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Properties of a Subclass of Multivalent Harmonic Functions Defined by a Linear Operator

Elif Yaşar[1] and Sibel Yalçın[2]

[1, 2]Department of Mathematics, Faculty of Arts and Sciences
Uludag University, Bursa, Turkey
[1]E-mail: elifyasar@uludag.edu.tr
[2]E-mail: syalcin@uludag.edu.tr

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Abstract

In this paper, we introduce a linear operator for harmonic multivalent functions by using harmonic convolution operator and generalized Saitoh operator. We investigate some properties of a new subclass of harmonic multivalent functions defined by using this operator.

Keywords: *harmonic, multivalent, linear operator, convolution, generalized Saitoh operator.*

1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain D we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. The harmonic function $f = h + \overline{g}$ is sense preserving and locally one to one in D if $|h'(z)| > |g'(z)|$ in D. See Clunie and Sheil-Small [6].

For $p \geq 1$, $n \in \mathbb{N}$, denote by $SH^{n,p}$ the class of functions $f = h + \overline{g}$ that are sense preserving, harmonic multivalent in the unit disk $U = \{z : |z| < 1\}$, where h and g defined by

$$h(z) = z^p + \sum_{k=n+p}^{\infty} A_k z^k, \quad g(z) = \sum_{k=n+p-1}^{\infty} B_k z^k, \quad |B_p| < 1, \tag{1}$$

which are analytic and multivalent functions in U.

Note that $SH^{n,p}$ reduces to $S^{n,p}$, the class of analytic multivalent functions, if the co-analytic part of $f = h + \overline{g}$ is identically zero.

For a_1, a_2, c_1, c_2 are positive real numbers, $\lambda \geq 0$, and $f = h + \overline{g}$ given by (1), we define the operator

$$\begin{aligned} L_p f(z) \quad : \quad &= L_p(a_1, c_1, a_2, c_2, \lambda) f(z) \\ &= D_\lambda f(z) \tilde{*} \left(\phi_1^p(a_1, c_1, z) + \overline{\phi_2^p(a_2, c_2, z)} \right) \\ &= H(z) * \phi_1^p(a_1, c_1, z) + \overline{G(z) * \phi_2^p(a_2, c_2, z)}, \end{aligned}$$

where

$$\begin{aligned} D_\lambda f(z) &= (1-\lambda)\left(h(z) + \overline{g(z)}\right) + \frac{\lambda}{p}\left(zh'(z) - \overline{zg'(z)}\right) \\ &= H(z) + \overline{G(z)}, \end{aligned}$$

$$\phi_1^p(a_1, c_1, z) = z^p + \sum_{k=n+p}^{\infty} \frac{(a_1)_{k-p}}{(c_1)_{k-p}} z^k, \; \phi_2^p(a_2, c_2, z) = \sum_{k=n+p-1}^{\infty} \frac{(a_2)_{k-p}}{(c_2)_{k-p}} z^k \;,$$

and $(x)_k$ denotes the Pochhammer symbol given by

$$(x)_k = \begin{cases} 1 & , \text{if } k = 0 \\ x(x+1)...(x+k-1) & , \text{if } k \in \mathbb{N} = \{1, 2, 3, ...\} . \end{cases}$$

If $f = h + \overline{g} \in SH^{n,p}$, then $L_p f(z) = L_p h(z) + \overline{L_p g(z)}$,
where

$$L_p h(z) = z^p + \sum_{k=n+p}^{\infty} \left[\lambda \left(\frac{k}{p} - 1 \right) + 1 \right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}} A_k z^k,$$

and

$$L_p g(z) = -\sum_{k=n+p-1}^{\infty} \left[\lambda \left(\frac{k}{p} + 1 \right) - 1 \right] \frac{(a_2)_{k-p}}{(c_2)_{k-p}} B_k z^k.$$

Remark 1. (i) For $f(z) \in SH^{1,1}$, $L_1(a_1, c_1, a_2, c_2, 0) f(z) = L(f)$ defined and studied by Ahuja [2]-[3],

(ii) *For* $f(z) \in SH^{1,1}$, $L_1(n+1, 1, n+1, 1, 0) f(z)$ *reduces to the Ruscheweyh derivative operator for harmonic functions [9],*

(iii) *For* $f(z) \in S^{n,p}$, $L_p(a_1, c_1, a_2, c_2, \lambda) f(z) = L_p(a, c, \lambda) f(z)$ *defined by Mahzoon [8],*

(iv) *For* $f(z) \in S^{1,p}$, $L_p(a_1, c_1, a_2, c_2, 0)f(z)$ *reduces to the Saitoh operator [10],*

(v) *For* $f(z) \in S^{1,1}$, $L_p(a_1, c_1, a_2, c_2, 0)f(z)$ *reduces to the Carlson-Shaffer operator [5].*

Let $SH_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$ denote the subclass of $SH^{n,p}$ consisting of functions $f = h + \overline{g} \in SH^{n,p}$ that satisfy the condition

$$Re\left\{\frac{z\left[L_p h(z)\right]' - \overline{z\left[L_p g(z)\right]'}}{L_p h(z) + \overline{L_p g(z)}}\right\} \geq \alpha p. \tag{2}$$

$$(\lambda \geq 0,\ 0 \leq \alpha < 1,\ p \in \mathbb{N},\ n \in \mathbb{N},\ z \in U)$$

Denote by $\overline{SH}^{n,p}$ the subclass of $SH^{n,p}$, consist of harmonic functions $f = h + \overline{g}$ where h and g are of the form

$$h(z) = z^p - \sum_{k=n+p}^{\infty} A_k z^k, \quad g(z) = -\sum_{k=n+p-1}^{\infty} B_k z^k, \ \ A_k,\ B_k \geq 0. \tag{3}$$

Define $\overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda) := SH_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda) \cap \overline{SH}^{n,p}$.
If $f = h + \overline{g} \in \overline{SH}^{n,p}$, then $L_p f(z) = L_p h(z) + \overline{L_p g(z)}$,
where

$$L_p h(z) = z^p - \sum_{k=n+p}^{\infty} \left[\lambda\left(\frac{k}{p} - 1\right) + 1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}} A_k z^k,$$

and

$$L_p g(z) = \sum_{k=n+p-1}^{\infty} \left[\lambda\left(\frac{k}{p} + 1\right) - 1\right] \frac{(a_2)_{k-p}}{(c_2)_{k-p}} B_k z^k.$$

By suitably specializing the parameters, the classes $SH_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH_0^{1,1}(1, 1, 1, 1, 0) = SH^*(0)$, is the class of univalent harmonic starlike functions; [4], [11], [12]

(ii) $SH_{\alpha}^{1,1}(1, 1, 1, 1, 0) = SH^*(\alpha)$, is the class of univalent harmonic starlike functions of order α; [7], [11], [12]

(iii)$SH_0^{1,1}(1, 1, 1, 1, 1) = KH(0)$, is the class of univalent harmonic convex functions; [4], [11], [12]

(iv)$SH_{\alpha}^{1,1}(1, 1, 1, 1, 1) = KH(\alpha)$, is the class of univalent harmonic convex functions of order α; [7], [11], [12]

(v) $SH_{\alpha}^{n,p}(1, 1, 1, 1, 0) = SH(p, \alpha)$, is the class of multivalent harmonic starlike functions; [1]

(vi) $SH_{\alpha}^{1,1}(n+1, 1, n+1, 1, 0) = RH(n, \alpha)$, is the class of univalent Ruscheweyh-type harmonic functions;[9].

2 Main Results

Theorem 1. Let $f(z) \in SH^{n,p}$ be given by (1). If

$$\sum_{k=n+p}^{\infty} \left[\lambda\left(\frac{k}{p}-1\right)+1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)\,|A_k| + \sum_{k=n+p-1}^{\infty} \left|\lambda\left(\frac{k}{p}+1\right)-1\right| \frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)\,|B_k| \le p(1-\alpha), \qquad (4)$$

$$\left(\lambda \ge 0,\ \frac{p-1}{p} \le \alpha < 1,\ p \in \mathbb{N},\ n \in \mathbb{N}\right)$$

then $f \in SH_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$.

Proof. In view of (2), we need to prove that $Re\,\{w\} \ge 0$, where

$$w = \frac{z\,[L_p h(z)]' - \overline{z\,[L_p g(z)]'} - \alpha p\left[L_p h(z) + \overline{L_p g(z)}\right]}{L_p h(z) + \overline{L_p g(z)}} := \frac{A(z)}{B(z)}.$$

Using the fact that $Rew \ge 0 \Leftrightarrow |1+w| \ge |1-w|$, it suffices to show that

$$|A(z)+B(z)| - |A(z)-B(z)| \ge 0.$$

Therefore we obtain

$$\begin{aligned}
& |A(z)+B(z)| - |A(z)-B(z)| \\
\ge\ & [1+p(1-\alpha)]\,|z|^p - \sum_{k=n+p}^{\infty} \left[\lambda\left(\frac{k}{p}-1\right)+1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p+1)\,|A_k|\,|z|^k \\
& - \sum_{k=n+p-1}^{\infty} \left|\lambda\left(\frac{k}{p}+1\right)-1\right| \frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p+1)\,|B_k|\,|z|^k \\
& - |1-p(1-\alpha)|\,|z|^p - \sum_{k=n+p}^{\infty} \left[\lambda\left(\frac{k}{p}-1\right)+1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p-1)\,|A_k|\,|z|^k \\
& - \sum_{k=n+p-1}^{\infty} \left|\lambda\left(\frac{k}{p}+1\right)-1\right| \frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p-1)\,|B_k|\,|z|^k \\
\ge\ & 2p(1-\alpha)\,|z|^p - 2\sum_{k=n+p}^{\infty} \left[\lambda\left(\frac{k}{p}-1\right)+1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)\,|A_k|\,|z|^k \\
& -2\sum_{k=n+p-1}^{\infty} \left|\lambda\left(\frac{k}{p}+1\right)-1\right| \frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)\,|B_k|\,|z|^k
\end{aligned}$$

$$> \ 2p(1-\alpha)\left|z\right|^{p}\times\left\{1-\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}\frac{(k-\alpha p)}{p(1-\alpha)}\left|A_k\right|\right.$$
$$\left.-\sum_{k=n+p-1}^{\infty}\left|\lambda\left(\frac{k}{p}+1\right)-1\right|\frac{(a_2)_{k-p}}{(c_2)_{k-p}}\frac{(k+\alpha p)}{p(1-\alpha)}\left|B_k\right|\right\}.$$

This last expression is non-negative by (4), and so the proof is complete.

Theorem 2. Let $f(z)$ be of the form (3). $f(z)\in\overline{SH}_{\alpha}^{n,p}(a_1,c_1,a_2,c_2,\lambda)$ if and only if

$$\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)A_k$$
$$+\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)B_k\leq p(1-\alpha). \qquad (5)$$
$$(\lambda\geq\frac{1}{2},\frac{p-1}{p}\leq\alpha<1,\ p\in\mathbb{N},\ n\in\mathbb{N})$$

Proof. The "if" part follows from Theorem 1 upon noting that $\overline{SH}_{\alpha}^{n,p}(a_1,c_1,a_2,c_2,\lambda)\subset SH_{\alpha}^{n,p}(a_1,c_1,a_2,c_2,\lambda)$. For the "only if" part, we show that $f\notin\overline{SH}_{\alpha}^{n,p}(a_1,c_1,a_2,c_2,\lambda)$ if the condition (5) does not hold.

Note that a necessary and sufficient condition for $f=h+\overline{g}$ given by (3), to be in $\overline{SH}_{\alpha}^{n,p}(a_1,c_1,a_2,c_2,\lambda)$ is that the condition (2) to be satisfied. This is equivalent to

$$Re\left\{\frac{p(1-\alpha)z^{p}-\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)A_kz^{k}}{z^{p}-\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}A_kz^{k}+\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}B_k\overline{z}^{k}}\right.$$
$$\left.-\frac{\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)B_k\overline{z}^{k}}{z^{p}-\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}A_kz^{k}+\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}B_k\overline{z}^{k}}\right\}\geq 0.$$

The above condition must hold for all values of z, $|z|=r<1$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$ we must have

$$\frac{p(1-\alpha) - \sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)A_k r^{k-p}}{1-\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}A_k r^{k-p}+\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}B_k r^{k-p}}$$

$$-\frac{\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)B_k r^{k-p}}{1-\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}A_k r^{k-p}+\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}B_k r^{k-p}} \ge 0. \tag{6}$$

If the condition (5) does not hold then the numerator of (6) is negative for r sufficiently close to 1. Hence there exists a $z_0 = r_0$ in $(0,1)$ for which the quotient in (6) is negative. This contradicts the required condition for $f \in \overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$ and so the proof is complete.

Next we determine the distortion bounds for the functions in $\overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$.

Theorem 3. Let $f \in \overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$. Also, let $a_1c_2 < a_2c_1$. Then for $|z| = r < 1$ we have

$$\begin{aligned}|f(z)| \;\le\; & (1+B_{n+p-1})\,r^p + \left[\frac{p(1-\alpha)(c_1)_n}{\left[\frac{\lambda n}{p}+1\right][n+p(1-\alpha)]\,(a_1)_n}\right.\\ & \left.-\frac{\left[\lambda\left(\frac{n-1}{p}+2\right)-1\right](a_2)_{n-1}(c_1)_n\,[n-1+p(1+\alpha)]}{\left[\frac{\lambda n}{p}+1\right](c_2)_{n-1}(a_1)_n\,[n+p(1-\alpha)]}B_{n+p-1}\right] r^{n+p},\end{aligned}$$

and

$$\begin{aligned}|f(z)| \;\ge\; & (1-B_{n+p-1})\,r^p - \left[\frac{p(1-\alpha)(c_1)_n}{\left[\frac{\lambda n}{p}+1\right][n+p(1-\alpha)]\,(a_1)_n}\right.\\ & \left.-\frac{\left[\lambda\left(\frac{n-1}{p}+2\right)-1\right](a_2)_{n-1}(c_1)_n\,[n-1+p(1+\alpha)]}{\left[\frac{\lambda n}{p}+1\right](c_2)_{n-1}(a_1)_n\,[n+p(1-\alpha)]}B_{n+p-1}\right] r^{n+p}.\end{aligned}$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in \overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$.

Taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\leq (1+B_{n+p-1})\,r^p + \sum_{k=n+p}^{\infty}(A_k+B_k)\,r^k \\
&\leq (1+B_{n+p-1})\,r^p + \sum_{k=n+p}^{\infty}(A_k+B_k)\,r^{n+p} \\
&= (1+B_{n+p-1})\,r^p + \frac{p(1-\alpha)(c_1)_n}{\left[\frac{\lambda n}{p}+1\right][n+p(1-\alpha)]\,(a_1)_n} \\
&\quad \times \sum_{k=n+p}^{\infty}\left[\frac{\lambda n}{p}+1\right]\frac{(a_1)_n}{(c_1)_n}\frac{n+p(1-\alpha)}{p(1-\alpha)}\,(A_k+B_k)\,r^{n+p}
\end{aligned}$$

$$\begin{aligned}
&\leq (1+B_{n+p-1})\,r^p + \left(\frac{p(1-\alpha)(c_1)_n}{\left[\frac{\lambda n}{p}+1\right][n+p(1-\alpha)]\,(a_1)_n}\right) \\
&\quad \times \left[\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}\frac{(k-\alpha p)}{p(1-\alpha)}A_k r^{n+p}\right. \\
&\quad \left. + \sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}\frac{(k+\alpha p)}{p(1-\alpha)}B_k r^{n+p}\right].
\end{aligned}$$

Using Theorem 2, we obtain

$$\begin{aligned}
|f(z)| &= (1+B_{n+p-1})\,r^p + \left[\frac{p(1-\alpha)(c_1)_n}{\left[\frac{\lambda n}{p}+1\right][n+p(1-\alpha)]\,(a_1)_n}\right. \\
&\quad \left. - \frac{\left[\lambda\left(\frac{n-1}{p}+2\right)-1\right](a_2)_{n-1}(c_1)_n\,[n-1+p(1+\alpha)]}{\left[\frac{\lambda n}{p}+1\right](c_2)_{n-1}(a_1)_n\,[n+p(1-\alpha)]}B_{n+p-1}\right] r^{n+p}.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 3.

Corollary 1. Let f of the form (3) be so that $f \in \overline{SH}_{\alpha}^{n,p}(a_1,c_1,a_2,c_2,\lambda)$. Also, let $a_1c_2 < a_2c_1$. Then

$$\left\{w : |w| < \left[\frac{\left[\frac{\lambda n}{p}+1\right][n+p(1-\alpha)](a_1)_n - p(1-\alpha)(c_1)_n}{\left[\frac{\lambda n}{p}+1\right][n+p(1-\alpha)](a_1)_n}\right.\right.$$
$$\left.\left. - \frac{\left[\frac{\lambda n}{p}+1\right](c_2)_{n-1}(a_1)_n[n+p(1-\alpha)] - \left[\lambda\left(\frac{n-1}{p}+2\right)-1\right](a_2)_{n-1}(c_1)_n[n-1+p(1+\alpha)]}{\left[\frac{\lambda n}{p}+1\right](c_2)_{n-1}(a_1)_n[n+p(1-\alpha)]}B_{n+p-1}\right]\right\} \subset f(U).$$

Theorem 4. Let f be given by (3). Then $f \in \overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$ if and only if

$$f(z) = \sum_{k=n+p-1}^{\infty} \left(x_k h_k(z) + y_k g_k(z) \right),$$

where

$$\begin{aligned}
h_{n+p-1}(z) &= z^p, \\
h_k(z) &= z^p - \frac{p(1-\alpha)}{\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)} z^k, \quad (k = n+p, n+p+1, ...) \\
g_k(z) &= z^p - \frac{p(1-\alpha)}{\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)} \overline{z}^k, \quad (k = n+p-1, n+p, ...) \\
x_{n+p-1} &\equiv x_p = 1 - \left(\sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k \right), \ x_k \geq 0, \ y_k \geq 0.
\end{aligned}$$

In particular, the extreme points of $\overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$ are $\{h_k\}$ and $\{g_k\}$.

Proof.

$$\begin{aligned}
f(z) &= \sum_{k=n+p-1}^{\infty} \left(x_k h_k(z) + y_k g_k(z) \right) \\
&= z^p - \sum_{k=n+p}^{\infty} \frac{p(1-\alpha)}{\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)} x_k z^k \\
&\quad - \sum_{k=n+p-1}^{\infty} \frac{p(1-\alpha)}{\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)} y_k \overline{z}^k.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} \left[\lambda\left(\frac{k}{p}-1\right)+1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}} \frac{(k-\alpha p)}{p(1-\alpha)} \frac{p(1-\alpha)}{\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}(k-\alpha p)} x_k \\
&+ \sum_{k=n+p-1}^{\infty} \left[\lambda\left(\frac{k}{p}+1\right)-1\right] \frac{(a_2)_{k-p}}{(c_2)_{k-p}} \frac{(k+\alpha p)}{p(1-\alpha)} \frac{p(1-\alpha)}{\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}(k+\alpha p)} y_k \\
= &\sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k = 1 - x_p \leq 1,
\end{aligned}$$

and so $f \in \overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$.

Conversely, if $f \in \overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$, then

$$A_k \leq \frac{(c_1)_{k-p}\,[p(1-\alpha)]}{\left[\lambda\left(\frac{k}{p}-1\right)+1\right](a_1)_{k-p}(k-\alpha p)},$$

and

$$B_k \leq \frac{(c_2)_{k-p}\,[p(1-\alpha)]}{\left[\lambda\left(\frac{k}{p}+1\right)-1\right](a_2)_{k-p}(k+\alpha p)}.$$

Set

$$x_k = \left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}\frac{(k-\alpha p)}{p(1-\alpha)}A_k,\ (k = n+p,\ n+p+1,...)$$

$$y_k = \left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}\frac{(k+\alpha p)}{p(1-\alpha)}B_k,\ (k = n+p-1,\ n+p,...)$$

and

$$x_p = 1 - \left(\sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k\right),$$

where $x_p \geq 0$. Then, as required, we obtain

$$f(z) = x_p z^p + \sum_{k=n+p}^{\infty} x_k h_k(z) + \sum_{k=n+p-1}^{\infty} y_k g_k(z).$$

Theorem 5. The class $\overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$ is closed under convex combinations.

Proof. Let $f_i \in \overline{SH}_{\alpha}^{n,p}(a_1, c_1, a_2, c_2, \lambda)$ for $i = 1, 2, ...$, where f_i is given by

$$f_i(z) = z^p - \sum_{k=n+p}^{\infty} A_{k_i} z^k - \sum_{k=n+p-1}^{\infty} B_{k_i}\overline{z}^k.$$

Then by (5),

$$\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}\frac{(k-\alpha p)}{p(1-\alpha)}A_{k_i} + \sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}\frac{(k+\alpha p)}{p(1-\alpha)}B_{k_i} < 1. \tag{7}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z^p - \sum_{k=n+p}^{\infty}\left(\sum_{i=1}^{\infty} t_i A_{k_i}\right)z^k - \sum_{k=n+p-1}^{\infty}\left(\sum_{i=1}^{\infty} t_i B_{k_i}\right)\overline{z}^k.$$

Then by (7),

$$\begin{aligned}
&\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}\frac{(k-\alpha p)}{p(1-\alpha)}\left(\sum_{i=1}^{\infty}t_iA_{k_i}\right)\\
&\quad+\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}\frac{(k+\alpha p)}{p(1-\alpha)}\left(\sum_{i=1}^{\infty}t_iB_{k_i}\right)\\
=&\sum_{i=1}^{\infty}t_i\left\{\sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}\frac{(k-\alpha p)}{p(1-\alpha)}A_{k_i}\right.\\
&\quad\left.+\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}\frac{(k+\alpha p)}{p(1-\alpha)}B_{k_i}\right\}\\
\leq&\sum_{i=1}^{\infty}t_i=1.
\end{aligned}$$

This is the condition required by (5) and so $\sum_{i=1}^{\infty}t_if_i(z)\in\overline{SH}_{\alpha}^{n,p}(a_1,c_1,a_2,c_2,\lambda)$.

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