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Classes of Minimal Words of Small Lengths in a Finitely Generated Free Group

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Abstract

Let F_n denote the free group of finite rank n. A word $w \in F_n$ is called minimal if no type 2 Whitehead automorphism can reduce its length. We explore the Whitehead algorithm in classifying minimal words of small lengths in F_n up to equivalence.

Keywords: *Whitehead automorphisms, Minimal words, Equivalence.*

1 Introduction

In 1936, J. H. C. Whitehead ([4], [5]) used topological means to introduce a theorem which can be used to decide whether two elements of a finitely generated free group are equivalent under an automorphism of the group. Twenty-two years later, E. S. Rapaport gave an algebraic proof of Whitehead's result, and her work was further simplified by Higgins and Lyndon in [1].

In this paper, the corresponding algorithm for classifying all minimal words of any given length in F_n up to equivalence will be introduced. Furthermore, we investigate the equivalence classes of minimal words of lengths $2, 3, 4$ and 5 in F_n, and conclude by establishing some results on the classification.

Definition 1.1.
- *For $w_1, w_2 \in F_n$, we denote* ***equivalence*** *by $\sim$, and write $w_1 \sim_\alpha w_2$ if $\alpha \in Aut(F_n)$ such that $\alpha(w_1) = w_2$.*

- *The set L_n of generators and inverses of F_n is defined as:* $L_n = \{x_1, \cdots, x_n, \bar{x_1}, \cdots, \bar{x_n}\}$ *with* $\bar{x_i} = x_i^{-1}$ *for* $1 \leq i \leq n$. *We define* L_n^* *as the set of words over* L_n.
- *A word* $w \in F_n$ *is said to be* ***minimal*** *if* $|w| \leq |\alpha(w)| \ \forall \ \alpha \in Aut(F_n)$.

Theorem 1.2 (Whitehead)**.** *If* $w_1, w_2 \in F_n$ *such that* $w_1 \sim w_2$*, and* w_2 *is minimal, then there exists a sequence* $\alpha_1, \alpha_2, \cdots, \alpha_l$ *of Whitehead automorphisms such that the following conditions are satisfied:*

- *(K1)* $\alpha_l \alpha_{l-1} \cdots \alpha_1(w_1) = w_2$ *;*
- *(K2)* $|\alpha_{i+1}\alpha_i \cdots \alpha_1(w_1)| \leq |\alpha_i \cdots \alpha_1(w_1)|$ *for* $0 \leq i \leq (l-1)$*, and with strict inequality unless* $\alpha_i \cdots \alpha_1(w_1)$ *is minimal.*

We define the Whitehead automorphisms described above as follows:

Definition 1.3 (Whitehead Automorphisms)**.**
- *A* ***type*** 1 *Whitehead automorphism,* α *is a permutation which acts on elements of* L_n*, and preserves inverses as follows:* $\alpha(\bar{x}) = \overline{\alpha(x)} \ \forall \ x \in L_n$.
- *A* ***type*** 2 *Whitehead automorphism,* β *is that which for a fixed* $a \in L_n$ *and* $\beta(a) = a$*,* β *carries each generator* x *of* L_n *(with* $x \neq a, \bar{a}$*) into one of* $x, xa, \bar{a}x$ *or* $\bar{a}xa$.

Remark 1.4. *As type* 1 *automorphisms are permutations, they do not decrease the length of a word in* F_n*. In the Whitehead theorem, the automorphisms in condition (K2) can only be type* 2 *Whitehead automorphisms (**the likely length decreasing ones**). We refer to either type* 1 *or* 2 *Whitehead automorphisms as Whitehead automorphisms.*

We denote the set of all Whitehead automorphisms of F_n by Ω_n, and the set of Whitehead automorphisms of types 1 and 2 by ${}_1\Omega_n$ and ${}_2\Omega_n$ respectively. $|\Omega_n| < \infty$. In particular, there are $2n(4^{n-1} - 1)$ non-trivial type 2 Whitehead automorphisms for any given F_n. On the other hand, $|{}_1\Omega_n| = 2^n n!$. See [3] for more details. Let β be a type 2 Whitehead automorphism as described in the definition above. We write $\beta = (A, a)$, where $A = \{a, y : \beta(y) = ya \text{ or } \beta(y) = \bar{a}ya\} \subseteq L_n$ with $y \neq a, \bar{a}$. So, if $x \mapsto \bar{a}x$, then $\bar{x} \in A$, and if $x \mapsto \bar{a}xa$, then $x, \bar{x} \in A$.

Proposition 1.5 ([2])**.** *Let* β *be a type* 2 *Whitehead automorphism (with* $a \in L_n$ *fixed). For each* $x \in L_n$*,* β *acts on* x *as follows:*

$$\beta(x) = (A,a)(x) := \begin{cases} x & \text{if } x, \bar{x} \notin A \\ xa & \text{if } x \in A \text{ and } \bar{x} \notin A \\ \bar{a}x & \text{if } x \notin A \text{ and } \bar{x} \in A \\ \bar{a}xa & \text{if } x, \bar{x} \in A. \end{cases}$$

The following is an immediate consequence of Proposition 1.5.

Corollary 1.6. (A, a) *never reduces length of a word* w *if both* a ***and*** $\bar{a}$ *are not in* w.

Notation 1.7. *Given a word* $w \in F_n$. *We denote the set of type* 2 *Whitehead automorphisms that do not reduce the length of* w *as described in Corollary 1.6 by* $\underline{{}_2\Omega_n}(w)$ *(or* $\underline{{}_2\Omega_n}$ *for short), and called the "bad type* 2 *Whitehead automorphisms". Similarly, the complement of* $\underline{{}_2\Omega_n}$ *in* ${}_2\Omega_n$ *will be denoted by* $\overline{{}_2\Omega_n}$*, and called the "maybe good type* 2 *Whitehead automorphisms". A subset of* $\overline{{}_2\Omega_n}$ *consisting of only the length reducing type* 2 *Whitehead automorphisms will be called the "good type* 2 *Whitehead automorphisms", and denoted by* $\overline{\overline{{}_2\Omega_n}}$.

McCool in [2] demonstrated that $Aut(F_n)$ is finitely presented, with the Whitehead automorphisms as the generators.

2 Main Results

From the Whitehead's theorem and its consequences studied in the last section, it is evident that the Whitehead algorithm is a powerful tool for determining equivalence classes of minimal words in a finitely generated free group.

Definition 2.1.

- *A word* $w \in F_n$ *is said to be* ***Whitehead reducible*** *if it is non-minimal; i.e. if there exist a type* 2 *Whitehead automorphism* β *such that* $|\beta(w)| < |w|$. *On the other hand,* w *is* ***Whitehead irreducible*** *if it is not Whitehead reducible.*

- *A word* $w \in F_n$ *is said to be* ***Whitehead equivalent*** *to another word* $w^* \in F_n$ *if there exist Whitehead automorphisms* $\gamma_1, \cdots, \gamma_k$ *such that* $\gamma_k \cdots \gamma_1(w) = w^*$.

From now onwards, whenever we mention reducible (irreducible), we mean Whitehead reducible (irreducible). Similar statement holds for equivalence.

Lemma 2.2. *Two irreducible words of different lengths are not equivalent.*

Proof: Let w and w^* be two irreducible words of different lengths. Without loss of generality, suppose for contradiction that w^* is gotten from w by Whitehead equivalence, and $|w| > |w^*|$. By Remark 1.4 and Notation 1.7, the automorphisms that induce the equivalence must involve a good type 2 Whitehead automorphism. This further implies that w is reducible; thus contradicting the hypothesis that w is irreducible.

2.1 Algorithm Developments

We introduce the Whitehead algorithm aimed at classifying all minimal words of any given length in F_n up to equivalence. First and foremost, construct algorithms for the set of Whitehead automorphisms ${}_1\Omega_n$ and ${}_2\Omega_n$ as well as a function "IsMin" for checking whether a word $w \in F_n$ is minimal or not. Find all the reduced words of length l in F_n, and call it R_{ln}. Finally, use the IsMin function to find all corresponding minimal words of length l in F_n from R_{ln}, and denote them by M_{ln}. Then follow the procedures below.

Algorithm 2.3. *For finding a minimal word equivalent to a word $w \in F_n$.*

- *Step 1. Use the IsMin function constructed to check whether w is minimal. If yes, return w and call it w^{last}; else proceed to Step 2.*
- *Step 2. Substitute $\beta(w)$ for w whenever $|\beta(w)| < |w|$ for $\beta \in {}_2\Omega_n$ (may take $\beta \in \overline{{}_2\Omega_2}$ for faster computation). Repeat until no further reduction is obtainable. Return the resulting irreducible word w^{last}.*

Algorithm 2.4. *For finding a list of minimal words equivalent to a word $w \in F_n$.*

- *Step 1. Use Algorithm 2.3 to find w^{last}, then create a singleton list "MinEquivElts" containing w^{last}.*
- *Step 2. For each $\beta \in {}_2\Omega_n$, if $\beta(w^{last}) = \underline{w}$, $|\underline{w}| = |w^{last}|$ and $\underline{w}$ is not already in the list MinEquivElts, then append $\underline{w}$ to MinEquivElts. Repeat the process for each $\alpha \in {}_1\Omega_n$.*

Algorithm 2.5. *For determining whether two elements $w, w^* \in F_n$ are equivalent.*

- *Step 1. Use Algorithm 2.3 to find w^{last} corresponding to w, and Algorithm 2.4 to find MinEquivElts for w^*.*
- *Step 2. Return true if w^{last} is contained in MinEquivElts, and false if otherwise.*

Algorithm 2.5 can be used to classify all minimal words of a certain length l in F_n up to equivalence. We shall investigate this in the next section.

2.2 Equivalence Classes of Minimal Words of Lengths 2, 3, 4 and 5

Definition 2.6.
- *(T1) A **cyclic structure** of a reduced word w is a cyclic representation of w.*

- *(T2) Two non-identity elements of a free group are said to be in the* ***same cyclic structure*** *(say (w)) if both words represent the same cyclic word.*

- *(T3) A reduced word is called* ***exact*** *if no other reduced word can be in its cyclic structure. In other words, if it is the only reduced word obtainable from its cyclic structure.*

Proposition 2.7. *Every exact word is minimal.*

Remark 2.8. *The converse of Proposition 2.7 is not necessarily true.*

Question 2.9. *Is the converse of Proposition 2.7 true for any word length?*

We answer as follows:

Lemma 2.10.
- *(U1) Every minimal word of length 2 or 3 is exact.*
- *(U2) There is no full characterization for minimal words of length $l \geq 4$ in $F_{n\geq 2}$.*

Corollary 2.11. *There is only one equivalence class of minimal words of lengths 2 and 3 in $F_{n\geq 2}$. Furthermore, the number of elements in such equivalence class is $2n$.*

Definition 2.12. *Let $L_n = \{f_1, f_1^{-1}, f_2, f_2^{-1}, \cdots, f_n, f_n^{-1}\}$. Given a well-ordering $\leq$ on L_n, we define a* ***well-ordering on*** *L_n^* as follows: if $a = a_1a_2\cdots a_l$ and $b = b_1b_2\cdots b_m$, then $a < b$ if and only if either $l < m$ or $l = m$ and $a_j = b_j$ for $j \leq i < l$ (with $a_{i+1} < b_{i+1}$).*

Take $L_2 = \{f_1, f_1^{-1}, f_2, f_2^{-1}\}$ with the ordering $f_1^{-1} < f_2^{-1} < f_1 < f_2$. We view the irreducible (minimal) words of length 2 in F_2 as $f_1^{-2} < f_2^{-2} < f_1^2 < f_2^2$. In the sequel, we take the representative of a class of equivalent minimal words to be the least element in that class with respect to the ordering defined in Definition 2.12.

Notation 2.13. *Let n denote the rank of a free group, l word length, M the number of minimal words of length l in F_n, N is the number of (distinct) equivalence classes of minimal words of length l in F_n, and Card the respective cardinality of each equivalence class.*

We give a summary of our results as follows:

n	l	M	N	Class representatives	Card
2	2	4	1	f_1^{-2}	4
2	3	4	1	f_1^{-3}	4
2	4	44	3	$f_1^{-4}, f_1^{-2}f_2^{-2}, f_1^{-1}f_2^{-1}f_1f_2$	$4, 32, 8$

2	5	164	4	$f_1^{-5}, f_1^{-3}f_2^{-2}, f_1^{-2}f_2^{-1}f_1^{-1}f_2, f_1^{-2}f_2^{-1}f_1f_2$	$4, 80, 40, 40$
2	6	436	10	$f_1^{-6}, f_1^{-4}f_2^{-2}, f_1^{-3}f_2^{-1}f_1^{-1}f_2, f_1^{-3}f_2^{-1}f_1f_2,$ $f_1^{-3}f_2^{-3}, f_1^{-2}f_2^{-1}f_1^{-2}f_2, f_1^{-2}f_2^{-1}f_1^{-1}f_2^2,$ $f_1^{-2}f_2^{-1}f_1^2f_2, f_1^{-2}f_2^{-2}f_1^{-1}f_2, f_1^{-2}f_2^{-2}f_1f_2$	$4, 120, 48, 48, 24, 24, 48,$ $48, 48, 24$
3	2	6	1	f_1^{-2}	6
3	3	6	1	f_1^{-3}	6
3	4	126	3	$f_1^{-4}, f_1^{-2}f_2^{-2}, f_1^{-1}f_2^{-1}f_1f_2$	$6, 96, 24$
3	5	486	4	$f_1^{-5}, f_1^{-3}f_2^{-2}, f_1^{-2}f_2^{-1}f_1^{-1}f_2, f_1^{-2}f_2^{-1}f_1f_2$	$6, 240, 120, 120$
3	6	3270	11	$f_1^{-6}, f_1^{-4}f_2^{-2}, f_1^{-3}f_2^{-1}f_1^{-1}f_2, f_1^{-3}f_2^{-1}f_1f_2,$ $f_1^{-3}f_2^{-3}, f_1^{-2}f_2^{-1}f_1^{-2}f_2, f_1^{-2}f_2^{-1}f_1^{-1}f_2^2,$ $f_1^{-2}f_2^{-2}f_1^{-1}f_2, f_1^{-2}f_2^{-2}f_1f_2, f_1^{-2}f_2^{-2}f_3^{-2},$ $f_1^{-2}f_2^{-1}f_1^2f_2$	$6, 360, 144, 144, 72, 72,$ $144, 144, 144, 1968, 72$
4	2	8	1	f_1^{-2}	8
4	3	8	1	f_1^{-3}	8
4	4	248	3	$f_1^{-4}, f_1^{-2}f_2^{-2}, f_1^{-1}f_2^{-1}f_1f_2$	$8, 192, 48$
4	5	968	4	$f_1^{-5}, f_1^{-3}f_2^{-2}, f_1^{-2}f_2^{-1}f_1^{-1}f_2, f_1^{-2}f_2^{-1}f_1f_2$	$8, 480, 240, 240$
5	2	10	1	f_1^{-2}	10
5	3	10	1	f_1^{-3}	10
5	4	410	3	$f_1^{-4}, f_1^{-2}f_2^{-2}, f_1^{-1}f_2^{-1}f_1f_2$	$10, 320, 80$
5	5	1610	4	$f_1^{-5}, f_1^{-3}f_2^{-2}, f_1^{-2}f_2^{-1}f_1^{-1}f_2, f_1^{-2}f_2^{-1}f_1f_2$	$10, 800, 400, 400$
6	2	12	1	f_1^{-2}	12
6	3	12	1	f_1^{-3}	12
6	4	612	3	$f_1^{-4}, f_1^{-2}f_2^{-2}, f_1^{-1}f_2^{-1}f_1f_2$	$12, 480, 120$
6	5	2412	4	$f_1^{-5}, f_1^{-3}f_2^{-2}, f_1^{-2}f_2^{-1}f_1^{-1}f_2, f_1^{-2}f_2^{-1}f_1f_2$	$12, 1200, 600, 600$

Let M_{ln} denote the set of all minimal words of length l in $F_{n\geq 2}$. It is evident that $|M_{2n}| = 2n = |M_{3n}|$. Moreover, there is only one equivalence class of trivial minimal words. The representatives can be seen as f_1^{-2} and f_1^{-3} for $l = 2$ and $l = 3$ respectively.

Question 2.14.
- *(V1) Is it possible to characterize all representatives of equivalence classes of minimal words of lengths $l \geq 4$ in $F_{n\geq 2}$?*
- *(V2) What can be said about $|M_{ln}|$ for $l \geq 4$?*

We answer (V1) and (V2) for $l = 4$ and $l = 5$ as follows:

Conjecture 2.15.
- *(W1) There are exactly three equivalence classes of minimal words of length 4 in $F_{n\geq 2}$. Their representatives can be seen as $f_1^{-4}, f_1^{-1}f_2^{-1}f_1f_2$ and $f_1^{-2}f_2^{-2}$ with cardinalities $2n$, $4n(n-1)$ and $16n(n-1)$ respectively.*
- *(W2) $|M_{4n}| = 2n(10n - 9)$.*

Conjecture 2.16. • *(X1) There are exactly four equivalence classes of minimal words of length* 5 *in* $F_{n\geq 2}$. *Their representatives are* f_1^{-5}, $f_1^{-3}f_2^{-2}$, $f_1^{-2}f_2^{-1}f_1^{-1}f_2$ *and* $f_1^{-2}f_2^{-1}f_1f_2$ *with cardinalities* $2n$, $40n(n-1)$, $20n(n-1)$ *and* $20n(n-1)$ *respectively.*

- *(X2)* $|M_{5n}| = 2n(40n-39)$.

2.3 Open Problems

One possible future work is to prove conjectures 2.15 and 2.16 as well as answer (V1) and (V2) for $l \geq 6$. Another important observation is that only f_1 and f_2 are enough to describe all the representatives of equivalent classes of minimal words of lengths 2, 3, 4 and 5 in $F_{n\geq 2}$. On the other hand, the two letters are not sufficient to describe all the representatives of equivalence classes of minimal words of length 6. Hence, it is important to investigate the following:

Question 2.17. *How many letters do we need to describe all the representatives of equivalence classes of minimal words of any given length?*

In conclusion, the role Whitehead automorphisms play in Automorphism of finitely generated free groups is comparable to the role prime numbers play in Number theory, and elements play in Chemistry. The Whitehead algorithm is a powerful tool for classifying all minimal words of any given length in a finitely generated free group.

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