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Some Results on Quantization of Signals and Pair Frames

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Abstract

In this paper we consider quantized frame expansions and pair frames. Especially one method of quantization called $\Sigma\Delta$ algorithm is studied, also we study pair frames and we obtain some properties of them. Moreover by using pair frames, duals and controlled frames, we get some upper bounds for the error obtained in $\Sigma\Delta$ algorithm.

Keywords: *Quantization of signals, Pair frame, Approximation error.*

1 Introduction

In 1946, Gabor [8] introduced a method for reconstructing functions (signals) using a family of elementary functions. Later in 1952, Duffin and Schaeffer [6] presented a similar tool in the context of nonharmonic Fourier series and this is the starting point of frame theory. After some decades, Daubechies, Grossmann and Meyer reintroduced frames in [5].

Let H be a separable Hilbert space and I be a finite or countable set. A sequence $F = \{f_i\}_{i\in I} \subseteq H$ is a *frame* for H, if there exist two constants $A, B > 0$, such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. In this case we say that F is an (A, B)-frame. If only the right-hand side inequality holds, then F is called a *Bessel sequence*. If $\|f_i\| = 1$, for each $i \in I$, then F is called a *normalized* frame. If $F = \{f_i\}_{i\in I}$ is a Bessel sequence, then the operator $S : H \longrightarrow H$, which is defined by $S(f) = \sum_{i\in I}\langle f, f_i\rangle f_i$ is bounded and called the *frame operator* of F. If F is an (A, B)-frame, then $A.Id_H \leq S \leq B.Id_H$. Now define $\tilde{f}_i = S^{-1}f_i$. Then for each $f \in H$, we have

$$\sum_{i\in I}\langle f, \tilde{f}_i\rangle f_i = f, \quad f = \sum_{i\in I}\langle f, f_i\rangle \tilde{f}_i.$$

The sequence $\widetilde{F} = \{\tilde{f}_i\}_{i\in I}$ is an $(\frac{1}{B}, \frac{1}{A})$-frame for H and called the *canonical dual frame* of F. We say that a Bessel sequence $\{g_i\}_{i\in I}$ is a *dual* for the Bessel sequence $\{f_i\}_{i\in I}$, if for each $f \in H$, we have

$$f = \sum_{i\in I}\langle f, f_i\rangle g_i,$$

or equivalently

$$f = \sum_{i\in I}\langle f, g_i\rangle f_i.$$

In signal processing, one of the primary goals is to obtain a digital representation of the signal which is suitable for storage, transmission and recovery. As we stated above if $\{f_i\}_{i\in I}$ is a frame for H, then every signal $f \in H$ has an expansion like $f = \sum_{i\in I} c_i f_i$. This expansion is not digital, since the coefficient sequence $\{c_i\}_{i\in I}$ is real or complex valued, therefore it is needed to reduce this sequence to a discrete, and preferably finite set. This step is called *quantization*, for more information, see [2]. In this paper, we consider one method of quantization called $\Sigma\Delta$ algorithm, we obtain some upper error bounds especially by considering pair frames, duals and controlled frames.

2 Quantized Frame Expansions

We recall the definition of pair frames, for more study see [7]:

Definition 2.1. *Let H be a Hilbert space, $F = \{f_i\}_{i\in I}, G = \{g_i\}_{i\in I} \subseteq H$ and $\{m_i\}_{i\in I} \subseteq \mathbb{C}$.*

(i) (m, G, F) *is called an* m-pair Bessel *if the operator $S_{mGF} : H \longrightarrow H$*

$$S_{mGF}f = \sum_{i\in I} m_i\langle f, f_i\rangle g_i,$$

is bounded. If $m_i = 1$, for each $i \in I$, then (G, F) is called a pair Bessel.

(ii) *Suppose that (m, G, F) is an m-pair Bessel. We say that (m, G, F) is an* m-pair frame *if S_{mGF} is invertible. If $m_i = 1$, for each $i \in I$, then (G, F) is called a* pair frame.

(iii) *Let (m, G, F) be an m-pair Bessel. We call it an* m-pair dual *if*

$$f = \sum_{i \in I} m_i \langle f, f_i \rangle g_i, \forall f \in H.$$

(iv) *Let (m, G, F) be an m-pair Bessel. Then it is called a* near identity pair frame *if there exists a nonzero element $a \in \mathbb{C}$ such that $\|Id_H - aS_{mGF}\| < 1$. In this case (m, G, F) is called an a-*near identity pair frame.

Lemma 2.2. *Let H be a Hilbert space. If $\{f_i\}_{i \in I}$ is an (A, B) frame with the frame operator S, then $A^\alpha.Id_H \leq S^\alpha \leq B^\alpha.Id_H$, for $\alpha \geq 0$ and $B^\alpha.Id_H \leq S^\alpha \leq A^\alpha.Id_H$, for $\alpha < 0$.*

Proof. First suppose that $\alpha \geq 0$. If $\lambda \in \sigma(S)$, since S is positive and $S \leq B.Id_H$, then $\lambda \in \mathbb{R}^+$ and $\lambda \leq \|S\| \leq B$. For $\lambda < A$, we have $S - \lambda.Id_H \geq (A - \lambda)Id_H$. Since $A - \lambda > 0$, then $S - \lambda.Id_H$ is invertible, therefore $\lambda \notin \sigma(S)$. Thus $\sigma(S) \subseteq [A, B]$. Now define $f(z) = B^\alpha - z^\alpha$. Since the spectrum of S does not contain zero and negative numbers, then f is analytic on $\sigma(S)$, hence

$$\sigma(B^\alpha.Id_H - S^\alpha) = \sigma(f(S)) = f(\sigma(S)) \qquad [\text{ by Theorem 4.10 in [4]}].$$

Since $\sigma(S) \subseteq [A, B]$ and $\alpha \geq 0$, then $f(\sigma(S)) \subseteq \mathbb{R}^+$. Hence $(B^\alpha.Id_H - S^\alpha)$ is a positive operator. It means that $S^\alpha \leq B^\alpha.Id_H$. Now define $g(z) = z^\alpha - A^\alpha$. We have

$$\sigma(S^\alpha - A^\alpha.Id_H) = \sigma(g(S)) = g(\sigma(S)).$$

Since $\sigma(S) \subseteq [A, B]$ and $\alpha \geq 0$, then $g(\sigma(S)) \subseteq \mathbb{R}^+$. Thus $S^\alpha - A^\alpha.Id_H$ is positive. For $\alpha < 0$, put $\eta = -\alpha$. By the first case, we have $A^\eta.Id_H \leq S^\eta \leq B^\eta.Id_H$. Now by Theorem 2.2.5 in [9], $B^\alpha.Id_H \leq S^\alpha \leq A^\alpha.Id_H$. □

Proposition 2.3. *Let $\{f_i\}_{i \in I}$ be an (A, B) frame, $\alpha \in \mathbb{R}$ and $m = \{m_i\}_{i \in I}$ with $m_i = 1$, for each $i \in I$. Then*

(i) *If $\alpha \geq 0$ and $\frac{B^{2\alpha+2}}{A^2(A^\alpha+1)^2} < 1$, then $(m, \{(Id_H + S^\alpha)^{-1}S^{-1}f_i\}_{i \in I}, \{f_i\}_{i \in I})$ is an 1-near identity pair frame.*

(ii) *If $\alpha < 0$ and $\frac{B^2 A^{2\alpha-2}}{(B^\alpha+1)^2} < 1$, then $(m, \{(Id_H + S^\alpha)^{-1}S^{-1}f_i\}_{i \in I}, \{f_i\}_{i \in I})$ is an 1-near identity pair frame.*

Proof. (i) Let $g_i = (Id_H + S^\alpha)^{-1}S^{-1}f_i$ and $h_i = f_i + S^\alpha f_i$. It is clear that $G = \{g_i\}_{i \in I}$ and $Z = \{h_i\}_{i \in I}$ are frames. Since the frame operator of $\{h_i\}_{i \in I}$ is $(Id_H + S^\alpha)S(Id_H + S^\alpha)$, then

$$\{(Id_H + S^\alpha)^{-1}S^{-1}(Id_H + S^\alpha)^{-1}(Id_H + S^\alpha)f_i\}_{i \in I} = \{g_i\}_{i \in I},$$

is the canonical dual frame of $\{h_i\}_{i\in I}$. Now for each $f \in H$, we have

$$\sum_{i\in I} |\langle f, f_i - h_i\rangle|^2 = \sum_{i\in I} |\langle S^\alpha f, f_i\rangle|^2 \leq B\|S^\alpha\|^2\|f\|^2 \leq B^{2\alpha+1}\|f\|^2,$$

the last inequality follows from the above lemma and Theorem 2.2.5 in [9]. Also we have

$$\begin{aligned}\sum_{i\in I} |\langle f, g_i\rangle|^2 &\leq B\|S^{-1}\|^2\|(Id_H + S^\alpha)^{-1}\|^2\|f\|^2 \\ &\leq \frac{B}{A^2(A^\alpha+1)^2}\|f\|^2.\end{aligned}$$

Note that the inequality $\|(Id_H + S^\alpha)^{-1}\| < \frac{1}{A^\alpha+1}$, follows from the above lemma. Now we have

$$\begin{aligned}\|(Id_H - S_{mGF})x\| = \|(S_{mGZ} - S_{mGF})x\| &= \sup_{\|y\|=1} |\langle (S_{mGZ} - S_{mGF})x, y\rangle| \\ &\leq \Big[B^{2\alpha+1}\Big(\frac{B}{A^2(A^\alpha+1)^2}\Big)\Big]^{\frac{1}{2}}\|x\|,\end{aligned}$$

and since $B^{2\alpha+1}(\frac{B}{A^2(A^\alpha+1)^2}) < 1$, then the result follows.

The proof of (ii) is similar to (i). □

Now we recall some definitions from [2]:
Let $K \in \mathbb{N}$ and $\delta > 0$. Given the midrise quantization alphabet

$$\mathcal{A}_K^\delta = \{(-K+\frac{1}{2})\delta, (-K+\frac{3}{2})\delta, \ldots, (-\frac{1}{2})\delta, (\frac{1}{2})\delta, \ldots, (K-\frac{1}{2})\delta\},$$

consisting of 2K elements, the 2K-level midrise uniform scalar quantizer with stepsize δ is defined by

$$Q(u) = argmin_{q\in\mathcal{A}_k^\delta}|u - q|.$$

Thus, $Q(u)$ is the element of the alphabet which is closest to u. If two elements of $\mathcal{A}_K^\delta$ are equally close to u, then let $Q(u)$ be the larger of these two elements, i.e., the one larger than u.

Definition 2.4. *Let $N \in \mathbb{N}$, $\delta > 0$, $\{c_i\}_{i=1}^N \subseteq \mathbb{R}$ and p be a permutation of $\{1, 2, \ldots, N\}$. The associated first order $\Sigma\Delta$ quantizer is defined by the iteration*

$$u_i = u_{i-1} + c_{p(i)} - q_i, \quad q_i = Q(u_{i-1} + c_{p(i)}),$$

where u_0 is a specified constant. The first order $\Sigma\Delta$ quantizer produces the quantized sequence $\{q_i\}_{i=1}^N$, and an auxiliary sequence $\{u_i\}_{i=0}^N$ of state variables.

So if $f \in \mathbb{R}^d$ has an expansion by using the sequence $\{c_{p(i)}\}_{i=1}^N$, then after quantization, it will be changed to a signal $\widetilde{f}$ which it's expansion uses $\{q_i\}_{i=1}^N$.

Definition 2.5. *Let $F = \{f_i\}_{i=1}^N$ be a finite frame for $\mathbb{R}^d$, and let p be a permutation of $\{1, \dots, N\}$. The* variation *of the frame F with respect to p is defined by*

$$\sigma(F, p) := \sum_{i=1}^{N-1} \|f_{p(i)} - f_{p(i+1)}\|.$$

It was shown in [2] that variation is an important tool to find a good approximation error. Note that if $f \in \mathbb{R}^d$ and $\widetilde{f}$ is quantized of f, then it is desirable that $\|f - \widetilde{f}\|$ gets small. In the rest of this paper, we obtain some upper bounds for $\|f - \widetilde{f}\|$ by using different concepts related to frame theory such as controlled frames which are considered in [1]:

Definition 2.6. *Let R be a bounded and invertible operator on H. A sequence $F = \{f_i\}_{i \in I} \subseteq H$ is called a frame controlled by T or $T-$controlled frame if there exist two positive numbers A and B such that*

$$A\|f\|^2 \leq \sum_{i \in I} \langle f, f_i \rangle \langle Rf_i, f \rangle \leq B\|f\|^2,$$

for each $f \in H$. It was proved in [1] that in this case F is a frame and the operator $S_R : H \longrightarrow H$ which is defined by

$$S_R(f) = \sum_{i \in I} \langle f, f_i \rangle Rf_i,$$

is bounded, positive and invertible.

Suppose that $H = \mathbb{R}^d$, $\{f_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ are two frames for H and T is an invertible operator such that

$$f = \sum_{i=1}^{N} c_{p(i)} T^{-1} g_{p(i)}, \quad c_{p(i)} = \langle f, f_{p(i)} \rangle,$$

where p is a permutation of $\{1, \dots, N\}$. Now suppose that $\widetilde{f}$ is quantized of f obtained by $\Sigma\Delta$ algorithm, so $\widetilde{f} = \sum_{i=1}^N q_i T^{-1} g_{p(i)}$.

Theorem 2.7. *Let $F = \{f_i\}_{i=1}^N$ and $G = \{g_i\}_{i=1}^N$ be two frames for $\mathbb{R}^d$, $f \in \mathbb{R}^d$ and for each $1 \leq i \leq N-1$, $|u_i| \leq \frac{\delta}{2}$. Then*

(i) $\|f - \widetilde{f}\| \leq \|T^{-1}\| \Big(\sigma(G, p) \frac{\delta}{2} + |u_N| \|g_{p(N)}\| + |u_0| \|g_{p(1)}\| \Big).$

(ii) *If F and G are duals, then*

$$\|f-\widetilde{f}\| \leq \sigma(G,p)\frac{\delta}{2}+|u_N|\|g_{p(N)}\|+|u_0|\|g_{p(1)}\|.$$

(iii) *If F is an R-controlled frame, then*

$$\|f-\widetilde{f}\| \leq \|{S_R}^{-1}R\|\Big(\sigma(F,p)\frac{\delta}{2}+|u_N|\|f_{p(N)}\|+|u_0|\|f_{p(1)}\|\Big).$$

(iv) *If (m,G,F) is an m-pair frame and $m=\{m_i\}_{i=1}^N$, then*

$$\|f-\widetilde{f}\| \leq \|S_{mGF}^{-1}\|\Big(\sigma(m.G,p)\frac{\delta}{2}+|u_N|\|m_{p(N)}.g_{p(N)}\|+|u_0|\|m_{p(1)}.g_{p(1)}\|\Big),$$

where $m.G=\{m_i g_i\}_{i=1}^N$.

(v) *If (m,G,F) is an m-pair dual and $m=\{m_i\}_{i=1}^N$, then*

$$\|f-\widetilde{f}\| \leq \sigma(m.G,p)\frac{\delta}{2}+|u_N|\|m_{p(N)}.g_{p(N)}\|+|u_0|\|m_{p(1)}.g_{p(1)}\|,$$

where $m.G=\{m_i g_i\}_{i=1}^N$.

(vi) *If (m,G,F) is an a-near identity pair frame, then*

$$\|f-\widetilde{f}\| \leq \|S_{mGF}^{-1}\|\Big(\sigma(m.G,p)\frac{\delta}{2}+|u_N|\|m_{p(N)}.g_{p(N)}\|+|u_0|\|m_{p(1)}.g_{p(1)}\|\Big).$$

Proof. (i) We have

$$\begin{aligned} f-\widetilde{f} &= \sum_{i=1}^{N}(c_{p(i)}-q_i)T^{-1}g_{p(i)} \\ &= \sum_{i=1}^{N}(u_i-u_{i-1})T^{-1}g_{p(i)} \\ &= \sum_{i=1}^{N-1}u_iT^{-1}(g_{p(i)}-g_{p(i+1)}) \\ &+ u_NT^{-1}g_{p(N)}-u_0T^{-1}g_{p(1)}, \end{aligned}$$

consequently

$$\begin{aligned} \|f-\widetilde{f}\| &\leq \sum_{i=1}^{N-1}\frac{\delta}{2}\|T^{-1}\|\|g_{p(i)}-g_{p(i+1)}\| \\ &+ |u_N|\|T^{-1}\|\|g_{p(N)}\|+|u_0|\|T^{-1}\|\|g_{p(1)}\| \\ &= \|T^{-1}\|\Big(\frac{\delta}{2}\sigma(G,p)+|u_N|\|g_{p(N)}\|+|u_0|\|g_{p(1)}\|\Big). \end{aligned}$$

(ii) If F and G are duals, then $T = Id_H$ and the result follows from part (i).
(iii) If F is an R-controlled frame, then

$$f = \sum_{i=1}^{N} c_{p(i)} S_R^{-1} R f_{p(i)}, \quad c_{p(i)} = \langle f, f_{p(i)} \rangle,$$

so $\{S_R^{-1} R f_i\}_{i=1}^N$ is a dual for F and by part (ii), we have

$$\begin{aligned} \|f - \widetilde{f}\| &\leq \sigma(\{S_R^{-1} R f_i\}_{i=1}^N, p)\frac{\delta}{2} + |u_N|\|S_R^{-1} R f_{p(N)}\| \\ &+ |u_0|\|S_R^{-1} R f_{p(1)}\| \\ &\leq \|S_R^{-1} R\|\Big(\sigma(F, p)\frac{\delta}{2} + |u_N|\|f_{p(N)}\| + |u_0|\|f_{p(1)}\|\Big). \end{aligned}$$

(iv) Let (m, G, F) be an m-pair frame and $m = \{m_i\}_{i=1}^N$. Then it is easy to see that $m.G$ is a Bessel sequence and

$$f = \sum_{i=1}^{N} c_{p(i)} S_{mGF}^{-1} m_{p(i)} g_{p(i)}, \quad c_{p(i)} = \langle f, f_{p(i)} \rangle,$$

and the result follows from part (i).
(v) If (m, G, F) is an m-pair dual, then $S_{mGF} = Id_H$ and the result follows from part (iv).
(vi) Since $\|Id_H - aS_{mGF}\| < 1$, then aS_{mGF} is invertible, so S_{mGF} is invertible and the result follows from part (i). □

The following corollary considers normalized frames which are very important in frame theory:

Corollary 2.8. *Suppose that $f \in \mathbb{R}^d$, F and G are two frames in $\mathbb{R}^d$ such that G is normalized and for each $1 \leq i \leq N-1$, $|u_i| \leq \frac{\delta}{2}$. Then*

(i) $\|f - \widetilde{f}\| \leq \|T^{-1}\|\Big(\sigma(G, p)\frac{\delta}{2} + |u_N| + |u_0|\Big)$.

(ii) *If F and G are duals, then*

$$\|f - \widetilde{f}\| \leq \sigma(G, p)\frac{\delta}{2} + |u_N| + |u_0|.$$

(iii) *If F is an R-controlled frame and F is normalized, then*

$$\|f - \widetilde{f}\| \leq \|S_R{}^{-1} R\|\Big(\sigma(F, p)\frac{\delta}{2} + |u_N| + |u_0|\Big).$$

(iv) *If (m,G,F) is an m-pair frame and* $m = \{m_i\}_{i=1}^N$, *then*

$$\|f - \widetilde{f}\| \le \|S_{mGF}^{-1}\|\Big(\sigma(m.G,p)\frac{\delta}{2} + |u_N||m_{p(N)}| + |u_0||m_{p(1)}|\Big),$$

where $m.G = \{m_i g_i\}_{i=1}^N$.

(v) *If* (m, G, F) *is an m-pair dual and* $m = \{m_i\}_{i=1}^N$, *then*

$$\|f - \widetilde{f}\| \le \sigma(m.G,p)\frac{\delta}{2} + |u_N||m_{p(N)}| + |u_0||m_{p(1)}|,$$

where $m.G = \{m_i g_i\}_{i=1}^N$.

(vi) *If* (m, G, F) *is an a-near identity pair frame, then*

$$\|f - \widetilde{f}\| \le \|S_{mGF}^{-1}\|\Big(\sigma(m.G,p)\frac{\delta}{2} + |u_N||m_{p(N)}| + |u_0||m_{p(1)}|\Big).$$

Corollary 2.9. *Suppose that* $f \in \mathbb{R}^d$, F *and* G *are two frames in* $\mathbb{R}^d$ *such that* G *is normalized and for each* $0 \le i \le N$, $|u_i| \le \frac{\delta}{2}$. *Then*

(i) $\|f - \widetilde{f}\| \le \|T^{-1}\|\frac{\delta}{2}\Big(\sigma(G,p) + 2\Big)$.

(ii) *If* F *and* G *are duals, then*

$$\|f - \widetilde{f}\| \le \frac{\delta}{2}\Big(\sigma(G,p) + 2\Big).$$

(iii) *If* F *is an R-controlled frame and* F *is normalized, then*

$$\|f - \widetilde{f}\| \le \|{S_R}^{-1}R\|\frac{\delta}{2}\Big(\sigma(F,p) + 2\Big).$$

(iv) *If (m,G,F) is an m-pair frame and* $m = \{m_i\}_{i=1}^N$, *then*

$$\|f - \widetilde{f}\| \le \|S_{mGF}^{-1}\|\frac{\delta}{2}\Big(\sigma(m.G,p) + |m_{p(N)}| + |m_{p(1)}|\Big),$$

where $m.G = \{m_i g_i\}_{i=1}^N$.

(v) *If* (m, G, F) *is an m-pair dual and* $m = \{m_i\}_{i=1}^N$, *then*

$$\|f - \widetilde{f}\| \le \frac{\delta}{2}\Big(\sigma(m.G,p) + |m_{p(N)}| + |m_{p(1)}|\Big),$$

where $m.G = \{m_i g_i\}_{i=1}^N$.

(vi) *If* (m, G, F) *is an a-near identity pair frame, then*

$$\|f - \widetilde{f}\| \leq \|S_{mGF}^{-1}\| \frac{\delta}{2} \Big(\sigma(m.G, p) + |m_{p(N)}| + |m_{p(1)}| \Big).$$

Note that parts (i) of the above results generalize Theorem 3.4 and Corollary 3.6 in [2].

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