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Generalized K-Iterated Function System

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Abstract

In 1969, Kannan [1] proved a new mapping which improved the Banach Contraction theorem. The purpose of this paper is to generalize the Kannan mapping and also to prove it, in space of fractals.

Keywords: *Kannan Mapping, Contraction Mapping, Iterated Function System, K-Iterated Function System.*

1 Introduction

Benoit B. Mandelbrot [3], in his book "The Fractal Geometry of Nature", introduce the fractal geometry which explain many details of nature. Fractal geometry has been found to be a very effective mean for modeling the infinite details found in nature. In 1985, Barnsley and Demko [7] gave the idea of Iterated Function Systems or IFS. Iterated Function Systems provide a convenient framework for the description, classification and communication of fractals. More precisely the most popular "fractal-based" algorithms for both representation and compression of computer images have involved some implementation of the method of Iterated Function Systems (IFS) on complete metric spaces. The basic concept of IFS is usually attributed to Hutchinson [6]. However Vrscay [5] have traced the idea back to Williams [4], who studied fixed point of infinite composition of contractive maps. Fractals are discovered as the fixed points of certain set maps. In 1994, Groller [13] Showed that use of nonlinear function incrases the flexibility when defining an IFS. Study in this field was further carried on by Frame and Angers [11].

In this paper, the results introduced by Kannan in metric space, has been generalized to pair of map and at the same time the metric space has been replaced by space of fractals. We generalize the notion of “Kannan Iterated Function System” (KIFS) [2].

2 Iterated Function System

A mapping $T : X \to X$ on a complete metric space (X, d) is called contractive or a contraction mapping if there is a constant $0 \leq s < 1$ such that

$$d(T(x), T(y)) \leq sd(x.y), \forall x, y \epsilon X \tag{1}$$

Any such number s is called a contractivity factor for T. Polish mathematician S. Banach [14]. Proved a very important result, regarding contraction mapping in 1922, known as Banach Contraction principle.

Theorem 2.1 *Let $T : X \to X$ be a contraction mapping on a complete metric space (X, d) with contractivity factor $'s'$. Then T possesses exactly one fixed point $x^* \epsilon X$ and moreover for any point $x \epsilon X$, the sequence $\{T^{on}(x) : n = 0, 1, 2....\}$converges to x^*. That is, $\lim_{n\to\infty} T^{on}(x) = x^*$, for each $x \epsilon X$.*

IFS generally employ contractive maps over a complete metric space (X, d) where the Banach’s celebrated result mentioned above guarantees the existence and uniqueness of the fixed point knowns as “attractor”. The main property of contraction mapping which is used in IFS is given by the following lemma:

Lemma 2.2 *Let $T : X \to X$ be a contraction mapping on the complete metric space (X, d) with contractivity factor $'s'$. Then T is continuous.*

We now discuss certain definitions required to understand Iterated Function System. Let (X, d) be a complete metric space and let $(H(X), h(d))$ denote the corresponding space of nonempty compact subsets, with Hausdorff metric $h(d)$.

Definition 2.3 *Let (X, d) be a complete metric space, $x \epsilon X$ and $B \epsilon H(X)$. Let $d\ (x, B)$ be the distance from the point x to the set B, where*

$$d(x, B) = min\{d(x, y) : y \epsilon B\}.$$

Definition 2.4 *Let* (X, d) *be a complete metric space, and* $A, B\epsilon H(X)$. *Let d* (A, B) *be the distance between the Set A and the Set B, where*

$$d(A, B) = max\{d(x, B) : x\epsilon A\}.$$

Definition 2.5 *Let* (X, d) *be a complete metric space. Then Hausdorff distance between two points* $A, B\epsilon H(X)$ *is defined as*

$$h(A, B) = d(A, B) \vee d(B, A)$$

Then the function $h(d)$ is the metric defined on the space $H(x)$.

Throughout this paper the notation $u \vee \vartheta$ means the maximum and $u\Lambda\vartheta$ denotes the minimum of the pair of real numbers u and ϑ.

Lemma 2.6 *Let* $T : X \to X$ *be a contraction mapping on the complete metric space* (X, d). *Then* T *maps* $H(X)$ *into itself.*

Lemma 2.7 *Let* $T : X \to X$ *be a contraction mapping on the complete metric space* (X, d) *with contractivity factor 's'. Then* $T : H(X) \to H(X)$ *defined by*

$$T(B) = \{T(x) : x\epsilon B\} \quad \forall B\epsilon H(X)$$

is a contraction mapping on $(H(X), h(d))$ with contractivity factor s.

Lemma 2.8 *Let* (X, d) *be a complete metric space. Let* $\{T_n : n = 1, 2, ...N\}$ *be contraction mappings on* $(H(X), h)$. *Let the contractivity factor for* T_n *be denoted by* $'s_n'$ *for each n. Define* $W : H(X) \to H(X) by$

$$W(B) = T_1(B) \cup T_2(B) \cup - - - \cup T_N(B)$$

$$= \cup_{n=1}^{N} T_n(B) \qquad \forall B\epsilon H(X)$$

Then W is a contraction mapping with contractivity factor $s = max\{s_n : n = 1, 2, - - -, N\}$.

In IFS, the contractivity maps act on the members of Hausdorff space, i.e. the compact subsets of X. Thus, an Iterated Function System is defined as follows:

A (hyperbolic) Iterated Function System consists of a complete metric space (X, d) together with a finite set of continuous contraction mappings $T_n : X \rightarrow X$ with respective contractivity factors $s_n \; for \, n = 1, 2....N$. The notation for the IFS is $\{X, T_n, n = 1, 2....N\}$and its contractivity factor if $s = max\{s_n : n = 1, 2, ...N\}$. Thus, the following theorem was given by Barnsley[12].

Theorem 2.9 *Let* $\{X, T_n, n = 1, 2, ...N\}$*be an IFS with contractivity factor* $'s'$. *Then the transformation* $W : H(X) \rightarrow H(X)$ *defined by*

$W(B) = \cup_{n=1}^{N} T_n(B)$ for all $B \epsilon H(X)$,is a contractivity mapping on the complete metric space $(H(X), h(d))$ with contractivity factor s.

That is

$h(W(B), W(C)) \leq sh(B, C) \forall B, C \epsilon H(X)$

Its unique fixed point, which is also called an attractor, $A \epsilon H(X)$ satisfies the condition

$A = W(A) = \cup_{n=1}^{N} T_n(A),$

and is given by $A = \lim_{n \rightarrow \infty} W^{on}(B)$ for any $B \epsilon H(X). W^{on}$ denotes the n-fold composition of W.

The contraction mappings used in IFS are typically affine maps. The fundamental property of an iterated function system is that it determines a unique attractor, which is usually a fractal. For a simple example, take F to be the middle third Cantor set. Let $S_1, S_2 : R \rightarrow R$ be given by

$$S_1(x) = \frac{1}{3}x \; ; \; S_2(x) = \frac{1}{3}x + \frac{2}{3}$$

Then $S_1(F)$ and $S_2(F)$ are just the left and right 'halves' of F , so that $F = S_1(F) \cup S_2(F)$; thus F is an attractor of the IFS consisting of the contractions $\{S_1, S_2\}$, the two mappings which represent the basic self-similarities of Cantor set.

3 Generalized K-Iterated Function System

In this section, we shall try to explore the possibility of improvement in IFS by replacing contraction condition by a more general condition known as Kannan condition.Kannan [1] introduced a mapping which was an improvement over contraction mapping, known as kannan mapping defined as follows :

A mapping T on a metric space (X, d) is called Kannan, if there exists α, $0 < \alpha < \frac{1}{2}$, such that $d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))]$ (2)

for all $x, y \epsilon X$.Kannan proved that if X is complete then every Kannan mapping has a fixed point, let us name α as K-contractivity factor of Kannan mapping T.

On the basis of definition of (hyperbolic) Iterated Function System given by Barnsley[12], we now introduce Generalized K-Iterated Function System;

A Generalized K- Iterated Function System consists of a complete metric space (X, d) together with a finite pair set of Kannan mappings $P_{2n+1}, Q_{2n+2} : X \to X$ with respective K- contractivity factors $\alpha_{2n+1}, \alpha_{2n+2}$ for each $n = 1, 2, ..., N$, where $\alpha = max\{\alpha_{2n+1}, \alpha_{2n+2} : n = 1, 2, ..., N\}$

First of all we state and prove the two propositions which will establish a relation between pair of Kannan mapping $P^{2m+1}, Q^{2m+2} : m = 1, 2, ..., N$ with respective K-contractivity factors $\alpha_{2m+1}, \alpha_{2m+2}$ for $m = 1, 2, ..., N$ and uniqueness of common fixed point of P, Q if it exists , respectively.

Proposition 3.1 *Let $P, Q : X \to X$ be pair of Kannan mapping with K-contractivity factors $'\alpha_1'$, $'\alpha_2'$ respectively where $'\alpha' = max\{\alpha_1, \alpha_2\}$, on a metric space (X, d) and $x \epsilon X$. Then P and Q satisfy the following conditions:*

$d(P^{2m+1}(x), P^{2m+2}(x)) \leq (\frac{\alpha}{1-\alpha})^{2m+1} d(x, P(x))$ and

$d(Q^{2m+2}(x), Q^{2m+3}(x)) \leq (\frac{\alpha}{1-\alpha})^{2m+2} d(x, Q(x))$.

Moreover, $lim_{m \to \infty} d(P^{2m+1}(x), P^{2m+2}(x)) = 0$

and $\quad lim_{m\to\infty} d(Q^{2m+2}(x), Q^{2m+3}(x)) = 0.$

Proof.

Since P is a Kannan contraction mapping , we have

$d(P^{2m+1}(x), P^{2m+2}(x)) \leq \alpha(d(P^{2m}(x), P^{2m+1}(x)) + d(P^{2m+1}(x), P^{2m+2}(x))).$

It follows that $d(P^{2m+1}(x), P^{2m+2}(x)) \leq \frac{\alpha}{1-\alpha} d(P^{2m}(x), P^{2m+1}(x))$

$$\leq \frac{\alpha}{1-\alpha}[\frac{\alpha}{1-\alpha} d(P^{2m-1}(x), P^{2m}(x))]$$

$$\leq (\frac{\alpha}{1-\alpha})^{2m+1} d(x, P(x)).$$

Taking limit as $m \to \infty$, we have

$lim_{m\to\infty} d(P^{2m+1}(x), P^{2m+2}(x)) \leq lim_{m\to\infty}(\frac{\alpha}{1-\alpha})^{2m+1} d(x, P(x)).$

Therefore, $lim_{m\to\infty} d(P^{2m+1}(x), P^{2m+2}(x)) = 0$, since $\frac{\alpha}{1-\alpha} < 1$.

Similarly, it can be established that for Kannan mapping Q;

$lim_{m\to\infty} d(Q^{2m+2}(x), Q^{2m+3}(x)) = 0.$

Proposition 3.2 *Let $P, Q : X \to X$ be a pair of Kannan mapping ,with K-contractivity $'\alpha_1', '\alpha_2'$ respectively where $'\alpha' = max\{\alpha_1, \alpha_2\}$, on a metric space (X, d).If P and Q possesses a common fixed point, then it is unique.*

Proof.

On the contrary, let x^* and y^*be two common fixed points of P and Q.Then $x^* = P(x^*); x^* = Q(x^*)$ for every $x^* \epsilon X$ and $y^* = P(y^*); y^* = Q(y^*)$ for every $y^* \epsilon X$, then

$$d(x^*, y^*) = d(Q(x^*), Q(y^*))$$

$$= d(P(x^*), P(y^*))$$

$$\leq \alpha[d(x^*, P(x^*)) + d(y^*, P(y^*))]$$

$$= \alpha[d(x^*, x^*) + d(y^*, y^*)]$$

$$= 0$$

Therefore $x^* = y^*$.

Using Propositions 3.1 and 3.2 we now prove the following theorem which is an extension of contraction mapping theorem for pair of Kannan mapping .

Theorem 3.3 *Let $P, Q : X \to X$ be a pair of Kannan mapping , with K-contractivity factors $'\alpha_1','\alpha_2'$ respectively and $'\alpha' = max\{\alpha_1, \alpha_2\}$, on a complete metric space (X, d). Then P and Q possesses common fixed point $x^*\epsilon X$ and moreover for any point $x\epsilon X$, the sequences $\{X, P^{2n+1}(x), n = 0, 1, 2........\}$ and $\{X, Q^{2n+2}(x), n = 0, 1, 2, ...\}$ converges to x^*.*

That is $lim_{n\to\infty} P^{2n+1}(x) = x^* = lim_{n\to\infty} Q^{2n+2}(x)$ for each $x\epsilon X$.

Proof.

Let $x\epsilon X$. Since P, Q be pair of Kannan mapping with K-contractivity factor $'\alpha_1','\alpha_2'$ respectively and $\alpha = max\{\alpha_1, \alpha_2\}$, we have $d(P^{2m+1}(x), P^{2m+2}(x)) \leq (\frac{\alpha}{1-\alpha})^{2m+1} d(x, P(x)) \forall m = 0, 1, 2,$ .... and
$d(Q^{2m+2}(x), Q^{2m+3}(x)) \leq (\frac{\alpha}{1-\alpha})^{2m+2} d(x, Q(x)) \forall m = 0, 1, 2........$
Then , for any fixed $x\epsilon X$, we get

$$d(P^{2n+1}(x), P^{2m+1}(x)) \leq s^{(2m+1)\Lambda(2n+1)} d(x, P^{|(2n+1)-(2m+1)|}(x)) \quad (3)$$

and $d(Q^{2n+2}(x), Q^{2m+2}(x)) \leq s^{(2m+2)\Lambda(2n+2)} d(x, Q^{|(2n+2)-(2m+2)|}(x))$ (4)

where m,n=0,1,2,.......... and $s := \frac{\alpha}{1-\alpha}$.In particular , let us take k = |(2n+1)-(2m+1)|, for k =0,1,2,........
we have $d(x, P^k(x)) \leq d(x, P(x)) + d(P(x), P^2(x)) + ... + d(P^{k-1}(x), P^k(x))$

$\leq (1 + s + s^2 + ... + s^{k-1})d(x, P(x))$

$\leq (\frac{1-s^k}{1-s})d(x, P(x)).$

On substituting in eqution (3), we obtain $d(P^{2n+1}(x), P^{2m+1}(x)) \leq \frac{s^{(2m+1)\Lambda(2n+1)}(1-s^k)}{1-s} d(x, P(x))$,it immediately follows that $\{P^{2n+1}(x)\}_{n=0}^{\infty}$ is a Cauchy sequence .Since X is a complete metric space , this Cauchy sequence has a limit $x^* \epsilon X$ and we have $lim_{n\to\infty} P^{2n+1}(x) = x^*$. (5)

Now to prove that x^* is a fixed point of P, we see that

$d(x^*, P(x^*)) \leq d(x^*, P^{2n+1}(x)) + d(P^{2n+1}(x), P(x^*))P^{2n+1}(x)).$

$\leq d(x^*, P^{2n+1}(x)) + \alpha[d(P^{2n}(x), P^{2n+1}(x)) + d(x^*, P(x^*))].$

$+d(x^*, P(x^*))].$

Taking limit as $n \to \infty$, on considering equation (5) and proposition(3.1), we get

$d(x^*, P(x^*)) \leq (1 + \alpha)d(x^*, P(x^*)).$

Hence $x^* = P(x^*)$.By proposition (3.2),$x^*$is a unique.

Similarly, we can prove that $lim_{n\to\infty} Q^{2n+2}(x) = x^*$ and $x^* = Q(x^*)$.

That is $lim_{n\to\infty} P^{2n+1}(x) = x^* = lim_{n\to\infty} Q^{2n+2}(x)$for each $x \epsilon X$.

This completes the proof .

Lemma 3.4 *Let $P, Q : X \to X$ be a pair of continuous Kannan mapping on the metric space (X, d) with K-contractivity factor $'\alpha_1','\alpha_2'$ respectively ,and $'\alpha' = max\{\alpha_1, \alpha_2\}$.*

Then $P, Q : H(X) \to H(X)$ defined by, $P(B) = \{P(x) : x \epsilon B\}$for every $B \epsilon H(x)$and $Q(B) = \{Q(y) : y \epsilon B\}$ for every $y \epsilon H(x)$ are a Kannan mapping on $(H(X), h(d))$ with contractivity factor $'\alpha'$.

Proof.

Since$P$ is a continuous mapping , therefore by Lemma 2.6 and [12], P maps $H(X)$ into itself.

Now consider $B, C\epsilon H(X)$.Then
$h(P(B), P(C)) = d(P(B), P(C)) \vee d(P(C), P(B)) \leq \alpha\{[d(B, P(B)) + d(C, P(C))] \vee [d(C, P(C)) + d(B, P(B))]\}$

$$= \alpha[d(B, P(B)) + d(C, P(C))]$$

$$\leq \alpha[h(B, P(B)) + h(C, P(C))]$$

Therefore,

$h(P(B), P(C)) \leq \alpha[h(B, P(B)) + h(C, P(C))]$.

Similarly it can be be established that for Kannan mapping Q; that is

$h(Q(B), Q(C)) \leq \alpha[h(B, Q(B)) + h(C, Q(C))]$.

This completes the proof.

Lemma 3.5 *Let (X, d) be a metric space. Let $p_n, q_n : n = 1, 2, ..., N$ be pair of continuous Kannan mapping on $(H(X), h(d))$. Let the K-contractivity factor for p_n, q_n be denoted by 'α_{2n+1}','α_{2n+2}' for each n, respectivly and $\alpha = max\{\alpha_{2n+1}, \alpha_{2n+2} : n = 1, 2, ..., N\}$. Define $P : H(X) \to H(X)$ by $P(B) = p_1(B) \cup p_3(B) \cup .... \cup p_{2N+1}(B)$*

$$P(B) = U_{n=0}^{2N+1} p_n(B) \text{ for each } B\epsilon H(X),$$

and $Q : H(X) \to H(X)$ by $Q(B) = q_2(B) \cup q_4(B)...... \cup q_{2N+2}(B)$

$$Q(B) = U_{n=0}^{2N+2} q_n(B) \text{ for each } B\epsilon H(X).$$

Then P and Q are pair of Kannan mapping with K-contractivity factor $'\alpha'$.

Proof.

We shall prove the theorem , using mathematical induction method and the properties of metric h.On the basis of induction it is true for N=1.Now, we demonstrate the fact for N=2.Let $B, C\epsilon H(X)$, we have

$$h(P(B),P(C)) = h(p_1(B)\cup p_3(B), p_1(C)\cup p_3(C))$$

$$\leq h(p_1(B),p_1(C)) \vee h(p_3(B),p_3(C))$$

$$\leq \alpha_1[h(B,p_1(B)) + h(C,p_1(C))] \vee \alpha_3[h(B,p_3(B)) + h(C,p_3(C))]$$

$$\leq (\alpha_1 \vee \alpha_3)[\{h(B,p_1(B)) \vee h(B,p_3(B))\} + \{h(C,p_1(C)) \vee h(C,p_3(C))\}]$$

$$= \alpha[h(B,p_1(B)\cup p_3(B)) + h(C,p_1(C)\cup p_3(C))].$$

Therefore ,

$h(P(B),P(C)) \leq \alpha[h(B,P(B) + h(C,P(C))]$.

By the condition of mathematical induction Lemma 3.5 is proved.

Similarly ;

$h(Q(B),Q(C)) \leq \alpha[h(B,Q(B)) + h(C,Q(C))]$ for each $B, C\epsilon H(X)$.

Thus, from all the above results and the definition of Gneralized K-Iterated Function System(GKIFS).We are in the position to present the following theorem for GKIFS.

Theorem 3.6 *Let* $\{X,(p_o),p_1,p_3,...........p_{2N+1}\}$, *and* $\{X,(q_o),q_2,q_4,...........q_{2N+2}\}$, *where* p_o, q_o *are the condensation mappings be a Generalized K-Iterated Function System with K-contractivity factor* $'\alpha'_{2n+1},'\alpha'_{2n+2}$ *respectively for each* n *and* $\alpha = max\{\alpha_{2n+1},\alpha_{2n+2}\}$.*Then the transformation* $P : H(X) \to H(X)$ *defined by* $P(B) = U_{n=0}^{2N+1} p_n(B)$ *for all* $B\epsilon H(X)$ *and the transformation* $Q : H(X) \to H(X)$ *defined by* $Q(B) = U_{n=0}^{2N+2} q_n(B)$ *for all* $B\epsilon H(X)$.

Then P and Q are pair of continuous Kannan Mapping on the complete metric space $(H(X), h(d))$ with contractivity factor α.

Its common fixed point, $A\epsilon H(X)$ satisfies the condition;

$$A = P(A) = U_{n=0}^{2N+1} p_n(A)$$

$$\text{and } A = Q(A) = U_{n=0}^{2N+2} q_n(A)$$

and is given by, $A = lim_{n\to\infty} P^{o(2n+1)}(B) = lim_{n\to\infty} Q^{o(2n+2)}(B)$ for any $B \epsilon H(X)$.

The common fixed point A,described above is called an attractor of the GKIFS.

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