Gen. Math. Notes, Vol. 14, No. 2, February 2013, pp.10-22
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

On S_{γ_1}-Open Sets and S_{γ_1}-Continuous in Bitopological Spaces

Hariwan Z. Ibrahim[1] **and Alan M. Omar**[2]

[1]Department of Mathematics, Faculty of Science,
University of Zakho, Kurdistan-Region, Iraq
E-mail: hariwan_math@yahoo.com
[2]Department of Mathematics, Faculty of Science,
University of Duhok, Kurdistan-Region, Iraq
E-mail: alan.m.omer@gmail.com

(Received: 16-11-12 / Accepted: 14-1-13)

Abstract

In this paper, we introduce and study the notions of S_{γ_1}-open sets, S_{γ_1}-continuous and 12-almost S_{γ_1}-continuous functions in bitopological space. We also investigated the fundamental properties of such functions.

Keywords: *γ-open, S_{γ_1}-open, S_{γ_1}-continuous, 12-almost S_{γ_1}-continuous.*

1 Introduction

Throughout this paper, (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. Let (X, τ) be a space and A a subset of X. An operation γ [10] on a topology τ is a mapping from τ in to power set $P(X)$ of X such that $V \subset \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [5] if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subset A$. Then, τ_γ denotes the set of all γ-open set in X. Clearly $\tau_\gamma \subset \tau$. Complements of γ-open sets are called γ-closed. The τ_γ-interior [9] of A is denoted by τ_γ-$Int(A)$ and defined to be the union of all γ-open sets of X contained in A. A topological X with an operation γ on τ is said to be γ-regular [5] if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U)$ contained in V. It is also to be noted that $\tau = \tau_\gamma$ if and only if X is a γ-regular space [5].

A subset A of X is said to be $\imath\jmath$-semi open [8] (resp., $\imath\jmath$-pre open [6], $\imath\jmath$-α-open [7], $\imath\jmath$-semi-preopen [11], $\imath\jmath$-regular open [12]) if $A \subseteq \jmath Cl(\imath Int(A))$ (resp., $A \subseteq \imath Int(\jmath Cl(A))$, $A \subseteq \imath Int(\jmath Cl(\imath Int(A)))$, $A \subseteq \jmath Cl(\imath Int(\jmath Cl(\ A)))$, $A = \imath Int(\jmath Cl(A)))$.

A point x of X is said to be $\imath\jmath$-δ-cluster point [4] of A if $A \cap U \neq \varphi$ for every $\imath\jmath$-reguler open set U containing x, the set of all $\imath\jmath$-δ-cluster points of A is called $\imath\jmath$-δ-closure of A, a subset A of X is said to be $\imath\jmath$-δ-closed if $\imath\jmath$-δ-cluster points of $A \subseteq A$, the complement of $\imath\jmath$-δ-closed set is $\imath\jmath$-δ-open. A point $x \in X$ is in the $\imath\jmath$-θ-closure [3] of A, denoted by $\imath\jmath$-$Cl_\theta(A)$, if $A \cap \jmath Cl(U) \neq \varphi$ for every $\imath$-open set U containing x. A subset A of X is said to be $\imath\jmath$-θ-closed if $A = \imath\jmath$-$Cl_\theta(A)$. A subset A of X is said to be $\imath\jmath$-θ-open if $X \setminus A$ is $\imath\jmath$-θ-closed.

The complement of an $\imath\jmath$-semi open (resp., $\imath\jmath$-pre open, $\imath\jmath$-α-open, $\imath\jmath$-semi-preopen, $\imath\jmath$-regular open) set is said to be $\imath\jmath$-semi closed (resp., $\imath\jmath$-pre closed, $\imath\jmath$-α-closed, $\imath\jmath$-semi-preclosed, $\imath\jmath$-regular closed).

Proposition 1.1 *Let Y be a subspace of a space (X, τ_1, τ_2). If A is a 21-semi closed subset in Y and Y is 21-semi closed in X, then A is a 21-semi closed in X.*

Remark 1.2 *[8] It is clear that the intersection of two $\jmath\imath$-semi closed sets is $\jmath\imath$-semi closed, and also every $\imath$-closed set is $\imath\jmath$-semi closed.*

Remark 1.3 *[5] If (X, τ_1) is a γ_1-T_1 space, then every singlton is γ_1-closed*

Proposition 1.4 *[5] Lel $\gamma : \tau \to p(X)$ be a regular operation on τ. If A and B are γ-open, then $A \cap B$ is γ-open.*

2 S_{γ_1}-Open Sets

Definition 2.1 *An γ_1-open subset A of a space X is called S_{γ_1}-open if for each $x \in A$, there exists a 21-semi closed set F such that $x \in F \subseteq A$.*

The family of all S_{γ_1}-open subsets of a bitopological space (X, τ_1, τ_2) is denoted by $S_{\gamma_1}O(X, \tau_1, \tau_2)$ or $S_{\gamma_1}O(X)$.

A subset B of a space X is called S_{γ_1}-closed if $X \setminus B$ is S_{γ_1}-open. The family of all S_{γ_1}-closed subsets of a bitopological space (X, τ_1, τ_2) is denoted by $S_{\gamma_1}C(X, \tau_1, \tau_2)$ or $S_{\gamma_1}C(X)$.

Proposition 2.2 *A subset A of a space X is S_{γ_1}-open if and only if A is γ_1-open and it is a union of 21-semi closed sets. That is, $A = \bigcup F_\alpha$ where A is γ_1-open and F_α is a 21-semi closed set for each α.*

Proof. Obvious.

It is clear from the definition that every S_{γ_1}-open subset of a space X is γ_1-open, but the converse is not true in general as shown by the following example.

Example 2.3 *Let $X = \{x, y, z\}$ with $\tau_1 = \{X, \varphi, \{x\}, \{x, y\}, \{x, z\}\}$ and $\tau_2 = \{X, \varphi, \{y\}, \{y, z\}\}$, define γ_1 on τ_1 by $\gamma_1(A) = A$ for all $A \in \tau_1$, the S_{γ_1}-open sets are $\{X, \varphi, \{x\}, \{x, z\}\}$ then $\{x, y\}$ is γ_1-open but not S_{γ_1}-open.*

Proposition 2.4 *Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of S_{γ_1}-open sets in a bitopological space X. Then $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is also S_{γ_1}-open.*

Proof. Since A_α is a S_{γ_1}-open set for each α, then A_α is γ_1-open and $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is γ_1-open [5], so for all $x \in A_\alpha$, there exists a 21-semi closed set F such that $x \in F \subseteq A_\alpha$ this implies that $x \in F \subseteq A_\alpha \subseteq \bigcup\{A_\alpha : \alpha \in \Delta\}$, then $x \in F \subseteq \bigcup\{A_\alpha : \alpha \in \Delta\}$, and hence $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is a S_{γ_1}-open set.

Remark 2.5 *The intersection of two S_{γ_1}-open sets need not be S_{γ_1}-open as can be seen from the following example:*

Example 2.6 *Let $X = \{x, y, z\}$ and $\tau_1 = \tau_2 = P(X)$. Define an operation γ_1 on τ_1 by*

$$\gamma_1(A) = \begin{cases} A & \text{if } A = \{x, y\} \text{ or } \{x, z\} \text{ or } \{y, z\} \\ X & \text{otherwise} \end{cases}$$

Clearly, $\tau_{\gamma_1} = \{\phi, \{x, y\}, \{x, z\}, \{y, z\}, X\}$. Let $A = \{x, y\}$ and $B = \{x, z\}$, then A and B are S_{γ_1}-open, but $A \cap B = \{x\}$ which is not S_{γ_1}-open.

Proposition 2.7 *If γ_1 is a regular operation on τ_1, then the intersection of two S_{γ_1}-open sets is S_{γ_1}-open.*

Proof. Let A and B be two S_{γ_1}-open sets, then A and B are γ_1-open sets. Since, γ_1 is regular this implies that $A \cap B$ is also an γ_1-open set, we have to prove that $A \cap B$ is S_{γ_1}-open, let $x \in A \cap B$ then $x \in A$ and $x \in B$, for all $x \in A$ there exists a 21-semi closed set F such that $x \in F \subseteq A$ and for all $x \in B$ there exists a 21-semi closed set E such that $x \in E \subseteq B$, and so that $x \in F \cap E \subseteq A \cap B$. Since the intersection of two 21-semi closed sets is 21-semi closed (by Remark 1.2), this shows that $A \cap B$ is S_{γ_1}-open set.

From propositions 2.4 and 2.7 for γ_1 is a regular operation on τ_1 we conclude that the family of all S_{γ_1}-open subsets of a space X is a topology on X.

Proposition 2.8 *A subset A of a space (X, τ_1, τ_2) is S_{γ_1}-open if and only if for each $x \in A$, there exists a S_{γ_1}-open set B such that $x \in B \subseteq A$.*

Proof. Assume that A is a S_{γ_1}-open set in (X, τ_1, τ_2), let $x \in A$. If we put $B = A$ then B is a S_{γ_1}-open set containing x such that $x \in B \subseteq A$.
Conversely, suppose that for each $x \in A$, there exists a S_{γ_1}-open set B_x such that $x \in B_x \subseteq A$, thus $A = \bigcup B_x$ where $B_x \in S_{\gamma_1}O(X)$ for each x, therefore A is S_{γ_1}-open.

Proposition 2.9 *If (X, τ_1) is a γ_1-T_1 space, then $S_{\gamma_2}O(X) = \tau_{\gamma_2}$, where γ_2 is an operation on τ_2.*

Proof. Let A be any subset of a space X and $A \in \tau_{\gamma_2}$, if $A = \varphi$, then $A \in S_{\gamma_2}O(X)$. If $A \neq \varphi$, let $x \in A$, since (X, τ_1) is a γ_1-T_1 space, then every singlton is γ_1-closed by Remark 1.2, implies that every singlton is 12-semi closed and hence $x \in \{x\} \subseteq A$. Therefore, $A \in S_{\gamma_2}O(X)$. Hence, $\tau_{\gamma_2} \subseteq S_{\gamma_2}O(X)$, but from definition of S_{γ_2}-open sets we have $S_{\gamma_2}O(X) \subseteq \tau_{\gamma_2}$. Thus $S_{\gamma_2}O(X) = \tau_{\gamma_2}$.

Remark 2.10 *Every S_{γ_1}-open set is S_1-open [1].*

The converse of the above Remark is not true in general as shown in the following example.

Example 2.11 *Let $X = \{x, y, z\}$ with $\tau_1 = \{X, \varphi, \{y\}, \{x, y\}, \{y, z\}\}$ and $\tau_2 = \{X, \varphi, \{y\}, \{y, z\}\}$, define γ_1 on τ_1 by $\gamma_1(A) = X$ for all $A \in \tau_1$, then $\{y\}$ is S_1-open set but not S_{γ_1}-open.*

Remark 2.12 *Let (X, τ_1, τ_2) be a space and $x \in X$. If $\{x\}$ is S_{γ_1}-open, then $\{x\}$ is 21-semi closed.*

Proposition 2.13 *Let (Y, σ_1, σ_2) be a subspace of a space (X, τ_1, τ_2). If $A \in S_{\gamma_1}O(Y)$ and $Y \in 21\text{-}SC(X)$, then for each $x \in A$, there exists a 21-semi closed set F in X such that $x \in F \subseteq A$.*

Proof. Let $A \in S_{\gamma_1}O(Y)$, then $A \in \sigma_1$ and for each $x \in A$, there exists a 21-semi closed set F in Y such that $x \in F \subseteq A$. Since $Y \in 21\text{-}SC(X)$, by Proposition 1.1, $F \in 21\text{-}SC(X)$, which completes the proof.

Proposition 2.14 *A subset B of a space X is S_{γ_1}-closed if and only if B is an γ_1-closed set and it is an intersection of 21-semi open sets.*

Proof. Obvious.

Proposition 2.15 *Let $\{B_\alpha : \alpha \in \Delta\}$ be a collection of S_{γ_1}-closed sets in a bitopological space X. Then $\bigcap\{B_\alpha : \alpha \in \Delta\}$ is S_{γ_1}-closed set.*

Proof. Follows from Proposition 2.4.

Definition 2.16 *Let* (X, τ_1, τ_2) *be a bitopological space and* $x \in X$. *A subset* N *of* X *is said to be* S_{γ_1}*-neighborhood of* x *if there exists a* S_{γ_1}*-open set* U *in* X *such that* $x \in U \subseteq N$.

Theorem 2.17 *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is* S_{γ_1}*-open if and only if it is a* S_{γ_1}*-neighborhood of each of its points.*

Proof. Let $A \subseteq X$ be a S_{γ_1}-open set, since for every $x \in A$, $x \in A \subseteq A$ and A is S_{γ_1}-open. This shows that A is S_{γ_1}-neighborhood of each of its points. **Conversely**, suppose that A is a S_{γ_1}-neighborhood of each of its points, then for each $x \in A$, there exists $B_x \in S_{\gamma_1}O(X)$ such that $x \in B_x \subseteq A$. Therefore $A = \bigcup\{B_x : x \in A\}$. Since each B_x is S_{γ_1}-open, it follows that A is a S_{γ_1}-open set.

Definition 2.18 *For any subset* A *in a space* X, *the* S_{γ_1}*-interior of* A, *denoted by* $S_{\gamma_1}Int(A)$, *is defined by the union of all* S_{γ_1}*-open sets which are contained in* A.

Remark 2.19 *Let* A *be any subset of a bitopological space. A point* $x \in A$ *is belongs to* $S_{\gamma_1}Int(A)$ *if and only if there exists an* S_{γ_1}*-open set* G *such that* $x \in G \subset A$.

Proposition 2.20 *Let* A *be any subset of a space* X. *If a point* x *is in the* S_{γ_1}*-interior of* A, *then there exists a* 21*-semi closed set* F *of* X *containing* x *such that* $F \subseteq A$.

Proof. Suppose that $x \in S_{\gamma_1}Int(A)$, then there exists a S_{γ_1}-open set U of X containing x such that $U \subseteq A$. Since U is a S_{γ_1}-open set, so there exists a 21-semi closed set F containing x such that $x \in F \subseteq U \subseteq A$. Hence, $x \in F \subseteq A$.

Definition 2.21 *For any subset* A *in a space* X, *the* S_{γ_1}*-closure of* A, *denoted by* $S_{\gamma_1}Cl(A)$, *is defined by the intersection of all* S_{γ_1}*-closed sets containing* A.

Corollary 2.22 *Let* A *be a set in a space* X. *A point* $x \in X$ *is in the* S_{γ_1}*-closure of* A *if and only if* $A \cap U \neq \varphi$ *for every* S_{γ_1}*-open set* U *containing* x.

Proof. Let $x \notin S_{\gamma_1}Cl(A)$. Then $x \notin \bigcap F$, where F is S_{γ_1}-closed with $A \subseteq F$. So $x \in X \setminus \bigcap F$ and $X \setminus \bigcap F$ is a S_{γ_1}-open set containing x and hence, $(X \setminus \bigcap F) \cap A \subseteq (X \setminus \bigcap F) \cap (\bigcap F) = \varphi$.
Conversely, suppose that there exists a S_{γ_1}-open set containing x with $A \cap U = \varphi$, then $A \subseteq X \setminus U$ and $X \setminus U$ is a S_{γ_1}-closed with $x \notin X \setminus U$. Hence, $x \notin S_{\gamma_1}Cl(A)$.

Proposition 2.23 *Let A be any subset of a space X and x is a point of X. If $A \cap F \neq \varphi$ for every 21-semi closed set F of X containing x, then the point x is in the S_{γ_1}-closure of A.*

Proof. Suppose that U is any S_{γ_1}-open set containing x, then by Definition 2.1 , there exists a 21-semi closed set F such that $x \in F \subseteq U$. So by hypothesis $A \cap F \neq \varphi$ which implies that $A \cap U \neq \varphi$ for every S_{γ_1}-open set U containing x. Therefore, by Corollary 2.22, $x \in S_{\gamma_1}Cl(A)$.

3 S_{γ_1}-Continuous and 12-Almost S_{γ_1}-Continuous

Definition 3.1 *A function $f : X \to Y$ is called S_{γ_1}-continuous at a point $x \in X$ if for each 1-open set V of Y containing $f(x)$, there exists a S_{γ_1}-open set U of X containing x such that $f(U) \subseteq V$. If f is S_{γ_1}-continuous at every point x of X, then it is called S_{γ_1}-continuous.*

Definition 3.2 *A function $f : X \to Y$ is called 12-almost S_{γ_1}-continuous at a point $x \in X$ if for each 1-open set V of Y containing $f(x)$, there exists a S_{γ_1}-open set U of X containing x such that $f(U) \subseteq 1Int(2ClV)$. If f is 12-almost S_{γ_1}-continuous at every point x of X, then it is called 12-almost S_{γ_1}-continuous.*

It is obvious from the definition that S_{γ_1}-continuity implies 12-almost S_{γ_1}-continuity. However, the converse is not true in general as it is shown in the following example.

Example 3.3 *Let $X = \{x, y, z\}$, $\tau_1 = \{X, \varphi, \{x\}, \{x, y\}\}$, $\tau_2 = \{X, \varphi, \{z\}, \{y, z\}\}$, $\sigma_1 = \{X, \varphi, \{x\}, \{z\}, \{x, z\}\}$, $\sigma_2 = \{X, \varphi, \{y, z\}\}$, and γ_1 defined on τ_1 by $\gamma_1(A) = A$ for all $A \in \tau_1$. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ f is 12-almost S_{γ_1}-continuous but not S_{γ_1}-continuous at z, because $\{z\}$ is a 1-open set in (X, σ_1, σ_2) containing $f(z) = z$, there exists no S_{γ_1}-open set U in (X, τ_1, τ_2) containing z such that $x \in f(U) \subseteq \{z\}$.*

Proposition 3.4 *Let X and Y be bitopological spaces. A function $f : X \to Y$ is S_{γ_1}-continuous if and only if the inverse image under f of every 1-open set in Y is a S_{γ_1}-open in X.*

Proof. Assume that f is S_{γ_1}-continuous and let V be any 1-open set in Y. We have to show that $f^{-1}(V)$ is S_{γ_1}-open in X.
If $f^{-1}(V) = \varphi$, there is nothing to prove. So let $f^{-1}(V) \neq \varphi$ and let $x \in f^{-1}(V)$ so that $f(x) \in V$. By S_{γ_1}-continuity of f, there exists an S_{γ_1}-open set U in X containing x such that $f(U) \subseteq V$, that is $x \in U \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is a S_{γ_1}-open set.

Conversely, let $f^{-1}(V)$ be S_{γ_1}-open in X for every 1-open set V in Y. To show that f is S_{γ_1}-continuous at $x \in X$, let V be any 1-open set in Y such that $f(x) \in V$ so that $x \in f^{-1}(V)$. By hypothesis $f^{-1}(V)$ is S_{γ_1}-open in X. If $f^{-1}(V) = U$, then U is a S_{γ_1}-open set in X containing x such that

$$f(U) = f(f^{-1}(V)) \subseteq V$$

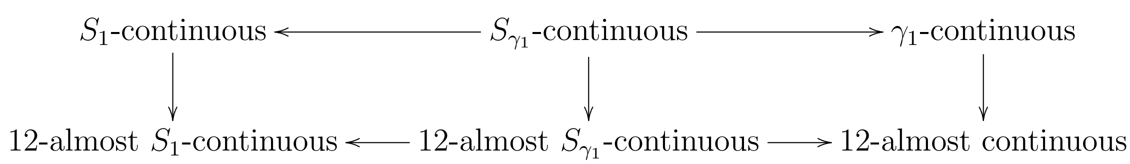
Hence f is a S_{γ_1}-continuous function. This completes the proof.

The proof of the following corollary follows directly from their definitions.

Corollary 3.5

1. *Every S_{γ_1}-continuous function is γ_1-continuous [2].*
2. *Every S_{γ_1}-continuous function is S_1-continuous [1].*
3. *Every* 12*-almost S_{γ_1}-continuous function is* 12*-almost S_1-continuous.*
4. *Every* 12*-almost S_{γ_1}-continuous function is* 12*-almost continuous.*

By Definition 3.1, Definition 3.2 and corollary 3.5, we obtain the following diagram.



In the sequel, it will be shown that none of the implications concerning S_{γ_1}-continuity and 12-almost S_{γ_1}-continuity is reversible.

Example 3.6 *Let $X = \{x, y, z, w\}$ with four topologies $\tau_1 = \{X, \varphi, \{z\}, \{x, w\}, \{x, z, w\}\}$, $\tau_2 = \{X, \varphi, \{y\}, \{x, y, w\}\}$, $\sigma_1 = \{X, \varphi, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $\sigma_2 = \{X, \varphi, \{w\}, \{x, y, z\}\}$, and γ_1 defined on τ_1 by $\gamma_1(A) = A$ for all $A \in \tau_1$. Then the family of S_{γ_1}-open subsets of X with respect to τ_1 and τ_2 is:*
$S_{\gamma_1}O(X) = \{X, \varphi, \{z\}, \{x, z, w\}\}$. We defined the function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ as follows $f(x) = y$, $f(y) = w$, $f(z) = x$, $f(w) = z$. Then f is γ_1-continuous but not S_{γ_1}-continuous, because $\{y, z\}$ is 1-open set in (X, σ_1, σ_2) containing $f(x) = y$, there exists no S_{γ_1}-open set U in (X, τ_1, τ_2) containing x such that $x \in f(U) \subseteq \{y, z\}$.

Example 3.7 *In Example 3.6, if we have $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is a function defined as follows $f(x) = x$, $f(y) = f(z) = w$, $f(w) = y$, then f is* 12*-almost continuous but not* 12*-almost S_{γ_1}-continuous, because $\{x\}$ is a 1-open set in (X, σ_1, σ_2) containing $f(x) = x$, there exists no S_{γ_1}-open set U in (X, τ_1, τ_2) containing x such that $x \in f(U) \subseteq$ 1-Int(2-$Cl\{x\}$) implies that $f(U) \subseteq \{x, y, z\}$.*

Proposition 3.8 *For a function $f : X \to Y$, the following statements are equivalent:*

1. f *is* S_{γ_1}*-continuous.*
2. $f^{-1}(V)$ *is a* S_{γ_1}*-open set in* X*, for each* 1*-open set* V *in* Y*.*
3. $f^{-1}(F)$ *is a* S_{γ_1}*-closed set in* X*, for each* 1*-closed set* F *in* Y*.*
4. $f(S_{\gamma_1}Cl(A)) \subseteq 1Cl(f(A))$*, for each subset* A *of* X*.*
5. $S_{\gamma_1}Cl(f^{-1}(B)) \subseteq f^{-1}(1Cl(B))$*, for each subset* B *of* Y*.*
6. $f^{-1}(1Int(B)) \subseteq S_{\gamma_1}Int(f^{-1}(B))$*, for each subset* B *of* Y*.*
7. $1Int(f(A)) \subseteq f(S_{\gamma_1}Int(A))$*, for each subset* A *of* X*.*

Proof. (1) ⇒ (2). Directly from Proposition 3.4.
(2) ⇒ (3). Let F be any 1-closed set of Y. Then $Y \setminus F$ is an 1-open set of Y. By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is S_{γ_1}-open set in X and hence $f^{-1}(F)$ is S_{γ_1}-closed set in X.
(3) ⇒ (4). Let A be any subset of X. Then $f(A) \subseteq 1Cl(f(A))$ and $1Cl(f(A))$ is 1-closed in Y. Hence $A \subseteq f^{-1}(1Cl(f(A)))$. By (3), we have $f^{-1}(1Cl(f(A)))$ is a S_{γ_1}-closed set in X. Therefore, $S_{\gamma_1}Cl(A) \subseteq f^{-1}(1Cl(f(A)))$. Hence $f(S_{\gamma_1}Cl(A)) \subseteq 1Cl(f(A))$.
(4) ⇒ (5). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (4), we have $f(S_{\gamma_1}Cl(f^{-1}(B))) \subseteq 1Cl(f(f^{-1}(B))) = 1Cl(B)$. Hence $S_{\gamma_1}Cl(f^{-1}(B)) \subseteq f^{-1}(1Cl(B))$.
(5) ⇒ (6). Let B be any subset of Y. Then apply (5) to $Y \setminus B$ is obtained $S_{\gamma_1}Cl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(1Cl(Y \setminus B)) \Leftrightarrow S_{\gamma_1}Cl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus 1Int(B)) \Leftrightarrow X \setminus S_{\gamma_1}Int(f^{-1}(B)) \subseteq X \setminus f^{-1}(1Int(B)) \Leftrightarrow f^{-1}(1Int(B)) \subseteq S_{\gamma_1}Int(f^{-1}(B))$. Therefore, $f^{-1}(1Int(B)) \subseteq S_{\gamma_1}Int(f^{-1}(B))$.
(6) ⇒ (7). Let A be any subset of X. Then $f(A)$ is a subset of Y. By (6), we have $f^{-1}(1Int(f(A))) \subseteq S_{\gamma_1}Int(f^{-1}(f(A))) = S_{\gamma_1}Int(A)$. Therefore, $1Int(f(A)) \subseteq f(S_{\gamma_1}Int(A))$.
(7) ⇒ (1). Let $x \in X$ and let V be any 1-open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (7), we have $1Int(f(f^{-1}(V))) \subseteq f(S_{\gamma_1}Int(f^{-1}(V)))$. Then $1Int(V) \subseteq f(S_{\gamma_1}Int(f^{-1}(V)))$. Since V is an 1-open set. Then $V \subseteq f(S_{\gamma_1}Int(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq S_{\gamma_1}Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is a S_{γ_1}-open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is S_{γ_1}-continuous.

Proposition 3.9 *For a function $f : X \to Y$, the following statements are equivalent:*

1. f *is* 12*-almost* S_{γ_1}*-continuous.*

2. *For each* $x \in X$ *and each* 12*-regular open set* V *of* Y *containing* $f(x)$*, there exists a* S_{γ_1}*-open* U *in* X *containing* x *such that* $f(U) \subseteq V$*.*

3. *For each* $x \in X$ *and each* 12*-*δ*-open set* V *of* Y *containing* $f(x)$*, there exists a* S_{γ_1}*-open* U *in* X *containing* x *such that* $f(U) \subseteq V$*.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and let V be any 12-regular open set of Y containing $f(x)$. By (1), there exists a S_{γ_1}-open set U of X containing x such that $f(U) \subseteq 1Int(2Cl(V))$. since V is 12-regular open, then $1Int(2Cl(V)) = V$. Therefore, $f(U) \subseteq V$.
(2) $\Rightarrow$ (3). Let $x \in X$ and let V be any 12-δ-open set of Y containing $f(x)$. Then for each $f(x) \in V$, there exists an 1-open set G containing $f(x)$ such that $G \subseteq 1Int(2Cl(G)) \subseteq V$. Since $1Int(2Cl(G))$ is 12-regular open set of Y containing $f(x)$. By (2), there exists a S_{γ_1}-open set U in X containing x such that $f(U) \subseteq 1Int(2Cl(G)) \subseteq V$. This completes the proof.
(3) $\Rightarrow$ (1). Let $x \in X$ and let V be any 1-open set of Y containing $f(x)$. Then $1Int(2Cl(V)$ is 12-δ-open set of Y containing $f(x)$. By (3), there exists a S_{γ_1}-open set U in X containing x such that $f(U) \subseteq 1Int(2Cl(V))$. Therefore, f is 12-almost S_{γ_1}-continuous.

Proposition 3.10 *For a function* $f : X \to Y$*, the following statements are equivalent:*

1. f *is* 12*-almost* S_{γ_1}*-continuous.*

2. $f^{-1}(1Int(2Cl(V)))$ *is a* S_{γ_1}*-open set in* X*, for each* 1*-open set* V *in* Y*.*

3. $f^{-1}(1Cl(2Int(F)))$ *is a* S_{γ_1}*-closed set in* X*, for each* 1*-closed set* F *in* Y*.*

4. $f^{-1}(F)$ *is a* S_{γ_1}*-closed set in* X*, for each* 12*-regular closed set* F *of* Y*.*

5. $f^{-1}(V)$ *is a* S_{γ_1}*-open set in* X*, for each* 12*-regular open set* V *of* Y*.*

6. $f^{-1}(G)$ *is a* S_{γ_1}*-open set in* X*, for each* 12*-*δ*-open set* G *of* Y*.*

Proof. (1) $\Rightarrow$ (2). Let V be any 1-open set in Y. We have to show that $f^{-1}(1Int(2Cl(V)))$ is S_{γ_1}-open set in X. Let $x \in f^{-1}(1Int(2Cl(V)))$. Then $f(x) \in 1Int(2Cl(V))$ and $1Int(2Cl(V))$ is an 12-regular open set in Y. Since f is 12-almost S_{γ_1}-continuous, then by Proposition 3.9, there exists a S_{γ_1}-open set U of X containing x such that $f(U) \subseteq 1Int(2Cl(V))$. Which implies that $x \in U \subseteq f^{-1}(1Int(2Cl(V)))$. Therefore, $f^{-1}(1Int\ (2Cl(V)))$ is a S_{γ_1}-open set in X.
(2) $\Rightarrow$ (3). Let F be any 1-closed set of Y. Then $Y \setminus F$ is an 1-open set of Y. By (2), $f^{-1}(1Int(2Cl(Y \setminus F)))$ is a S_{γ_1}-open set in X and $f^{-1}(1Int(2Cl(Y \setminus F))) =$

$f^{-1}(1Int(Y \setminus 2Int(F))) = f^{-1}(Y \setminus 1Cl(2Int(F))) = X \setminus f^{-1}(1Cl(2Int(F)))$ is a S_{γ_1}-open set in X and hence $f^{-1}(1Cl(2Int(F)))$ is S_{γ_1}-closed set in X.
(3) $\Rightarrow$ (4). Let F be any 12-regular closed set of Y. Then F is an 1-closed set of Y. By (3), $f^{-1}(1Cl(2Int(F)))$ is S_{γ_1}-closed set in X. Since F is 12-regular closed set, then $f^{-1}(1Cl(2Int(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a S_{γ_1}-closed set in X.
(4) $\Rightarrow$ (5). Let V be any 12-regular open set of Y. Then $Y \setminus V$ is an 12-regular closed set of Y and by (4), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is a S_{γ_1}-closed set in X and hence $f^{-1}(V)$ is a S_{γ_1}-open set in X.
(5) $\Rightarrow$ (6). Let G be any 12-δ-open set in Y, $G = \bigcup\{V_\alpha : \alpha \in \Delta\}$ where V_α is 12-regular open. Then $f^{-1}(G) = \bigcup\{f^{-1}(V_\alpha)\}$, from (5) we have $f^{-1}(V_\alpha)$ is a S_{γ_1}-open set, then $f^{-1}(G) = \bigcup\{f^{-1}(V_\alpha)\}$ is a S_{γ_1}-open.
(6) $\Rightarrow$ (1). Let $x \in X$ and let V be any 12-δ-open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (6), we have $f^{-1}(V)$ is a S_{γ_1}-open set in X. Therefore, we obtain $f(f^{-1}(V)) \subseteq V$. Hence by Proposition 3.9, f is 12-almost S_{γ_1}-continuous.

Proposition 3.11 *For a function $f : X \to Y$, the following statements are equivalent:*

1. *f is 12-almost S_{γ_1}-continuous.*
2. *$f(S_{\gamma_1}Cl(A)) \subseteq 12Cl_\delta(f(A))$, for each subset A of X.*
3. *$S_{\gamma_1}Cl(f^{-1}(B)) \subseteq f^{-1}(12Cl_\delta(B))$, for each subset B of Y.*
4. *$f^{-1}(F)$ is S_{γ_1}-closed set in X, for each 12-δ-closed set F of Y.*
5. *$f^{-1}(V)$ is S_{γ_1}-open set in X, for each 12-δ-open set V of Y.*
6. *$f^{-1}(12Int_\delta(B)) \subseteq S_{\gamma_1}Int(f^{-1}(B))$, for each subset B of Y.*
7. *$12Int_\delta(f(A)) \subseteq f(S_{\gamma_1}Int(A))$, for each subset A of X.*

Proof. (1) $\Rightarrow$ (2). Let A be a subset of X. Since $12Cl_\delta(f(A))$ is an 12-δ-closed set in Y, then $Y \setminus 12Cl_\delta(f(A))$ is 12-δ-open, from Proposition 3.10, $f^{-1}(Y \setminus 12Cl_\delta(f(A)))$ is S_{γ_1}-open, which implies that $X \setminus f^{-1}(12Cl_\delta(f(A)))$ is also S_{γ_1}-open, so $f^{-1}(12Cl_\delta(f(A)))$ is S_{γ_1}-closed set in X. Since $A \subseteq f^{-1}(12Cl_\delta(f(A)))$, so $S_{\gamma_1}Cl(A) \subseteq f^{-1}(12Cl_\delta(f(A)))$. Therefore, $f(S_{\gamma_1}Cl(A)) \subseteq 12Cl_\delta(f(A))$ is obtained.
(2) $\Rightarrow$ (3). Let B be a subset of Y. We have $f^{-1}(B)$ is a subset of X. By (2), we have $f(S_{\gamma_1}Cl(f^{-1}(B))) \subseteq 12Cl_\delta(f(f^{-1}(B))) = 12Cl_\delta(B)$. Hence $S_{\gamma_1}Cl(f^{-1}(B)) \subseteq f^{-1}(12Cl_\delta(B))$.
(3) $\Rightarrow$ (4). Let F be any 12-δ-closed set of Y. By (3), we have $S_{\gamma_1}Cl(f^{-1}(F)) \subseteq f^{-1}(12Cl_\delta(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is a S_{γ_1}-closed set in X.

(4) $\Rightarrow$ (5). Let V be any 12-δ-open set of Y. Then $Y \setminus V$ is an 12-δ-closed set of Y and by (4), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is a S_{γ_1}-closed set in X. Hence $f^{-1}(V)$ is a S_{γ_1}-open set in X.
(5) $\Rightarrow$ (6). For each subset B of Y. We have $12Int_\delta(B) \subseteq B$. Then $f^{-1}(12Int_\delta(B)) \subseteq f^{-1}(B)$. By (5), $f^{-1}(12Int_\delta(B))$ is a S_{γ_1}-open set in X. Then $f^{-1}(12Int_\delta(B)) \subseteq S_{\gamma_1}Int(f^{-1}(B))$.
(6) $\Rightarrow$ (7). Let A be any subset of X. Then $f(A)$ is a subset of Y. By (6), we obtain that $f^{-1}(12Int_\delta(f(A))) \subseteq S_{\gamma_1}Int(f^{-1}(f(A)))$. Hence $f^{-1}(12Int_\delta(f(A))) \subseteq S_{\gamma_1}Int(A)$, which implies that $12Int_\delta(f(A)) \subseteq f(S_{\gamma_1}\ Int(A))$.
(7) $\Rightarrow$ (1). Let $x \in X$ and V be any 12-reguler open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (7), we get $12Int_\delta(f(f^{-1}(V))) \subseteq f(S_{\gamma_1}Int(f^{-1}(V)))$ implies that $12Int_\delta(V) \subseteq f(S_{\gamma_1}Int(f^{-1}(V)))$. Since V is 12-reguler open set and hence 12-δ-open set, then $V \subseteq f(S_{\gamma_1}Int(f^{-1}(V)))$ this implies that $f^{-1}(V) \subseteq S_{\gamma_1}Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is a S_{γ_1}-open set in X which contains x and clearly $f(\ f^{-1}(V)) \subseteq V$. Hence, by Proposition 3.9, f is 12-almost S_{γ_1}-continuous.

Proposition 3.12 *For a function $f : X \to Y$, the following statements are equivalent:*

1. *f is 12-almost S_{γ_1}-continuous.*
2. *$S_{\gamma_1}Cl(f^{-1}(V)) \subseteq f^{-1}(1Cl(V))$, for each 21-$\beta$-open set V of Y.*
3. *$f^{-1}(1Int(F)) \subseteq S_{\gamma_1}Int(f^{-1}(F))$, for each 21-$\beta$-closed set F of Y.*
4. *$f^{-1}(1Int(F)) \subseteq S_{\gamma_1}Int(f^{-1}(F))$, for each 21-semi closed set F of Y.*
5. *$S_{\gamma_1}Cl(f^{-1}(V)) \subseteq f^{-1}(1Cl(V))$, for each 21-semi open set V of Y.*

Proof. (1) $\Rightarrow$ (2). Let V be any 21-β-open set of Y. It follows that $1Cl(V)$ is an 12-reguler closed set in Y. Since f is 12-almost S_{γ_1}-continuous. Then by Proposition 3.10, $f^{-1}(V)$ is a S_{γ_1}-closed set in X. Therefore, we obtain $S_{\gamma_1}Cl(f^{-1}(V)) \subseteq f^{-1}(1Cl(V))$.
(2) $\Rightarrow$ (3). Let F be any 21-β-closed set of Y. Then $Y \setminus F$ is a 21-β-open set of Y and by (2), we have $S_{\gamma_1}Cl(f^{-1}(Y \setminus F)) \subseteq f^{-1}(1Cl(Y \setminus F)) \Leftrightarrow S_{\gamma_1}Cl(X \setminus f^{-1}(F)) \subseteq f^{-1}(Y \setminus 1Int(F)) \Leftrightarrow X \setminus S_{\gamma_1}Int(f^{-1}(F)) \subseteq X \setminus f^{-1}(1Int(F))$. Therefore, $f^{-1}(1Int(F)) \subseteq S_{\gamma_1}Int(f^{-1}(F))$.
(3) $\Rightarrow$ (4). This is obvious since every 21-semi closed set is 21-β-closed set.
(4) $\Rightarrow$ (5). Let V be any 21-semi open set of Y. Then $Y \setminus V$ is 21-semi closed set and by (4), we have $f^{-1}(1Int(Y \setminus V)) \subseteq S_{\gamma_1}Int(f^{-1}(Y \setminus V)) \Leftrightarrow f^{-1}(Y \setminus 1Cl(V)) \subseteq S_{\gamma_1}Int(X \setminus f^{-1}(V)) \Leftrightarrow X \setminus f^{-1}(1Cl(V)) \subseteq X \setminus S_{\gamma_1}Cl(f^{-1}(V))$. Therefore, $S_{\gamma_1}Cl(f^{-1}(V)) \subseteq f^{-1}(1Cl(V))$.
(5) $\Rightarrow$ (1). Let F be any 12-reguler closed set of Y. Then F is a 21-semi open

set of Y. By (5), we have $S_{\gamma_1}Cl(f^{-1}(F)) \subseteq f^{-1}(1Cl(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is a S_{γ_1}-closed set in X. Therefore, by Proposition 3.10, f is 12-almost S_{γ_1}-continuous.

Proposition 3.13 *A function $f : X \to Y$ is 12-almost S_{γ_1}-continuous if and only if $f^{-1}(V) \subseteq S_{\gamma_1}Int(f^{-1}(1Int(2Cl(V))))$ for each 1-open set V of Y.*

Proof. Let V be any 1-open set of Y. Then $V \subseteq 1Int(2Cl(V))$ and $1Int(2Cl(V))$ is 12-reguler open set in Y. Since f is 12-almost S_{γ_1}-continuous, by Proposition 3.10, $f^{-1}(1Int(2Cl(V)))$ is a S_{γ_1}-open set in X and hence we obtain that $f^{-1}(V) \subseteq f^{-1}(1Int(2Cl(V))) = S_{\gamma_1}Int(f^{-1}(1Int(2Cl(V))))$.
Conversely, Let V be any 12-regular open set of Y. Then V is 1-open set of Y. By hypothesis, we have $f^{-1}(V) \subseteq S_{\gamma_1}Int(f^{-1}(1Int(2Cl(V)))) = S_{\gamma_1}Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is a S_{γ_1}-open set in X and hence by Proposition 3.10, f is 12-almost S_{γ_1}-continuous.

From Proposition 3.13, the following result is obtained.

Corollary 3.14 *A function $f : X \to Y$ is 12-almost S_{γ_1}-continuous if and only if $S_{\gamma_1}Cl(f^{-1}(1Cl(2Int(F)))) \subseteq f^{-1}(F)$ for each 1-closed set F of Y.*

Proposition 3.15 *Let $f : X \to Y$ is an 12-almost S_{γ_1}-continuous function and let V be any 1-open subset of Y. If $x \in S_{\gamma_1}Cl(f^{-1}(V)) \setminus f^{-1}(V)$, then $f(x) \in S_{\gamma_1}Cl(V)$.*

Proof. Let $x \in X$ be such that $x \in S_{\gamma_1}Cl(f^{-1}(V)) \setminus f^{-1}(V)$ and suppose $f(x) \notin S_{\gamma_1}Cl(V)$. Then there exists a S_{γ_1}-open set H containing $f(x)$ such that $H \cap V = \varphi$. Then $2Cl(H) \cap V = \varphi$ implies $1Int(2Cl(H)) \cap V = \varphi$ and $1Int(2Cl(H))$ is an 12-regular open set. Since f is 12-almost S_{γ_1}-continuous, by Proposition 3.10, there exists a S_{γ_1}-open set U in X containing x such that $f(U) \subseteq 1Int(2Cl(H))$. Therefore, $f(U) \cap V = \varphi$. However, since $x \in S_{\gamma_1}Cl(f^{-1}(V))$, $U \cap f^{-1}(V) \neq \varphi$ for every S_{γ_1}-open set U in X containing x, so that $f(U) \cap V \neq \varphi$. We get a contradiction. It follows that $f(x) \in S_{\gamma_1}Cl(V)$.

References

[1] A.B. Khelaf and A.M. Omar, $S_\imath$-open sets and $S_\imath$-continuity in bitopological spaces, *Tamkang Journal of Mathematics*, 43(2012), 81-97.

[2] C.K. Basu, B.M.U. Afsan and M.K. Ghosh, A class of functions and separation axioms with respect to an operation, *Hacettepe Journal of Mathematics and Statistics*, 38(2) (2009), 103-118.

[3] C.G. Kariofillis, On pairwise almost compactness, *Ann. Soc. Sci. Bruxelles.*, 100(1986), 129-137.

[4] F.H. Khedr and A.M. Alshibani, On pairwise super continuous mapping in bitopological spaces, *Internat. J. Math. and Math. Sci.*, 14(4) (1991), 715-722.

[5] H. Ogata, Operation on topological spaces and associated topology, *Math. Japonica*, 36(1991), 175-184.

[6] M. Jelic, A decomposition of pairwise continuity, *J. Inst. Math. Comput. Sci. Math. Ser.*, 3(1990), 25-29.

[7] M. Jelic, Feebly p-continuous mappings, V International Meeting on Topology in Italy (Italian) (Lecce, 1990/Otranto, 1990), *Rend. Circ. Mat. Palermo. (2) Suppl.*, 24(1990), 387-395.

[8] S.N. Maheshwari and R. Prasad, Semi open sets and semi continuous function in bitopological spaces, *Math. Notae.*, 26(1977/78), 29-37.

[9] G.S.S. Krishnan, A new class of semi open sets in a topological space, *Proc. NCMCM*, Allied Publishers, New Delhi, (2003), 305-311.

[10] S. Kasahara, Operalion-compact spaces, *Math. Japonica,* 24(1979), 97-105.

[11] F.H. Khedr and S.M. Al-Areefi, Precontinuity and semi-pre-continuity in bitopological spaces, *Indian. J. pure. appl. Math.*, 23(9) (1992), 625-633.

[12] S. Bose and D. Sinha, Almost open, Almost closed, θ-continuous and almost compact mapping in bitopological spaces, *Bull. Calcutta. Math. Soc.*, 73(1981), 345-354.