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Caristi-Type Fixed Point Theorems in a 2-Banach Space

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Abstract

This paper deals with some fixed point Theorems in 2-Banach spaces where mappings involved are of Caristi-type and results arrived at take care of those so far known.

Keywords: *2-Banach space, Common fixed point.*

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1 Introduction

The concept of 2 - Banach spaces has been initiated by S. Gähler [1] and these spaces have subsequently been studied by showing the existence of fixed point of contractive mappings. As in other spaces, the fixed point theory of mappings has been developed in such spaces also. Perhaps Iseki [2] obtained for the first time basic results on fixed point of mappings in 2 - Banach spaces. After the works of Iseki, fixed point results are obtained in such spaces by Khan et.al [3] and Cho et.al [4]. In the present paper we establish common fixed point theorem and coincidence point results for a pair of non linear mappings in 2 - Banach space, which mainly generalize the results of Cho et. al [4]. Also we have proved here an analogue of result of Seigal [5] in a setting of 2-Banach spaces.

2 Preliminary Notes

Definition 2.1 *Let X be a real linear space and $\|.,.\|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions :*

(i) $\|(x, y)\| = 0$ *if and only if* x *and* y *are linearly dependent in* X*,*

(ii) $\|(x, y)\| = \|(y, x)\|$*, for all* $x, y \in X$*,*

(iii) $\|x, ay\| = |a| \, \|x, y\|$*,* a *being real,* $x, y \in X$

(iv) $\|x, y + z\| \le \|x, y\| + \|x, z\|$*, for all* $x, y, z \in X$

$\|.,.\|$ *is called a 2 - norm and the pair* $(X, \|.,.\|)$ *is called a linear 2-normed space.*

Some of the basic properties of 2-norms are that they are non-negative satisfying $\|x, y + ax\| = \|x, y\|$*, for all* $x, y \in X$ *and all real numbers* a*.*

Definition 2.2 *A sequence* $\{x_n\}$ *in a linear 2-normed space* $(X, \|.,.\|)$ *is called a Cauchy sequence if* $\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$ *for all* y *in* X*.*

Definition 2.3 *A sequence* $\{x_n\}$ *in a linear 2-normed space* $(X, \|.,.\|)$ *is said to be convergent if there is a point* x *in* X *such that* $\lim_{n\to\infty} \|x_n - x, y\| = 0$ *for all* y *in* X*. If* $\{x_n\}$ *converges to* x*, we write* $\{x_n\} \to x$ *as* $n \to \infty$*.*

Definition 2.4 *A linear 2 - normed space* X *is said to be complete if every Cauchy sequence is convergent to an element of* X*. We then call* X *to be a 2 - Banach space.*

Definition 2.5 *Let* X *be a 2-Banach space and* T *be a self mapping of* X*.* T *is said to be continuous at* x *if for every sequence* $\{x_n\}$ *in* X*,* $\{x_n\} \to x$ *as* $n \to \infty$ *implies* $\{T(x_n)\} \to T(x)$ *as* $n \to \infty$

3 Main Results

Theorem 3.1 *Let* T *be a continuous self map of a 2 - Banach space* X*. Suppose that for any* u *in* X*, there exists a function* $\phi_u : X \to [0, \infty)$ *such that*

$$\|T(x) - T(y), u\| \le \phi_u(x) - \phi_u(T(x)) + \phi_u(y) - \phi_u(T(y)) \text{ for all } x, y \in X.$$

Then T *has a unique fixed point in* X*.*

Proof. Let $x_0 \in X$ be fixed. Let $x_n = T^n(x_0), n = 1, 2, \ldots$

$$\text{Now } \|x_n - x_{n+1}, u\| = \|T(x_{n-1}) - T(x_n), u\| \leq \phi_u(x_{n-1}) - \phi_u(x_n) + \phi_u(x_n)$$

$$-\phi_u(x_{n+1}) = \phi_u(x_{n-1}) - \phi_u(x_{n+1}) \text{ for all } u \text{ in } X, n = 1, 2, \ldots$$

$$\text{Therefore, } \sum_{k=1}^{n} \|x_k - x_{k+1}, u\| = \|x_1 - x_2, u\| + \|x_2 - x_3, u\|$$

$$+ \ldots + \|x_n - x_{n+1}, u\| \leq \phi_u(x_0) + \phi_u(x_1)$$

Hence $\sum_{k=1}^{\infty} \|x_k - x_{k+1}, u\| < \infty$ implying that $\|x_k - x_{k+1}, u\| \to 0$ as $k \to \infty$. Thus for any two positive integers m and n with $m > n$ and for all $u \in X$, $\|x_n - x_m, u\| \leq \|x_n - x_{n+1}, u\| + \|x_{n+1} - x_{n+2}, u\| + \ldots + \|x_{m-1} - x_m, u\| \to 0$ as $n, m \to \infty$. Treating the case $n > m$ in a similar way we see that $\{x_n\}$ is a cauchy sequence in X and since X is complete, there exists a point $z \in X$ such that $\{x_n\} \to z$ as $n \to \infty$. Since T is continuous, $\{T(x_n)\} \to T(z)$ as $n \to \infty$.

Now for any $u \in X$, $\|z - T(z), u\| \leq \|z - x_n, u\| + \|x_n - T(z), u\| = \|z - x_n, u\| + \|T(x_{n-1}) - T(z), u\| \to 0$ as $n \to \infty$, showing that $z = T(z)$

If possible let z_1 be another fixed point of T, then for all $u \in X$,

$$\|z - z_1, u\| = \|T(z) - T(z_1), u\| \leq \phi_u(z) - \phi_u(T(z)) + \phi_u(z_1) - \phi_u(T(z_1)) = 0,$$

implying that $z = z_1$

Theorem 3.2 *Let F and G be two continuous self mappings of a 2-Banach space X. Suppose that for any $u \in X$, there exists a function $\phi_u : X \to [0, \infty)$ such that*

$\|F(x) - G(y), u\| \leq \phi_u(x) - \phi_u(F(x)) + \phi_u(y) - \phi_u(G(y))$ *for all $x, y \in X$. Then F and G have a unique common fixed point in X.*

Proof. For $x_0 \in X$, $y_0 \in X$,

$$\text{take } x_1 = F(x_0), x_2 = F(x_1), \ldots, x_{n+1} = F(x_n) \text{ and}$$

$$y_1 = G(y_0), y_2 = G(y_1), \ldots, y_{n+1} = G(y_n).$$

$$\text{Now } \sum_{k=1}^{n} \|x_k - y_k, u\| = \sum_{k=1}^{n} \|F(x_{k-1}) - G(y_{k-1}), u\|$$

$$\leq \sum_{k=1}^{n} \{\phi_u(x_{k-1}) - \phi_u(x_k) + \phi_u(y_{k-1}) - \phi_u(y_k)\} \leq \phi_u(x_0) + \phi_u(y_0)$$

Again $\|x_{k+1} - y_k, u\| = \|F(x_k) - G(y_{k-1}), u\| \leq \phi_u(x_k) - \phi_u(x_{k+1}) + \phi_u(y_{k-1}) - \phi_u(y_k)$ for all $u \in X, k = 1, 2, \ldots$

Therefore $\sum_{k=1}^{n}\|x_{k+1}-y_k,u\| \leq \phi_u(x_1)+\phi_u(y_0)$. Consequently

$\sum_{k=1}^{n}\|x_k-x_{k+1},u\| \leq \sum_{k=1}^{n}\|x_k-y_k,u\| + \sum_{k=1}^{n}\|x_{k+1}-y_k,u\| \leq \phi_u(x_0)+\phi_u(x_1)+2\phi_u(y_0)$. Hence $\sum_{k=1}^{\infty}\|x_k-x_{k+1},u\| < \infty$, implying that $\|x_n-x_{n+1},u\| \to 0$ as $n\to\infty$. Thus for any two positive integers m and n, with $m>n$ and for $u\in X$,

$$\|x_n-x_m,u\| \leq \|x_n-x_{n+1},u\| + \|x_{n+1}-x_{n+2},u\| + \dots$$

$$+\|x_{m-1}-x_m,u\| \to 0 \text{ as } m,n\to\infty$$

Treating the case $n>m$ similarly, we see that $\{x_n\}$ is Cauchy in X.

Similarly we can show that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, let $z=\lim_n x_n$ and $z'=\lim_n y_n, z, z'\in X$,

i.e. for each $u\in X, \|x_n-z,u\|\to 0$ as $n\to\infty$ and $\|y_n-z',u\|\to 0$ as $n\to\infty$. As F and G is continuous, $\|F(x_n)-F(z),u\|\to 0$ as $n\to\infty$ and $\|G(y_n)-G(z'),u\|\to 0$ as $n\to\infty$.
Therefore $\|z-F(z),u\| \leq \|z-x_{n+1},u\|+\|x_{n+1}-F(z),u\| = \|z-x_{n+1},u\| + \|F(x_n)-F(z),u\|\to 0$ as $n\to\infty$ implying that $z=F(z)$. Similarly, it can be shown that $z'=G(z')$.
Now for each $u\in X, \|z-z',u\| = \|F(z)-G(z'),u\| \leq \phi_u(z)-\phi_u(F(z)) + \phi_u(z')-\phi_u(G(z'))=0$, implying that z is a common fixed point of F and G.
If possible let $w\in X$ be another fixed point of F and G i.e. $F(w)=G(w)=w$.
Now $\|w-z,u\| = \|F(w)-G(z),u\| \leq \phi_u(w)-\phi_u(F(w))+\phi_u(z)-\phi_u(G(z))=0$, showing that z is the unique common fixed point of F and G.

Theorem 3.3 *Let $\{F_\alpha\}_{\alpha\in\triangle}$ be a family of continuous self mappings of a 2-Banach space X. Suppose that for any u in X, there exists function $\phi_u : X\to[0,\infty)$ such that for all $x,y\in X, \|F_\alpha(x)-F_\beta(y),u\| \leq \phi_u(x)-\phi_u(F_\alpha(x)) + \phi_u(y)-\phi_u(F_\beta(y))$ then there exists a unique $z\in X$ satisfying $F_\alpha(z)=z$ for all $\alpha\in\triangle$.*

Proof. Let us take F_α and F_β in place of F and G respectively in Theorem 3.2, an application of which gives a unique $z\in X$ to satisfy $F_\alpha(z)=F_\beta(z)=z$. For any other member F_γ, uniqueness of z gives $F_\gamma(z)=z$ and the proof is complete

An application (Seigel-like Theorem). Let $\{F_\beta\}_{\beta\in\triangle}$ be a family of continuous self mappings of a 2-Banach space X. Suppose that for any u in X, there exists a function $\phi : X\to[0,\infty)$ such that for all $x,y\in X, \|x-F_\beta(y),u\| \leq$

$\phi(y) - \phi(F_\beta(y))$ then there exists a unique $z \in X$ satisfying $F_\beta(z) = z$ for all $\beta \in \triangle$.

Proof. Take F_β any member of the given family. Apply Theorem 3.2 in respect of the pair (I, F_β), I being the identity mapping on X. So there is a unique $z \in X$ with $F_\beta(z) = z$. If $F_\gamma(\neq F_\beta)$ is another member of the family, we have for $u \in X$, $\|F_\gamma(z) - F_\beta(z), u\| \leq \phi(z) - \phi(F_\beta(z)) = 0$, that means $F_\gamma(z) = z$. Hence Theorem is proved.

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