Gen. Math. Notes, Vol. 17, No. 1, July, 2013, pp.8-15
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

λ-Almost Summable and Statistically (V, λ)-Summable Sequences

Fatih Nuray[1] and Bunyamin Aydin[2]

[1]Afyon Kocatepe University, Afyonkarahisar, Turkey
E-mail: fnuray@aku.edu.tr
[2]Necmettin Erbakan University, Konya, Turkey
E-mail: bunyaminaydin63@hotmail.com

(Received: 1-12-12 / Accepted: 3-5-13)

Abstract

King [3] introduced and examined the concepts of almost A-summable sequence, almost conservative matrix and almost regular matrix. In this paper, we introduce and examine the concepts of λ-almost A-summable sequence, λ-almost conservative matrix and λ-almost regular matrix. Also we introduce statistically (V, λ)-summable sequence.

Keywords: *λ-sequence, λ-almost convergence, λ-almost conservative matrix, λ-almost regular matrix, λ-statistical convergence*

1 Introduction and Background

Let $A = (a_{lk})$ be an infinite matrix of complex numbers and $x = (x_k)$ be a sequence of complex numbers. The sequence $\{A_l(x)\}$ defined by

$$A_l(x) = \sum_{k=1}^{\infty} a_{lk} x_k$$

is called A-transform of x whenever the series converges for $l = 1, 2, 3, \dots$. The sequence x is said to be A-summable to L if $\{A_l(x)\}$ converges to L.

Let ℓ_∞ denote the linear space of bounded sequences. A sequence $x \in \ell_\infty$

is said to be almost convergent to L if

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}x_{k+i}=L$$

uniformly in i.

The matrix A is said to be conservative if $x\in c$ implies that the A-transform of x is convergent. The matrix A is said to be regular if the A-transform of x is convergent to the limit of x for each $x\in c$, where c is the linear spaces of convergent sequences.

In the theory of summability and its applications one is usually interested in conservative or regular matrices. In [3], King introduced almost conservative and almost regular matrices.

A sequence $x\in\ell_\infty$ is said to be almost A-summable to L if the A-transform of x is almost convergent to L. The matrix A is said to be almost conservative if $x\in c$ implies that the A-transform of x is almost convergent. The matrix A is said to be almost regular if the A-transform of x almost convergent to the limit of x for each $x\in c$.

Let $\lambda=(\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞, and $\lambda_{n+1}-\lambda_n\le 1$, $\lambda_1=1$. The generalized de la Vallée-Poussin mean is defined by

$$v_n=v_n(x)=\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n}x_k:=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k.$$

L. Leindler in [4] defined a sequence $x=(x_k)$ to be (V,λ)-summable to number L if $v_n(x)\to L$ as $n\to\infty$. If $\lambda_n=n$, then (V,λ)-summability is reduced to $(C,1)$-summability. We write

$$[V,\lambda]=\{x=(x_k):\qquad \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k-L|=0,\quad for\quad some\quad L\}$$

for set of sequences $x=(x_k)$ which are strongly (V,λ)-summable to L, that is, $x_k\to L[V,\lambda]$.

2 λ-Almost Conservative and λ-Almost Regular Matrices

In this section we introduce λ-almost conservative and λ-almost regular matrices.

Definition 2.1 *A sequence $x=(x_k)$ is said to be λ-almost convergent to number L if*

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}x_{k+i}=L$$

uniformly in i.

Definition 2.2 *The matrix A is said to be λ-almost conservative if $x \in c$ implies that A-transform of x is λ-almost convergent.*

Definition 2.3 *The matrix A is said to be λ-almost regular if $x \in c$ implies that A-transform of x is λ-almost convergent to the limit of x for each $x \in c$.*

Theorem 2.4 *Let $A = (a_{lk})$ be an infinite matrix. Then the matrix A is λ-almost conservative if and only if*
(i)

$$\sup_n \sum_{k=0}^{\infty} \frac{1}{\lambda_n} | \sum_{j \in I_n} a_{l+j,k}| < \infty, \quad l = 0, 1, 2, ...,$$

(ii) there exists an $\xi \in \mathcal{C}$, the set of complex numbers, such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{j \in I_n} \sum_{k=0}^{\infty} a_{l+j,k} = \xi$$

uniformly in l, and
(iii) there exists an $\xi_k \in \mathcal{C}$, k=0,1,2,... such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{j \in I_n} a_{l+j,k} = \xi_k$$

uniformly in l.

Proof. *Suppose that A is λ-almost conservative. Fix $l \in \mathcal{N}$,the set of natural numbers. Let*

$$t_{nl} = \frac{1}{\lambda_n} \sum_{j \in I_n} s_{j+l}(x)$$

where $s_{j+l}(x) = \sum_{k=0}^{\infty} a_{j+l,k} x_k$. It is clear that $s_{j+l} \in c^*, j, n = 1, 2, ...$ Hence $t_{nl} \in c^*$, where c^* is the continuous dual of c. Since A is λ-almost conservative $\lim_{n\to\infty} t_{nl} = t(x)$ uniformly in l. It follows that $\{t_{nl}(x)\}$ is bounded for $x \in c$ and fixed l. Hence $\{\|t_{nl}\|\}$ is bounded by the uniform bounded principle. For each $q \in \mathcal{N}$, define the sequence $y = y(l, n)$ by

$$y_k = \{ \begin{array}{cc} sgn \sum_{j \in I_n} a_{j+l,k}, & 0 \le k \le q \\ 0, & q < k. \end{array}$$

Then $y \in c$, $\|y\| = 1$, and

$$|t_{nl}(y)| = \frac{1}{\lambda_n} \sum_{k=0}^{q} | \sum_{j \in I_n} a_{j+l,k}|.$$

Hence $|t_{nl}(y)| \leq \|t_{nl}\|\|y\| = \|t_{nl}\|$. *Therefore* $\frac{1}{\lambda_n}\sum_{k=0}^{q}|\sum_{j\in I_n} a_{j+l,k}| \leq \|t_{nl}\|$, *so that (i) follows.*

Since $e = (1, 1, ...)$ *and* $e_k = (0, 0, ..., 0, 1, 0, 0, ...)$ *(with 1 in rank k) are convergent sequences,* $\lim_n t_{nl}(e)$ *and* $\lim_n t_{nl}(e_k)$ *must exit uniformly in* l. *Hence (ii) and (iii) hold.*

Now assume that (i), (ii) and (iii) hold. Fix l *and* $x \in c$. *Then*

$$t_{nl}(x) = \frac{1}{\lambda_n}\sum_{j\in I_n}\sum_{k=0}^{\infty} a_{l+j,k}x_k = \frac{1}{\lambda_n}\sum_{k=0}^{\infty}\sum_{j\in I_n} a_{l+j,k}x_k$$

so that

$$t_{nl}(x) \leq \frac{1}{\lambda_n}\sum_{k=0}^{\infty}|\sum_{j\in I_n} a_{l+j,k}x_k|\|x\|, n = 1, 2, ...$$

Therefore $t_{nl}(x) \leq K_l\|x\|$ *by (i), where* K_l *is a constant independent of* n. *Hence* $t_{nl} \in c^*$, *and the sequence* $\{\|t_{nl}\|\}$ *is bounded for each* l. *(ii) and (iii) imply that* $\lim_n t_{nl}(e)$ *and* $\lim_n t_{nl}(e_k)$ *exist for* $l, k = 0, 1, 2, ...$ *Since* $\{e, e_0, e_1, e_2, ...\}$ *is a fundamental set in c it follows from an elementary result of functional analysis that* $\lim_n t_{nl}(x) = t_l(x)$ *exists and* $t_l \in c$. *Therefore*

$$t_l(x) = \lim_k x_k[t_l(e) - \sum_{k=0}^{\infty} t_l(e_k)] + \sum_{k=0}^{\infty} x_k t_l(e_k),$$

But $t_l(e) = \xi$ *and* $t_l(e_k) = \xi_k$, *k=0,1,2,... ,by (ii) and (iii), respectively. Hence* $\lim_n t_{nl}(x) = t_l(x)$ *exists for each* $x \in c$, $l = 0, 1, 2, ...$, *with*

$$t(x) = \lim_k x_k[\xi - \sum_{k=0}^{\infty} \xi_k] + \sum_{k=0}^{\infty} \xi_k x_k. \tag{1}$$

Since $t_{kl} \in c^*$ *for each n and l, it has the form*

$$t_{nl}(x) = \lim_k x_k[t_{nl}(e) - \sum_{k=0}^{\infty} t_{nl}(e_k)] + \sum_{k=0}^{\infty} x_k t_{nl}(e_k), \tag{2}$$

It is easy to see from (1) and (2) that convergence of $\{t_{nl}(x)\}$ *to* $t(x)$ *is uniform in* l, *since* $\lim_n t_{nl}(e) = \xi$ *and* $\lim_n t_{nl}(e_k) = \xi_k$ *uniformly in* l. *Therefore A is λ-almost conservative and the theorem is proved.*

Theorem 2.5 *Let* $A = (a_{lk})$ *be an infinite matrix. Then the matrix A is λ-almost regular if and only if*
(iv)

$$\sup_n \sum_{k=0}^{\infty}\frac{1}{\lambda_n}|\sum_{j\in I_n} a_{l+j,k}| < \infty, \quad l = 0, 1, 2, ...,$$

(v)

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{j\in I_n}\sum_{k=0}^{\infty}a_{l+j,k}=1$$

uniformly in l, and
(vi)

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{j\in I_n}a_{l+j,k}=0$$

uniformly in l, k=0,1,2,... .

Proof. *Suppose that A is λ-almost regular. Then A is λ-almost conservative so that (iv) must hold. (v) and (vi) must hold since A-transform of the sequences e_k and e must be λ-almost convergent to) 0 and 1, respectively.*

Now suppose that (iv), (v) and (vi) hold. Then A is λ-almost conservative. Therefore $\lim_n t_{nl}(x)=t(x)$ uniformly in l for each $x\in c$. The representation (1) gives $t(x)=\lim_k x_k$. Hence A is λ-almost regular. This proves the theorem.

3 Statistically (V,λ)-Summable Sequences

The concept of statistical convergence was introduced by Fast [1]. In [8] Schoenberg gave some basic properties of statistical convergence and also studied the concept as a summability method. A sequence $x=(x_k)$ is said to be statistically convergent to the number L if for every $\epsilon>0$,

$$\lim_{p\to\infty}\frac{1}{p}|\{k\le p:\quad |x_k-L|\ge\epsilon\}|=0,$$

where the vertical bars denote the number of elements in the enclosed set. In this case we write $st-\lim x_k=L$. $\lim x_k=L$ implies $st-\lim x_k=L$, so statistical convergence may be considered as a regular summability method. This was observed by Schoenberg [8] along with the fact that the statistical limit is a linear functional on some sequence space. If x is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that x_k satisfies P for almost all k. In [2], Fridy proved that if x is a statistically convergent sequence then there is a convergent sequence y such that $x_k=y_k$ almost all k.

The concept of statistically summable $(C,1)$ sequence was introduced by Moricz[5]. A sequence $x=(x_k)$ is said to be statistically summable $(C,1)$ to L if $\frac{1}{n}\sum_{k=1}^{n}x_k$ is statistically convergent to L.

In[6], Mursaleen introduced the concept of λ-statistical convergence. A sequence $x=(x_k)$ is said to be λ-statistically convergent to the number L if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k\in I_n:\quad |x_k-L|\ge\epsilon\}|=0.$$

In this case we write $st_\lambda - \lim x_k = L$. In [7], Savas introduced the concept of almost λ-statistical convergence. A sequence $x = (x_k)$ is said to be almost λ-statistically convergent to the number L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \quad |x_{k+i} - L| \geq \epsilon\}| = 0.$$

uniformly in i.

In this section, we introduce the concept of statistically (V, λ)-summable sequence.

Definition 3.1 *A sequence $x = (x_k)$ is said to be statistically (V, λ)-summable to the number L if $v_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k$ is statistically convergent to L, i.e.,*

$$\lim_{p\to\infty} \frac{1}{p} |\{n \leq p : \quad |\frac{1}{\lambda_n} \sum_{k\in I_n} x_k - L| \geq \epsilon\}| = 0.$$

If $\lambda_n = n$, then statistically (V, λ)-summability is reduced to the statistically summablity $(C, 1)$.

Theorem 3.2 *If $x \in \ell_\infty$ and $st_\lambda - \lim x_k = L$ then $x = (x_k)$ is statistically (V, λ)-summable to the number L, i.e., $st - \lim v_n(x) = L$.*

Proof. *Without loss of generality we may assume that $L = 0$. This means that if $\epsilon > 0$ and if we denote by N_{λ_n} the number of $k \in I_n$ for which $|x_k| \geq \epsilon$, then*

$$\lim_{n\to\infty} \frac{N_{\lambda_n}}{\lambda_n} = 0. \tag{3}$$

Since (x_k) is bounded, we say $|x_k| \leq M$ for all k. Now

$$|\frac{1}{\lambda_n} \sum_{k\in I_n} x_k| \leq \frac{N_{\lambda_n} M + (\lambda_n - N_{\lambda_n})\epsilon}{\lambda_n}$$

$$= \frac{(M - \epsilon)N_{\lambda_n} + \lambda_n \epsilon}{\lambda_n} = \epsilon + (M - \epsilon)\frac{N_{\lambda_n}}{\lambda_n}$$

where, by (3), the right side less than 2ϵ provided that n is sufficiently large. Thus $\lim v_n(x) = 0$. Since $\lim x_k = L$ implies $st - \lim x_k = L$, we have $st - \lim v_n(x) = 0$.

Theorem 3.3 *If x statistically (V, λ)-summable to the number L and $\triangle v_n = O(\frac{1}{n})$, then x is (V, λ)-summable to the number L, where $\triangle v_n = v_n - v_{n+1}$.*

Proof. *Assume that x statistically (V, λ)-summable to the number L. Then $st - \lim v_n = L$ and we can choose a sequence w such that $\lim w_n = L$ and $v_n = w_n$ for almost all n. For each n, write $n = m(n) + p(n)$, where $m(n) =$*

$max\{i \leq n : \quad v_i = w_i\}$; if the set $\{i \leq n : \quad v_i = w_i\}$ is empty, take $m(n) = -1$. This can occur for at most a finite number of n. We assert that

$$\lim_n \frac{p(n)}{m(n)} = 0. \tag{4}$$

For, if $\frac{p(n)}{m(n)} > \epsilon > 0$, then

$$\frac{1}{n}|\{i \leq n : \quad v_i \neq w_i\}| \leq \frac{1}{m(n)+p(n)} p(n) \leq \frac{p(n)}{\frac{p(n)}{\epsilon} + p(n)} = \frac{\epsilon}{1+\epsilon}$$

so if $\frac{p(n)}{m(n)} \geq \epsilon$ for infinitely many n, we would contradict $v_n = w_n$ for almost all n. Thus (4) holds. Now consider that difference between $w_{m(n)}$ and v_n. Since $\triangle v_n = O(\frac{1}{n})$ there is a constant K such that $|\triangle v_n| \leq \frac{K}{n}$ for all n. Therefore

$$|w_{m(n)} - v_n| = |v_{m(n)} - v_{m(n)+p(n)}| \leq \sum_{i=m(n)}^{m(n)+p(n)-1} |\triangle v_i| \leq \frac{p(n)K}{m(n)}.$$

By (4), the last expression tends to 0 as $n \to \infty$, and since $\lim_{n\to\infty} w_n = L$, we conclude that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} x_k = L.$$

References

[1] H. Fast, Sur la convergence statistique, *Colloq. Math.*, 2(1952), 241-244.

[2] J.A. Fridy, On statistical convergence, *Analysis*, 5(1985), 301-313.

[3] J.P. King, Almost summable sequences, *Proc. Amer. Math. Soc.*, 17(1966), 1219-1225.

[4] L. Leindler, Über die verallgemeinerte de la Vallée-Poussinsche summierbarkeit allgemenier orthogonalreihen, *Acta Math. Acad. Sci. Hungar.*, 16(1965), 375-387.

[5] F. Moricz, Tauberian conditions, under which statistical convergence follows from statistical summability (C, 1), *J. Math. Anal. Appl.*, 275(2002), 277-287.

[6] Mursaleen, λ-Statistical convergence, *Math. Slovaca*, 50(1) (2000), 111-115.

[7] E. Savas, Strong almost convergence and almost λ-statistical convergence, *Hokkaido Math. J.*, 29(3) (2000), 531-536.

[8] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, 66(1959), 361-375.