The additive sub group $V \subset S$ is said to be Jordan ideal of S if whenever $v \in V, \alpha \in \Gamma, a \in S$ then $(V,a)_\alpha = V\alpha a + a\alpha V$ is in V.*

Definition 2.12 *A $\Gamma-$ regular ring R is called a Simple $\Gamma-$ regular ring if $R\Gamma R \neq 0$ and its ideals are 0 and R.*

Definition 2.13 *Let A be an ideal in $\Gamma-$ regular ring R. Then, the set R/A is defined by $R/A = \{x + a\alpha c / x \in R, a, c \in A, \alpha \in \Gamma\}$ and*

1. *$(x + a\alpha c) + (y + a\alpha c) = (x + y) + a\alpha c$;*

2. *$(x + a\alpha c)\alpha(y + a\alpha c) = x\alpha y + a\alpha c$ under the operation $(+, \bullet)$.*

Then, the set $(R/A, +, \bullet)$ form a $\Gamma-$ regular ring R.

Definition 2.14 *Let R be a $\Gamma-$ regular ring. The centre of R written as Z is the set of those elements in R, that is $Z = \{m \in R / m\alpha x = x\alpha m\}$ for all $x \in R$ and $\alpha \in \Gamma$.*

Definition 2.15 *Let R be a $\Gamma-$ regular ring and let R_{mn} and Γ_{nm} denote respectively, the sets of $m \times n$ matrices with entries from R and the sets of $n \times m$ matrices with entries from Γ. Then, the set R_{mn} is a Γ_{nm} regular ring and multiplication is defined by $(a_{ij})(\alpha_{ji})(b_{ij}) = (c_{ij})$ where $(c_{ij}) = \sum_p \sum_q a_{ip}\alpha_{pq}b_{qj}$. If $m = n$, then R_n is a Γ_n- ring.*

Definition 2.16 *Let R be a $\Gamma-$ regular ring. Then, R is called a division $\Gamma-$ regular ring if it has an identity element and its only non-zero ideal is itself.*

3 Lie and Jordan Structure

In this section, we have developed some characterization of Lie and Jordan Structures in Simple $\Gamma-$ regular ring.

Theorem 3.1 *Let R be a $\Gamma-$ regular ring and $A \neq 0$ is a right ideal of R. For given $a \in A$, $(a\alpha)^n a = 0$ for all $\alpha \in \Gamma$ and for fixed integer n. Then, R has a non-zero nilpotent ideal.*

Proof: To prove this Theorem by using Mathematical induction on n. Let $a \neq 0 \in A$ satisfying $a\alpha a = 0$ and let us suppose that $B = a\Gamma A \neq 0$. If $x \in R$, then $[(a + a\alpha x)\alpha]^n[a + a\alpha x] = 0$. Since it is in A, we obtain $[(a\alpha x)\alpha]^{n-1}(a\alpha x)\alpha a = 0$. Thus, $[(a\alpha x)\alpha]^{n-1}(a\alpha x)\Gamma A = 0$.

Let $T = \{x \in A/x\Gamma A = 0\}$ of course T is an ideal of A. Moreover, let $y \in B \Rightarrow (y\alpha)^{n-1}y \in T$. Therefore $\bar{B} = B/T$ every element satisfies $(y\alpha)^{n-1}y = 0$. By our induction hypothesis, $\bar{B}$ has a nilpotent ideal $\bar{U} \neq 0$. Let U be its inverse image in B. Since $(\bar{U}T)^k\bar{U} = 0$, $(U\Gamma)^k U \subset T$. Hence, $(U\Gamma)^{k+1}U \subset T\Gamma B = 0$. Also, since $\bar{U} \neq 0$, U is not a sub-set of T and hence $U \supset U\Gamma B \neq 0$. But $U\Gamma B = U\Gamma a\Gamma B \neq 0$ is a nil-potent ideal of R.

Suppose that $a \in A$ satisfying $a\alpha a = 0 \Rightarrow a\Gamma A = 0$. For any $x \in A$, $(x\alpha)^n x = 0$, we have $(x\alpha)^{n-1}x\alpha x = 0$ and so $(x\alpha)^{n-1}x\Gamma A = 0$.

Let $W = \{x \in A/x\Gamma A = 0\}$, W is an ideal of A. If $W = A$, then $A\Gamma A = 0$ and would provide us a nilpotent right ideal. If $W = A$, then $\bar{A} = A/W$, $(\bar{x}\alpha)^n\bar{x} = 0$. Our induction gives us a nilpotent ideal $\bar{V} \neq 0 \in \bar{A}$. If V is the inverse image of $\bar{V} \in A$ then $V\Gamma A \neq 0 \subset V$ and is nilpotent. Since, V is nilpotent, again we have seen that R must have a non-zero nilpotent right ideal.

If R has a non-zero nilpotent right ideal and it has almost trivially a non - zero nilpotent ideal. •

Our first objective will be to determine the Lie and Jordan ideals of the $\Gamma-$ regular ring R itself in the case R is restricted to a Simple $\Gamma-$ regular ring.

Theorem 3.2 *If U is a Jordan ideal of R, then $x\alpha(a\alpha b + b\alpha a) - (a\alpha b + b\alpha a)\alpha x \in U$ for all $a, b \in U$ and $x \in R$ and $\alpha \in \Gamma$.*

Proof: Since $a, b \in U$ and $\alpha \in \Gamma$ for any $x \in R$, we have $a\alpha(x\alpha b - b\alpha x) + (x\alpha b - b\alpha x)\alpha a \in U$. But $a\alpha(x\alpha b - b\alpha x) + (x\alpha b - b\alpha x)\alpha a = \{(a\alpha x - x\alpha a)\alpha b + b\alpha(a\alpha x - x\alpha a)\} + \{x\alpha(a\alpha b + b\alpha a) - (a\alpha b + b\alpha a)\alpha x\}$. The left side and the first term on the right side are in U. Hence $x\alpha(a\alpha b + b\alpha a) - (a\alpha b + b\alpha a)\alpha x \in U$ •

Theorem 3.3 *Let R be a $\Gamma-$ regular ring in which $2x = 0 \Rightarrow x = 0$ and suppose further that R has no non-zero nilpotent ideal of R contains a non-zero(associative) ideal of R.*

Proof: Let $U \neq 0$ be a Jordan ideal of R and suppose that $a, b \in R$. By Theorem 3.2, for any $x \in R$ and $\alpha \in \Gamma$,

We have $x\alpha c - c\alpha x$ where $c = a\alpha b + b\alpha a \in U$. $\rightarrow 3.31$

However, since $c \in U$, $x\alpha c + c\alpha x \in U$. $\rightarrow 3.32$

Adding 3.31 and 3.32, we get $2x\alpha c \in U$ for all x. Hence, for $y \in R$, $(2x\alpha c)\alpha y + y\alpha(2x\alpha c) \in U$. Since $2y\alpha x\alpha c \in U$, we obtain $2x\alpha c\alpha y \in U$ i.e., $2R\Gamma c\Gamma R \subset U$. Now $2R\Gamma c\Gamma R$ is an ideal of R so we do unless $2R\Gamma c\Gamma R = 0$. If $2R\Gamma c\Gamma R = 0$, by our assumption $R\Gamma c\Gamma R = 0$. Since R has no nilpotent ideals this forces $c = 0$, that is given $a, b \in U$ then $a\alpha b + b\alpha a = 0$.

Let $a \neq 0 \in U$, then for any $x \in R$, $\alpha \in \Gamma$ and $b = a\alpha x + x\alpha a \in U$. Hence, $a\alpha(a\alpha x + x\alpha a) + (a\alpha x + x\alpha a)\alpha a = 0$. that is $a\alpha a\alpha x + x\alpha a\alpha a + 2a\alpha x\alpha a = 0$. Now, for $a \in U$ and $a\alpha a = 0$, this reduces to $2a\alpha x\alpha a = 0$ for all $x \in R$, $\alpha \in \Gamma$ and so $a\Gamma R\Gamma a = 0$. But $a\Gamma R \neq 0$ is a nilpotent right ideal of R. This is a contradiction to our assumption. Inotherwords, we have shown that U contains a non-zero ideal of R. •

Lemma 3.4 *Let R be a $\Gamma-$ regular ring with no non-zero nilpotent ideals in which $2x = 0 \Rightarrow x = 0$. Suppose that $U \neq 0$ is both a Lie ideal and $\Gamma-$ regular ring of R. Then, either $U \subset Z$ or U contains a non-zero ideal of R.*

Proof: Let us first suppose that U has a $\Gamma-$ regular ring is not commutative. Then, for some $x, y \in U$ and $\alpha \in \Gamma$, we have $x\alpha y - y\alpha x \neq 0$. For any $m \in R$ and $\beta \in \Gamma$ we have $x\beta(y\alpha m) - (y\alpha m)\beta x \in U$ that is $(x\alpha y - y\alpha x)\beta m + y\beta(x\alpha m - m\alpha x) \in U$. The second memeber of this is in U since both y and $(x\alpha m - m\alpha x)$ are in U (U is both Lie ideal and sub $\Gamma-$ regular ring). The net result of all this is that $(x\alpha y - y\alpha x)\Gamma R \subset U$. But then for some $m, s \in R$ and $\alpha, \beta \in \Gamma$, we have $((x\alpha y - y\alpha x)\alpha m)\beta s - s\beta((x\alpha y - y\alpha x)\alpha m) \in U \Rightarrow R\Gamma(x\alpha y - y\alpha x)\Gamma R = 0$, then $R\Gamma(x\alpha y - y\alpha x)\Gamma R\Gamma(x\alpha y - y\alpha x)\Gamma R = 0$. This is a contradiction to our assumption. We have shown that the result is correct if U is a sub $\Gamma-$ regular ring of R is not commutative. So, by using sub-lemma 3.5 a must be in Z as follows. •

Sub-Lemma 3.5 *Let R be a $\Gamma-$ regular ring with no non-zero nilpotent ideals in which $2x = 0 \Rightarrow x = 0$. If $a \in R$ commutes with $a\alpha x - x\alpha a$ for all $x \in R$, $\alpha \in \Gamma$ then a is in Z.*

Proof: Suppose that U is commutative, we want to show that it lies in Z. Given $a \in U$, $x \in R$ then $a\alpha x - x\alpha a \in U$. Now for $x, y \in R$ we have $a\alpha c - c\alpha a$ where $c = (a\alpha(x\alpha y - y\alpha x)\alpha a - a\alpha(x\alpha y - y\alpha x)\alpha a)$.

Expanding $a\alpha(x\alpha y - y\alpha x)\alpha a$ as $(a\alpha x - x\alpha a)\alpha y + x\alpha(a\alpha y - y\alpha a)$ using this and commutes with $(a\alpha x - x\alpha a)$ and $(a\alpha y - y\alpha a)$ yields
$2(a\alpha x - x\alpha a)\beta\alpha(a\alpha y - y\alpha a) = 0$ for all $x, y \in R$ and $\beta \in \Gamma$. Since $2m = 0$ forces $m = 0$ we obtain $(a\alpha x - x\alpha a)\beta(a\alpha y - y\alpha a) = 0$. In this, put $y = a\alpha x$

this results in $(a\alpha x - x\alpha a)\Gamma R\Gamma(a\alpha x - x\alpha a) = 0$. Since R has no nilpotent, we conclude that $(a\alpha x - x\alpha a) = 0$ and so a must be in Z. •

Theorem 3.6 *Let R be a Simple $\Gamma-$ regular ring of characteristic $\neq 2$. Then any Lie ideal of R which is also a sub $\Gamma-$ regular ring if R must either be R itself or it contained in Z.*

Proof: Lemma 3.4 immediately gives the result of the Theorem. •

Definition 3.7 *If U is a Lie ideal of R, let $T(U) = \{x \in R/[x, R]_\Gamma \subset U\}$.*

Lemma 3.8 *For any $\Gamma-$ regular ring R, if U is a Lie ideal of R. Then, $T(U)$ is both a sub $\Gamma-$ regular ring and a Lie ideal of R. Moreover $U \subset T(U)$.*

Proof: If U is a Lie ideal of R then $U \subset T(U)$. Since $[T(U), M]_\Gamma \subset U \subset T(U)$ must be a Lie ideal of R. Suppose that $a, b \in T(U)$ and $m \in R$ then $(a\alpha b)\alpha m - m\alpha(a\alpha b) = a\alpha(b\alpha m) - (b\alpha m)\alpha a + b\alpha(m\alpha a) - (m\alpha a)\alpha b$. Since $a, b \in T(U)$, the right side of $a\alpha(b\alpha m) - (b\alpha m)\alpha a + b\alpha(m\alpha a) - (m\alpha a)\alpha b \in U$ and therefore $[a\alpha b, R]_\Gamma \subset U$ that is $a\alpha b \in T(U)$. •

Theorem 3.9 *Let R be a Simple $\Gamma-$ regular ring of characteristic $\neq 2$ and let U be a Lie ideal of R. Then, either $U \subset Z$ or $U \supset [R, R]_\Gamma$.*

Proof: By Theorem 3.6 and Lemma 3.8, $T(U)$ is a both a sub $\Gamma-$ regular ring and a Lie ideal of R. Therefore, $T(U) \subset Z$ or $T(U) = R$. If $T(U) = R$, then by the Definition 3.7, we have $[R, R]_\Gamma \subset U$. If $T(U) \subset Z$ and $U \subset T(U)$, we obtain $U \subset Z$. •

Corollary 3.10 *If R has a non-commutative Simple $\Gamma-$ regular ring of characteristic $\neq 2$, then the sub $\Gamma-$ regular ring generated by $[R, R]_\Gamma$ is R.*

Proof: Any additive sub-group containing $[R, R]_\Gamma$ is trivially a Lie ideal of R. Hence, the sub $\Gamma-$ regular ring is generated by $[R, R]_\Gamma$ is a Lie ideal of R. Hence, by Theorem 3.6, it equals to R or is in Z. If it is in Z, then $[R, R]_\Gamma \subset Z$. Thus, for $a \in R$, a commutates with all $a\alpha a$. In $a\alpha a$, $\alpha \in \Gamma$ then by the Sub-Lemma 3.5, we get $a \in Z$, that is $R \subset Z$. Since R to be non-commutative, that is ruled out hence the corollary. •

In Theorem 3.6, R has a Simple $\Gamma-$ regular ring of characteristic $\neq 2$. Now, we should like to settle the problem when R has characteristic 2, Theorem 3.6 fail?

Suppose that R has a Simple $\Gamma-$ regular ring of characteristic 2 and that U is a Lie ideal and sub $\Gamma-$ regular ring of R, we obtain $U \neq R$and U is not a subset of Z. As in the proof of Lemma 3.4, we obtain U as a sub $\Gamma-$ regular ring of R must be commutative. That is given $u, v \in U$, then $u\alpha v + v\alpha u = 0$ for all $\alpha \in \Gamma$.

Let $a \in U$ then $a\alpha s + s\alpha a \in U$ for all $s \in R$ and $\alpha \in \Gamma$. Hence, $a\alpha(a\alpha s + s\alpha a) = (a\alpha s + s\alpha a)\alpha a$. This says that $a\alpha a \in Z$. Since, for any $m \in R$, we have $a\alpha m + m\alpha a \in U$, also $(a\alpha m + m\alpha a)\alpha(a\alpha m + m\alpha a) \in Z$. If $Z = 0$, then $a\alpha a = 0$. that is $(a\alpha m + m\alpha a)\alpha(a\alpha m + m\alpha a) \in Z = 0$ from which we get $((a\alpha m)\alpha)^2(a\alpha m) = 0$. But $a\Gamma R$ is a right ideal of R in which every element in the form $((a\alpha m)\alpha)^2(a\alpha m) = 0$. By Theorem 3.1, R would have a nilpotent ideal, that is R would be nilpotent which is impossible for a Simple $\Gamma-$ regular ring.

Therefore, we assume that $Z \neq 0$ and that there is an element $a \in U$, $a \notin Z$ such that $a\alpha a \neq 0 \in Z$ and $(a\alpha m + m\alpha a)\alpha(a\alpha m + m\alpha a) \in Z$ for all $m \in R$ and $\alpha \in \Gamma$.

Theorem 3.11 *Let R be a Simple $\Gamma-$ regular ring of characteristic 2 and suppose that there exist an element $a \in R$, $a \notin R$ such that for all $a\alpha a \in Z$, $\alpha \in \Gamma$ and $[(a\alpha x + x\alpha a)\alpha]^3(a\alpha x + x\alpha a) \in Z$ for all $x \in R$ and $\alpha \in \Gamma$. Then, R is a 4 - dimensional over Z.*

Proof: If $Z = 0$, then both $a\alpha a = 0$ and $[(a\alpha x + x\alpha a)\alpha]^3(a\alpha x + x\alpha a) = 0$. Hence, $[(a\alpha x)\alpha]^4[a\alpha x] = a\alpha[(a\alpha x + x\alpha a)\alpha]^3(a\alpha x + x\alpha a)\alpha x = 0$ for all $x \in R$. But then the right ideal $a\Gamma R$ satisfies $(u\alpha)^4 u = 0$ for all elements of $u \in a\Gamma R$, by Theorem 3.1, this is not possible in a simple $\Gamma-$ regular ring.

Suppose that $Z \neq 0$, hence $1 \in R$. If $a\alpha a = 0$, then $b = a + 1$ satisfies $b\alpha b = 1$ and $[(b\alpha x + x\alpha b)\alpha]^3[b\alpha x + x\alpha b] \in Z$ for all $x \in R$. Therefore, we may assume that $a\alpha a = p \neq 0 \in Z$. Let $\bar{Z} = Z(\sqrt{P})$, then $\bar{R} = R \otimes Z \neq \bar{Z}$ is simple. Moreover in $\bar{R}$, we have $[(a\alpha\bar{x} + \bar{x}\alpha a)\alpha]^3(a\alpha\bar{x} + \bar{x}\alpha a) \in \bar{Z}$ for all $\bar{x} \in \bar{R}$.

Since, $dim\bar{R}/Z = dimR/Z$, to prove the theorem it is enough to do so in $\bar{R}$. Also $b = a/q$ where $q \in \bar{Z}$, then $q\alpha q = p$ satisifes $b\alpha b = 1$ and $[(b\alpha\bar{x} + \bar{x}\alpha b)\alpha]^3[b\alpha\bar{x} + \bar{x}\alpha b] \in Z$. Hence without loss of generality we may suppose that $a \in R$, $a \neq Z$, $a\alpha a = 1$ and $[(a\alpha x + x\alpha a)\alpha]^3(a\alpha x + x\alpha a) \in Z$ for all $x \in R$.

Now R is a dense $\Gamma-$ regular ring of linear $\Gamma-$ regular transformations on a vector space V over a division $\Gamma-$ regular ring Δ (Since $Z \neq 0$ and R is simple). Since $(a+1)\alpha(a+1) = 0$, $(a+1) \neq 0$, V must be more than 1 - dimensional over Δ. Since $a \neq 1$ it is immediate that there is a $v \in V$ such that v, $v\alpha a$ are linearly $\Gamma-$ regular independent over Δ.

If for some $w \in V, v, v\alpha a$ and $w\alpha(1+a)$ are linearly $\Gamma-$ regular independent over Δ, then the sub $\Gamma-$ regular space V_0 spanned by these is invariant under a and a induces the linear $\Gamma-$ regular transformations $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on V_0. By density of R on V, there is an $x \in R$ which includes $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ on V_0. Hence, $(a\alpha x + x\alpha a)$ induces $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ on V_0. But $[(a\alpha x + x\alpha a)\alpha]^3(a\alpha x + x\alpha a) \in Z$. Yet does not induces a scalar on V_0. Since it induces $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, we have that for all $w \in V$ such that $v, v\alpha a, w$ are linearly $\Gamma-$ regular independent over Δ. If V is more than 2-dimensional over Δ, there is a $w \in V$ such that $v, v\alpha a, w$ are linearly $\Gamma-$ regular independent over Δ. By the above, $w\alpha a$ is in the sub $\Gamma-$ regular space V, they span. The matrix of a on V is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ p & q & r \end{pmatrix}$. By density there is an $x \in R$ which induces $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ on V_1. But $(a\alpha x + x\alpha a)$ induces $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & p & 0 \end{pmatrix}$. We hve $[(a\alpha x + x\alpha a)\alpha]^3(a\alpha x + x\alpha a)$ is not a scalar.

Thus, we must have that V is 2-dimensional over Δ. All the remains is to show that Δ is commutative. Let $a = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then $a\Gamma_2 a = I_2$ where Γ_2 is the set of all 2×2 matrices of $\Gamma-$ regular ring over Δ and I_2 is the identity matrix. Now, we have $a\Gamma_2 a = I_2$ becomes $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It yields

1. $p\alpha_{11}p + q\alpha_{21}p + p\alpha_{12}r + q\alpha_{22}r = 1$
2. $p\alpha_{11}q + q\alpha_{21}q + p\alpha_{12}s + q\alpha_{22}s = 0$
3. $r\alpha_{11}p + s\alpha_{21}p + r\alpha_{12}r + s\alpha_{22}r = 0$
4. $r\alpha_{11}p + s\alpha_{21}p + r\alpha_{12}q + s\alpha_{22}s = 1.$

In particular not both $p, r = 0$. If $t \in \Delta$, then using $x = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$ and $[(a\Gamma_2 x + x\Gamma_2 a)\Gamma]^3(a\Gamma_2 x + x\Gamma_2 a) \in Z$. Now
$a\Gamma_2 x + x\Gamma_2 a = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} t\alpha_{11}p + t\alpha_{22}r & p\alpha_{11}t + q\alpha_{21}t + t\alpha_{12}q + t\alpha_{22}r \\ 0 & r\alpha_{11}t + s\alpha_{22}t \end{pmatrix}$. Therefore,

$[(a\Gamma_2 x + x\Gamma_2 a)\Gamma]^3(a\Gamma_2 x + x\Gamma_2 a) \in Z$. This gives for all $t \in \Delta$, 4 times of $(t\alpha_{11}p + t\alpha_{22}r)$ and $(r\alpha_{11}t + s\alpha_{22}t)$ are in Z. If $p \neq 0$, then $(t\alpha_{11}p + t\alpha_{22}r)$ runs through as t does, so every $x \in \Delta$ would satisfy $(x\Gamma_2)^3 x \in Z$. But a non-commutative division $\Gamma-$ regular ring cannot be purely inseparable over its centre. This $p \neq 0$ implies Δ is commutative. Similarly, $r \neq 0$ implies Δ is commutative. Since, one of these must hold we get that Δ is commutative and so R is 4 - dimensional over Z. •

Theorem 3.12 *If R is a simple $\Gamma-$ regular ring and if U is a Lie ideal of R, then either $U \subset Z$ or $U \supset [R,R]_\Gamma$ except R is of characteristic 2 and is 4-dimensional over its centre.* •

Corollary 3.13 *If R is a simple non-commutative $\Gamma-$ regular ring, then the sub $\Gamma-$ regular ring generated by $[R,R]_\Gamma$ is R.* •

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