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Tensor Commutation Matrices and Some Generalizations of the Pauli Matrices

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Abstract

In this paper, some tensor commutation matrices are expressed in termes of the generalized Pauli matrices by tensor products (of the Pauli matrices). This expression and some other relations in terms of other generalizations of the Pauli matrices make us to notice that there should be another generalization of the Pauli matrices, which generalizes the generalization of the Pauli matrices by tensor product.

Keywords: *Tensor product, Tensor commutation matrices, Pauli matrices, Generalized Pauli matrices, Kibler matrices, Nonions.*

1 Introduction

The tensor product of matrices is not commutative in general. However, a tensor commutation matrix (TCM) $n \otimes p$, $S_{n\otimes p}$ commutes the tensor product $A \otimes B$ for any $A \in \mathbf{C}^{n\times n}$ and $B \in \mathbf{C}^{p\times p}$ as the following

$$S_{n\otimes p}(A \otimes B) = (B \otimes A)S_{n\otimes p}$$

The tensor commutation matrices (TCMs) are useful in quantum theory and for solving matrix equations. In quantum theory $S_{2\otimes 2}$ can be expressed in the following way (Cf. for example[1, 9, 2])

$$S_{2\otimes 2} = \frac{1}{2}I_2 \otimes I_2 + \frac{1}{2}\sum_{i=1}^{3} \sigma_i \otimes \sigma_i \tag{1}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices, I_2 is the 2×2 unit-matrix.

$$S_{2\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_{3\otimes 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The Gell-Mann matrices are a generalization of the Pauli matrices. The TCM $n \otimes n$ can be expressed in terms of $n \times n$ Gell-Mann matrices, under the following expression [5]

$$S_{n\otimes n} = \frac{1}{n}I_n \otimes I_n + \frac{1}{2}\sum_{i=1}^{n^2-1} \Lambda_i \otimes \Lambda_i \tag{2}$$

This expression of $S_{n\otimes n}$ suggests us the topic of generalizing the formula (1) to an expression in terms of some generalized Pauli matrices.
For the calculus, we have used SCILAB, a mathematical software for numerical analysis.

2 Some Generalizations of the Pauli Matrices

In this section we give some generalizations of the Pauli matrices other than the Gell-Mann matrices.

2.1 Kibler Matrices

Let $j = exp(\frac{2\pi i}{3})$, the Kibler matrices are

$k_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $k_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $k_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,

$k_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}$, $k_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}$, $k_5 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix}$,

$k_6 = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}$, $k_7 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}$ $k_8 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}$, are the 3×3 Kibler matrices [4].

The Kibler matrices are traceless and

$$\frac{1}{3} I_3 \otimes I_3 + \frac{1}{3} \sum_{i=1}^{8} \ k_i \otimes \ k_i = P \tag{3}$$

with $P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ a permutation matrix.

2.2 The Nonions

The nonions matrices are [10]

$q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}$

$q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}$, $q_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $q_5 = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}$,

$q_6 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}$, $q_7 = \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $q_8 = \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The nonions are traceless and

$$\frac{1}{3} I_3 \otimes I_3 + \frac{1}{3} \sum_{i=1}^{8} \ q_i \otimes \ q_i = P \tag{4}$$

2.3 Generalization by Tensor Products of Pauli Matrices

There are also some generalization of the Pauli matrices constructed by tensor products (of the Pauli matrices), namely $(\sigma_i \otimes \sigma_j)_{0\leq i,j\leq 3}$, $(\sigma_i \otimes \sigma_j \otimes \sigma_k)_{0\leq i,j,k\leq 3}$, $(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n})_{0\leq i_1,i_2,\ldots,i_n\leq 3}$ (Cf. for example, [7, 8]). The elements of the set $(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n})_{0\leq i_1,i_2,\ldots,i_n\leq 3}$ satisfy the following properties (Cf. for example, [7, 8])

$$\Sigma_j^+ = \Sigma_j \ (hermitian) \tag{5}$$

$$\Sigma_j^2 = I_{2^n} \ (Square\ root\ of\ unity) \tag{6}$$

$$Tr\Sigma_j^+\Sigma_k = 2^n\delta_{jk} \ (Orthogonal) \tag{7}$$

3 Expression of a Tensor Commutation Matrix in Terms of the Generalized Pauli Matrices by Tensor Products

Definition 3.1 *For $n \in N^*, n \geq 2$, we call tensor commutation matrix $n \otimes n$ the permutation matrix $S_{n\otimes n}$ such that*

$$S_{n\otimes n}(a \otimes b) = b \otimes a$$

for any $a,b \in \boldsymbol{C}^{n\times 1}$.

The relations (1), (2), (3) and (4) suggest us that there should be a generalization of the Pauli matrices $(s_i)_{0\leq i\leq n^2-1}$ such that

$$S_{n\otimes n} = \frac{1}{n} I_n \otimes I_n + \frac{1}{n}\sum_{i=1}^{n^2-1} s_i \otimes s_i \tag{8}$$

We would like to look for matrices $(s_i)_{0\leq i\leq 8}$ which satisfy the relation (8). The TCMs $S_{4\otimes 4}$, $S_{8\otimes 8}$ can be expressed respectively in terms of the generalized Pauli matrices $(\sigma_i \otimes \sigma_j)_{0\leq i,j\leq 3}$, $(\sigma_i \otimes \sigma_j \otimes \sigma_k)_{0\leq i,j,k\leq 3}$ in the following way.

$$S_{4\otimes 4} = \frac{1}{4} I_4 \otimes I_4 + \frac{1}{4}\sum_{i=1}^{15} s_i \otimes s_i$$

where $s_1 = \sigma_0 \otimes \sigma_1$, $s_2 = \sigma_0 \otimes \sigma_2$, $\ldots$, $s_{13} = \sigma_3 \otimes \sigma_1$, $s_{14} = \sigma_3 \otimes \sigma_2$, $s_{15} = \sigma_3 \otimes \sigma_3$.

$$S_{8\otimes 8} = \frac{1}{8} I_8 \otimes I_8 + \frac{1}{8}\sum_{i=1}^{63} S_i \otimes S_i$$

where $S_1 = \sigma_0 \otimes \sigma_0 \otimes \sigma_1$,..., $S_{63} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3$ That is to say, this generalization by the tensor products satisfy the relation (8). So we think that (8) should be true for $n = 2^p$, $p \in N$, $p \geq 2$. For proving it, we give the following lemma which is the generalization of a proposition in [6].

Lemma 3.2 *If* $\sum_{j=1}^{m} M_j \otimes N_j = \sum_{i=1}^{n} A_i \otimes B_i$ *then*
$\sum_{j=1}^{m} M_j \otimes K \otimes N_j = \sum_{i=1}^{n} A_i \otimes K \otimes B_i$

Proof. Let $K = (K_{j_2}^{j_1}) \in \mathbf{C}^{q\times s}$, $M_j = M_{(j)k_2}^{k_1} \in \mathbf{C}^{p\times r}$, $A_i = A_{(i)k_2}^{k_1} \in \mathbf{C}^{p\times r}$, $N_j = N_{(j)l_2}^{l_1} \in \mathbf{C}^{t\times u}$ and $B_i = B_{(j)l_2}^{l_1} \in \mathbf{C}^{t\times u}$

$$\sum_{j=1}^{m} M_{(j)k_2}^{k_1} N_{(j)l_2}^{l_1} = \sum_{i=1}^{n} A_{(i)k_2}^{k_1} B_{(j)l_2}^{l_1}$$

$$K_{j_2}^{j_1} \sum_{j=1}^{m} M_{(j)k_2}^{k_1} N_{(j)l_2}^{l_1} = K_{j_2}^{j_1} \sum_{i=1}^{n} A_{(i)k_2}^{k_1} B_{(j)l_2}^{l_1}$$

$$\sum_{j=1}^{m} M_{(j)k_2}^{k_1} K_{j_2}^{j_1} N_{(j)l_2}^{l_1} = \sum_{i=1}^{n} A_{(i)k_2}^{k_1} K_{j_2}^{j_1} B_{(j)l_2}^{l_1}$$

$\sum_{j=1}^{m} M_{(j)k_2}^{k_1} K_{j_2}^{j_1} N_{(j)l_2}^{l_1}$ and $\sum_{i=1}^{n} A_{(i)k_2}^{k_1} K_{j_2}^{j_1} B_{(j)l_2}^{l_1}$
are respectively the elements of the $k_1 j_1 l_1$ row and $k_2 j_2 l_2$ colomn the $\sum_{j=1}^{m} M_j \otimes K \otimes N_j$ and $\sum_{i=1}^{n} A_i \otimes K \otimes B_i$. That is true for any $k_1, j_1, l_1, k_2, j_2, l_2$.
Hence, $\sum_{j=1}^{m} M_j \otimes K \otimes N_j = \sum_{i=1}^{n} A_i \otimes K \otimes B_i$ ■

Proposition 3.3 *For any* $n \in N^*, n \geq 2$,

$$S_{2^n \otimes 2^n} = \frac{1}{2^n} \sum_{i_1, i_2, \ldots, i_n = 0}^{3} (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}) \otimes (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n})$$

Proof. Let us prove it by reccurence. According to the relation (1), the proposition is true for $n = 1$. Suppose that it is true for a $n \in N^*, n \geqslant 2$. Let us take $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which form a basis of the $\mathbf{C}$-vector space $\mathbf{C}^{2\times 1}$.

It is sufficient to prove

$$\frac{1}{2^{n+1}}\sum_{j_1,j_2,\ldots,j_{n+1}=0}^{3}(\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_{n+1}})\otimes\ (\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_{n+1}})(e_{\alpha_1}\otimes\cdots\otimes\ e_{\alpha_{n+1}})$$
$$\otimes\ (e_{\beta_1}\otimes\cdots\otimes\ e_{\beta_{n+1}})=(e_{\beta_1}\otimes\cdots\otimes\ e_{\beta_{n+1}})\otimes(e_{\alpha_1}\otimes\cdots\otimes\ e_{\alpha_{n+1}})$$

$$\frac{1}{2^{n}}\sum_{j_1,j_2,\ldots,j_{n}=0}^{3}(\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_{n}})\otimes(\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_{n}})(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{n}})\otimes(e_{\beta_1}\otimes\cdots\otimes e_{\beta_{n}})$$
$$=(e_{\beta_1}\otimes\cdots\otimes\ e_{\beta_{n}})\otimes(e_{\alpha_1}\otimes\cdots\otimes\ e_{\alpha_{n}})$$

that is

$$\frac{1}{2^{n}}\sum_{j_1,j_2,\ldots,j_{n}=0}^{3}((\sigma_{j_1}e_{\alpha_1})\otimes\cdots\otimes(\sigma_{j_n}e_{\alpha_n}))\otimes\ ((\sigma_{j_1}e_{\beta_1})\otimes\cdots\otimes(\sigma_{j_n}e_{\beta_n}))$$
$$=(e_{\beta_1}\otimes\cdots\otimes\ e_{\beta_{n}})\otimes(e_{\alpha_1}\otimes\cdots\otimes\ e_{\alpha_{n}})$$

According to the lemma above

$$\frac{1}{2^{n}}\frac{1}{2}\sum_{j_1,j_2,\ldots,j_n,j_{n+1}=0}^{3}[(\sigma_{j_1}e_{\alpha_1})\otimes\cdots\otimes(\sigma_{j_{n+1}}e_{\alpha_{n+1}})]\otimes[(\sigma_{j_1}e_{\beta_1})\otimes\cdots\otimes(\sigma_{j_{n+1}}e_{\beta_{n+1}})]$$
$$=\frac{1}{2}\sum_{j_{n+1}=0}^{3}(e_{\beta_1}\otimes\cdots\otimes\ e_{\beta_n})\otimes(\sigma_{j_{n+1}}e_{\alpha_{n+1}})\otimes(e_{\alpha_1}\otimes\cdots\otimes\ e_{\alpha_n})\otimes(\sigma_{j_{n+1}}e_{\beta_{n+1}})$$

from the relation (1)

$$S_{2\otimes 2}(\alpha_{n+1}\otimes\beta_{n+1})=\frac{1}{2}\sum_{j_{n+1}=0}^{3}(\sigma_{j_{n+1}}e_{\alpha_{n+1}})\otimes(\sigma_{j_{n+1}}e_{\beta_{n+1}})=e_{\beta_{n+1}}\otimes e_{\alpha_{n+1}}$$

According again to the lemma above

$$\frac{1}{2}\sum_{j_{n+1}=0}^{3}(e_{\beta_1}\otimes\cdots\otimes e_{\beta_n})\otimes(\sigma_{j_{n+1}}e_{\alpha_{n+1}})\otimes(e_{\alpha_1}\otimes\cdots\otimes\ e_{\alpha_n})\otimes(\sigma_{j_{n+1}}e_{\beta_{n+1}})$$
$$=(e_{\beta_1}\otimes\cdots\otimes e_{\beta_n}\otimes e_{\beta_{n+1}})\otimes(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}\otimes e_{\alpha_{n+1}})$$

■

Conclusion and Discussion

We have calculated the left hand side of the relations (3) and (4), respectively for the 3×3 Pauli matrices of Kibler and the nonions in expecting to have the formula (8), for $n = 3$. Instead of $S_{3\otimes 3}$ we have the permutation matrix P as result. However, that makes us to think that there should be other 3×3 Pauli matrices which would satisfy the relation (8). These 3×3 Pauli matrices should not be the normalized Gell-Mann matrices in [3], because the 4×4 matrices which satisfy the relation (8) above are not the 4×4 normalized Gell-Mann matrices in [3], even though these normalized Gell-Mann matrices satisfy (8). The relation (8) is satisfied by the generalized Pauli matrices by tensor products, but only for $n = 2^k$. However, there is no 3×3 matrix, formed by zeros in the diagonal which satisfy both the relations (5) and (6). Thus, the 3×3 Pauli matrices which should satisfy (8), if there exist, do not satisfy both the relations (5),(6) and (7) like the generalized Pauli matrices by tensor products.

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