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On Some Separation Axioms via β-γ-Open Sets

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Abstract

In this paper, we introduce and investigate some weak separation axioms by using the notion of β-γ-open sets.

Keywords: *β-γ-open; β-γ-closure;β-γ-generalized closed.*

1 Introduction

The notion of R_0 topological spaces is introduced by Shanin [6] in 1943. Later, Davis [3] rediscovered it and studied some properties of this weak separation axiom. In the same paper, Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker than T_2. The notion of γ-open sets was introduced by Ogata [5]. In this paper, we offer some new separation axioms by utilizing β-γ-open sets and β-γ-closure operator. We also characterize their fundamental properties.

2 Preliminaries

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denotes a topological spaces on which no separation axioms is assumed unless explicitly stated. Let A be a subset of a topological space X. The closure of A is denoted by $Cl(A)$. A subset A of a topological space (X, τ) is said to be β-open [2] if $A \subseteq Cl(Int(Cl(A)))$. The complement of a β-open set is said to be β-closed.

Definition 2.1 *[1] Let (X, τ) be a topological space. An operation γ on the topology τ is a mapping from τ into the power set $P(X)$ of X such that $V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V.*

Definition 2.2 *[5] A subset A of a topological space (X, τ) is called γ-open set if for each $x \in A$ there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. The Complement of a γ-open set is called γ-closed.*

In the remainder of this section all the definitions, results and notations are from [4].

Definition 2.3 *Let $\gamma : \beta O(X) \to P(X)$ be a mapping satisfying the following property, $V \subseteq \gamma(V)$ for each non-empty subset $V \in \beta O(X)$ and $\gamma(\phi) = \phi$. The mapping γ is called an operation on $\beta O(X)$.*

Definition 2.4 *Let (X, τ) be a topological space and $\gamma : \beta O(X) \to P(X)$ an operation on $\beta O(X)$. A non-empty set A of X is called a β-γ-open set of (X, τ) if for each point $x \in A$, there exists a β-open set U containing x such that $\gamma(U) \subseteq A$.*
The complement of a β-γ-open set is called β-γ-closed in (X, τ) and the family of all β-γ-open (resp., β-γ-closed) sets of (X, τ) is denoted by $\beta O(X)_\gamma$ (resp., $\beta C(X)_\gamma$).

Definition 2.5 *A point $x \in X$ is in β-γ-closure of a set $A \subseteq X$, if $\gamma(U) \cap A \neq \phi$ for each β-open set U containing x.*
Following [4], the β-γ-closure of A is denoted by $\beta O(X)_\gamma$-$Cl(A)$.

Theorem 2.6 *For a point $x \in X$, $x \in \beta O(X)_\gamma$-$Cl(A)$ if and only if for every β-γ-open set V of X containing x such that $A \cap V \neq \phi$.*

Definition 2.7 *A topological space (X, τ) with an operation γ on $\beta O(X)$ is said to be*

1. *β-γ-T_0' if for any pair of distinct points x and y of X there exists a β-γ-open set U in X containing x but not y or a β-γ-open set V in X containing y but not x.*

2. *β-γ-T_1' if for any pair of distinct points x and y of X there exists a β-γ-open set U in X containing x but not y and a β-γ-open set V in X containing y but not x.*

3. *β-γ-T_2' if for any pair of distinct points x and y of X there exist disjoint β-γ-open sets U and V in X containing x and y, respectively.*

Theorem 2.8 *A topological space (X, τ) with an operation γ on $\beta O(X)$ is β-γ-T_1' if and only if the singletons are β-γ-closed sets.*

3 β-γ-Generalized Closed Sets

In this section, we introduce the concept of β-γ-generalized closed set and give some properties of this set. In the end of this section, we define the concept of β-γ-$T'_{\frac{1}{2}}$.

Definition 3.1 *A subset A of the space (X,τ) is said to be β-γ-generalized closed (Briefly. β-γ-g.closed) if $\beta O(X)_\gamma$-$Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a β-γ-open set in (X,τ). The complement of a β-γ-g.closed set is called a β-γ-g.open set.*

It is clear that every β-γ-closed subset of X is also a β-γ-g.closed set. The following example shows that a β-γ-g.closed set need not be β-γ-closed.

Example 3.2 *Consider $X = \{a,b,c\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$. Define an operation γ on $\beta O(X)$ by*

$$\gamma(A) = \begin{cases} A & \text{if } A = \{b\} \text{ or } \{a,c\} \\ X & \text{otherwise} \end{cases}$$

Now, if we let $A = \{a\}$, since the only β-γ-open supersets of A are $\{a,c\}$ and X, then A is β-γ-g.closed. But it is easy to see that A is not β-γ-closed.

Theorem 3.3 *A subset A of (X,τ) is β-γ-g.closed if and only if $\beta O(X)_\gamma$-$Cl(\{x\}) \cap A \neq \phi$, holds for every $x \in \beta O(X)_\gamma$-$Cl(A)$.*

Proof. Let U be a β-γ-open set such that $A \subseteq U$ and let $x \in \beta O(X)_\gamma$-$Cl(A)$. By assumption, there exists a $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ and $z \in A \subseteq U$. It follows from Theorem 2.6, that $U \cap \{x\} \neq \phi$, hence $x \in U$, this implies $\beta O(X)_\gamma$-$Cl(A) \subseteq U$. Therefore A is β-γ-g.closed.
Conversely, suppose that $x \in \beta O(X)_\gamma$-$Cl(A)$ such that $\beta O(X)_\gamma$-$Cl(\{x\}) \cap A = \phi$. Since, $\beta O(X)_\gamma$-$Cl(\{x\})$ is β-γ-closed, therefore $X \setminus \beta O(X)_\gamma$-$Cl(\{x\})$ is a β-γ-open set in X. Since $A \subseteq X \setminus (\beta O(X)_\gamma$-$Cl(\{x\}))$ and A is β-γ-g.closed implies that $\beta O(X)_\gamma$-$Cl(A) \subseteq X \setminus \beta O(X)_\gamma$-$Cl(\{x\})$ holds, and hence $x \notin \beta O(X)_\gamma$-$Cl(A)$. This is a contradiction. Therefore $\beta O(X)_\gamma$-$Cl(\{x\}) \cap A \neq \phi$.

Theorem 3.4 *A set A of a space X is β-γ-g.closed if and only if $\beta O(X)_\gamma$-$Cl(A) \setminus A$ does not contain any non-empty β-γ-closed set.*

Proof. Necessity. Suppose that A is β-γ-g.closed set in X. We prove the result by contradiction. Let F be a β-γ-closed set such that $F \subseteq \beta O(X)_\gamma$-$Cl(A) \setminus A$ and $F \neq \phi$. Then $F \subseteq X \setminus A$ which implies $A \subseteq X \setminus F$. Since A is β-γ-g.closed and $X \setminus F$ is β-γ-open, therefore $\beta O(X)_\gamma$-$Cl(A) \subseteq X \setminus F$,

that is $F \subseteq X \setminus \beta O(X)_\gamma\text{-}Cl(A)$. Hence $F \subseteq \beta O(X)_\gamma\text{-}Cl(A) \cap (X \setminus \beta O(X)_\gamma\text{-}Cl(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $\beta O(X)_\gamma\text{-}Cl(A) \setminus A$ does not contains any non-empty β-γ-closed set in X.
Sufficiency. Let $A \subseteq U$, where U is β-γ-open in (X, τ). If $\beta O(X)_\gamma\text{-}Cl(A)$ is not contained in U, then $\beta O(X)_\gamma\text{-}Cl(A) \cap X \setminus U \neq \phi$. Now, since $\beta O(X)_\gamma\text{-}Cl(A) \cap X \setminus U \subseteq \beta O(X)_\gamma\text{-}Cl(A) \setminus A$ and $\beta O(X)_\gamma\text{-}Cl(A) \cap X \setminus U$ is a non-empty β-γ-closed set, then we obtain a contradication and therefore A is β-γ-g.closed.

Corollary 3.5 *If a subset A of X is β-γ-g.closed set in X, then $\beta O(X)_\gamma\text{-}Cl(A) \setminus A$ dose not contain any non-empty γ-closed set in X.*

Proof. Follows from that every γ-open set is β-γ-open. The converse of the above corollary is not true in general as it is shown in the following example.

Example 3.6 *Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, X\}$. Define an operation γ on $\beta O(X)$ by $\gamma(A) = A$. If we let $A = \{a, c\}$ then A is not β-γ-g.closed, since $A \subseteq \{a, c\} \in \beta O(X)_\gamma$ and $\beta O(X)_\gamma\text{-}Cl(A) = X \nsubseteq \{a, c\}$, where $\beta O(X)_\gamma\text{-}Cl(A) \setminus A = \{b\}$ dose not contain any non-empty γ-closed set in X.*

Theorem 3.7 *If A is a β-γ-g.closed set of a space X , then the following are equivalent:*

1. *A is β-γ-closed.*
2. *$\beta O(X)_\gamma\text{-}Cl(A) \setminus A$ is β-γ-closed.*

Proof. (1) $\Rightarrow$ (2). If A is a β-γ-g.closed set which is also β-γ-closed, then by Theorem 3.4, $\beta O(X)_\gamma\text{-}Cl(A) \setminus A = \phi$ which is β-γ-closed.
(2) $\Rightarrow$ (1). Let $\beta O(X)_\gamma\text{-}Cl(A) \setminus A$ be β-γ-closed set and A be β-γ-g.closed. Then by Theorem 3.4, $\beta O(X)_\gamma\text{-}Cl(A) \setminus A$ does not contain any non-empty β-γ-closed subset. Since $\beta O(X)_\gamma\text{-}Cl(A) \setminus A$ is β-γ-closed and $\beta O(X)_\gamma\text{-}Cl(A) \setminus A = \phi$, this shows that A is β-γ-closed.

Theorem 3.8 *For a space (X, τ), the following are equivalent:*

1. *Every subset of X is β-γ-g.closed.*
2. *$\beta O(X)_\gamma = \beta C(X)_\gamma$.*

Proof. (1) $\Rightarrow$ (2). Let $U \in \beta O(X)_\gamma$. Then by hypothesis, U is β-γ-g.closed which implies that $\beta O(X)_\gamma\text{-}Cl(U) \subseteq U$, so, $\beta O(X)_\gamma\text{-}Cl(U) = U$, therefore $U \in \beta C(X)_\gamma$. Also let $V \in \beta C(X)_\gamma$. Then $X \setminus V \in \beta O(X)_\gamma$, hence by hypothesis $X \setminus V$ is β-γ-g.closed and then $X \setminus V \in \beta C(X)_\gamma$, thus $V \in \beta O(X)_\gamma$ according above we have $\beta O(X)_\gamma = \beta C(X)_\gamma$.
(2) $\Rightarrow$ (1). If A is a subset of a space X such that $A \subseteq U$ where $U \in \beta O(X)_\gamma$, then $U \in \beta C(X)_\gamma$ and therefore $\beta O(X)_\gamma\text{-}Cl(U) \subseteq U$ which shows that A is β-γ-g.closed.

Proposition 3.9 *If A is γ-open and β-γ-g.closed then A is β-γ-closed.*

Proof. Suppose that A is γ-open and β-γ-g.closed. As every γ-open is β-γ-open and $A \subseteq A$, we have $\beta O(X)_\gamma$-$Cl(A) \subseteq A$, also $A \subseteq \beta O(X)_\gamma$-$Cl(A)$, therefore $\beta O(X)_\gamma$-$Cl(A) = A$. That is A is β-γ-closed.

Theorem 3.10 *If a subset A of X is β-γ-g.closed and $A \subseteq B \subseteq \beta O(X)_\gamma$-$Cl(A)$, then B is a β-γ-g.closed set in X.*

Proof. Let A be β-γ-g.closed set such that $A \subseteq B \subseteq \beta O(X)_\gamma$-$Cl(A)$. Let U be a β-γ-open set of X such that $B \subseteq U$. Since A is β-γ-g.closed, we have $\beta O(X)_\gamma$-$Cl(A) \subseteq U$. Now $\beta O(X)_\gamma$-$Cl(A) \subseteq \beta O(X)_\gamma$-$Cl(B) \subseteq \beta O(X)_\gamma$-$Cl[\beta O(X)_\gamma$-$Cl(A)] = \beta O(X)_\gamma$-$Cl(A) \subseteq U$. That is $\beta O(X)_\gamma$-$Cl(B) \subseteq U$, where U is β-γ-open. Therefore B is a β-γ-g.closed set in X. The converse of the above theorem is not true in general as it is seen from the following example.

Example 3.11 *Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Define an operation γ on $\beta O(X)$ by $\gamma(A) = A$. Let $A = \{b\}$ and $B = \{b, c\}$. Then A and B are β-γ-g.closed sets in (X, τ). But $A \subseteq B \nsubseteq \beta O(X)_\gamma$-$Cl(A)$.*

Proposition 3.12 *Let γ be an operation on $\beta O(X)$. Then for each $x \in X$, $\{x\}$ is β-γ-closed or $X \setminus \{x\}$ is β-γ-g.closed in (X, τ).*

Proof. Suppose that $\{x\}$ is not β-γ-closed, then $X \setminus \{x\}$ is not β-γ-open. Let U be any β-γ-open set such that $X \setminus \{x\} \subseteq U$, implies $U = X$. Therefore $\beta O(X)_\gamma$-$Cl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is β-γ-g.closed.

Definition 3.13 *A space (X, τ) is said to be β-γ-$T'_{\frac{1}{2}}$ if every β-γ-g.closed set is β-γ-closed.*

Theorem 3.14 *The following statements are equivalent for a topological space (X, τ) with an operation γ on $\beta O(X)$:*

1. *(X, τ) is β-γ-$T'_{\frac{1}{2}}$.*
2. *Each singleton $\{x\}$ of X is either β-γ-closed or β-γ-open.*

Proof. (1) $\Rightarrow$ (2). Suppose $\{x\}$ is not β-γ-closed. Then by Proposition 3.12, $X \setminus \{x\}$ is β-γ-g.closed. Now since (X, τ) is β-γ-$T'_{\frac{1}{2}}$, $X \setminus \{x\}$ is β-γ-closed i.e. $\{x\}$ is β-γ-open.
(2) $\Rightarrow$ (1). Let A be any β-γ-g.closed set in (X, τ) and $x \in \beta O(X)_\gamma$-$Cl(A)$. By (2) we have $\{x\}$ is β-γ-closed or β-γ-open. If $\{x\}$ is β-γ-closed then $x \notin A$ will imply $x \in \beta O(X)_\gamma$-$Cl(A) \setminus A$, which is not possible by Theorem 3.4. Hence $x \in A$. Therefore, $\beta O(X)_\gamma$-$Cl(A) = A$, i.e. A is β-γ-closed. So, (X, τ) is β-γ-$T'_{\frac{1}{2}}$. On the other hand, if $\{x\}$ is β-γ-open then as $x \in \beta O(X)_\gamma$-$Cl(A)$, $\{x\} \cap A \neq \phi$. Hence $x \in A$. So A is β-γ-closed.

Definition 3.15 *Let A be a subset of a topological space (X, τ) and γ an operation on $\beta O(X, \tau)$. The union of all β-γ-open sets contained in A is called the β-γ-interior of A and denoted by $\beta O(X)_\gamma$-$Int(A)$.*

Proposition 3.16 *A subset A of X is β-γ-g.open if and only if $F \subseteq \beta O(X)_\gamma$-$Int(A)$ whenever $F \subseteq A$ and F is β-γ-closed in (X, τ).*

Proof. Obvious.

4 β-γ-Regular and β-γ-Normal Spaces

Definition 4.1 *A topological space (X, τ) with an operation γ on $\beta O(X)$ is called β-γ-regular if for each β-γ-closed set F of X not containing x, there exist disjoint β-γ-open sets U and V such that $x \in U$ and $F \subseteq V$. A β-γ-regular β-γ-T_1' space is called a β-γ-T_3' space.*

Remark 4.2 *Every β-γ-T_3' space is β-γ-T_2'.*

The following result contains some characterizations of β-γ-regular spaces.

Proposition 4.3 *The following are equivalent for a topological space (X, τ) with an operation γ on $\beta O(X)$:*

1. *X is β-γ-regular.*
2. *For each $x \in X$ and each β-γ-open set U containing x, there exists a β-γ-open set V containing x such that $x \in V \subseteq \beta O(X)_\gamma$-$Cl(V) \subseteq U$.*
3. *For each β-γ-closed set F of X, $\cap\{\beta O(X)_\gamma$-$Cl(V)$: $F \subseteq V$, $V \in \beta O(X)_\gamma\} = F$.*
4. *For each A subset of X and each $U \in \beta O(X)_\gamma$ with $A \cap U \neq \phi$, there exists a $V \in \beta O(X)_\gamma$ such that $A \cap V \neq \phi$ and $\beta O(X)_\gamma$-$Cl(V) \subseteq U$.*
5. *For each nonempty subset A of X and each β-γ-closed subset F of X with $A \cap F = \phi$, there exists $V, W \in \beta O(X)_\gamma$ such that $A \cap V \neq \phi$, $F \subseteq W$ and $W \cap V = \phi$.*
6. *For each β-γ-closed set F and $x \in F$, there exists $U \in \beta O(X)_\gamma$ and a β-γ-g.open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.*
7. *For each $A \subseteq X$ and each β-γ-closed set F with $A \cap F = \phi$, there exists $U \in \beta O(X)_\gamma$ and a β-γ-g.open set V such that $A \cap U \neq \phi$, $F \subseteq V$ and $U \cap V = \phi$.*

8. *For each β-γ-closed set F of X, $F = \cap\{\beta O(X)_\gamma\text{-}Cl(V)\colon F \subseteq V,\ V$ is β-γ-g.open$\}$.*

Proof. (1) $\Rightarrow$ (2). Let $x \notin X \setminus U$, where U is any β-γ-open set containing x. Then there exists $G, V \in \beta O(X)_\gamma$ such that $(X \setminus U) \subseteq G$, $x \in V$ and $G \cap V = \phi$. Therefore $V \subseteq (X \setminus G)$ and so $x \in V \subseteq \beta O(X)_\gamma\text{-}Cl(V) \subseteq (X \setminus G) \subseteq U$.
(2) $\Rightarrow$ (3). Let $X \setminus F$ be any β-γ-open set containing x. Then by (2) there exists a β-γ-open set U containing x such that $x \in U \subseteq \beta O(X)_\gamma\text{-}Cl(U) \subseteq (X \setminus F)$. So, $F \subseteq X \setminus \beta O(X)_\gamma\text{-}Cl(U) = V$, $V \in \beta O(X)_\gamma$ and $V \cap U = \phi$. Then by Theorem 2.6, $x \notin \beta O(X)_\gamma\text{-}Cl(V)$. Thus $F \supseteq \cap\{\beta O(X)_\gamma\text{-}Cl(V)\colon F \subseteq V, V \in \beta O(X)_\gamma\}$.
(3) $\Rightarrow$ (4). Let $U \in \beta O(X)_\gamma$ with $x \in U \cap A$. Then $x \notin (X \setminus U)$ and hence by (3) there exists a β-γ-open set W such that $X \setminus U \subseteq W$ and $x \notin \beta O(X)_\gamma\text{-}Cl(W)$. We put $V = X \setminus \beta O(X)_\gamma\text{-}Cl(W)$, which is a β-γ-open set containing x and hence $V \cap A \neq \phi$. Now $V \subseteq (X \setminus W)$ and so $\beta O(X)_\gamma\text{-}Cl(V) \subseteq (X \setminus W) \subseteq U$.
(4) $\Rightarrow$ (5). Let F be a set as in the hypothesis of (5). Then $(X \setminus F)$ is β-γ-open and $(X \setminus F) \cap A \neq \phi$. Then there exists $V \in \beta O(X)_\gamma$ such that $A \cap V \neq \phi$ and $\beta O(X)_\gamma\text{-}Cl(V) \subseteq (X \setminus F)$. If we put $W = X \setminus \beta O(X)_\gamma\text{-}Cl(V)$, then $F \subseteq W$ and $W \cap V = \phi$.
(5) $\Rightarrow$ (1) Let F be a β-γ-closed set not containing x. Then by (5), there exist $W, V \in \beta O(X)_\gamma$ such that $F \subseteq W$ and $x \in V$ and $W \cap V = \phi$.
(1) $\Rightarrow$ (6). Obvious.
(6) $\Rightarrow$ (7). For $a \in A$, $a \notin F$ and hence by (6) there exists $U \in \beta O(X)_\gamma$ and a β-γ-g.open set V such that $a \in U$, $F \subseteq V$ and $U \cap V = \phi$. So, $A \cap U \neq \phi$.
(7) $\Rightarrow$ (1) Let $x \notin F$, where F is β-γ-closed. Since $\{x\} \cap F = \phi$, by (7) there exists $U \in \beta O(X)_\gamma$ and a β-γ-g.open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \phi$. Now put $V = \beta O(X)_\gamma\text{-}Int(W)$. Using Proposition 3.16 of β-γ-g.open sets we get $F \subseteq V$ and $V \cap U = \phi$.
(3) $\Rightarrow$ (8). We have $F \subseteq \bigcap\{\beta O(X)_\gamma\text{-}Cl(V)\colon F \subseteq V$ and V is β-γ-g.open $\} \subseteq \bigcap\{\beta O(X)_\gamma\text{-}Cl(V)\colon F \subseteq V$ and V is β-γ-open $\} = F$.
(8) $\Rightarrow$ (1). Let F be a β-γ-closed set in X not containing x. Then by (8) there exists a β-γ-g.open set W such that $F \subseteq W$ and $x \in X \setminus \beta O(X)_\gamma\text{-}Cl(W)$. Since F is β-γ-closed and W is β-γ-g.open, $F \subseteq \beta O(X)_\gamma\text{-}Int(W)$. If $V = \beta O(X)_\gamma\text{-}Int(W)$, then $F \subseteq V$, $x \in U = X \setminus \beta O(X)_\gamma\text{-}Cl(V)$ and $U \cap V = \phi$.

Definition 4.4 *A topological space (X, τ) with an operation γ on $\beta O(X)$, is said to be β-γ-normal if for any pair of disjoint β-γ-closed sets A, B of X, there exist disjoint β-γ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. A β-γ-normal β-γ-T_1' space is called a β-γ-T_4' space.*

We give several characterizations of β-γ-normal spaces in the following proposition.

Proposition 4.5 *For a topological space (X, τ) with an operation γ on $\beta O(X)$, the following are equivalent:*

1. *X is β-γ-normal.*

2. *For each pair of disjoint β-γ-closed sets A, B of X, there exist disjoint β-γ-g.open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

3. *For each β-γ-closed A and any β-γ-open set V containing A, there exists a β-γ-g.open set U such that $A \subseteq U \subseteq \beta O(X)_\gamma$-$Cl(U) \subseteq V$.*

4. *For each β-γ-closed set A and any β-γ-g.open set B containing A, there exists a β-γ-g.open set U such that $A \subseteq U \subseteq \beta O(X)_\gamma$-$Cl(U) \subseteq \beta O(X)_\gamma$-$Int(B)$.*

5. *For each β-γ-closed set A and any β-γ-g.open set B containing A, there exists a β-γ-open set G such that $A \subseteq G \subseteq \beta O(X)_\gamma$-$Cl(G) \subseteq \beta O(X)_\gamma$-$Int(B)$.*

6. *For each β-γ-g.closed set A and any β-γ-open set B containing A, there exists a β-γ-open set U such that $\beta O(X)_\gamma$-$Cl(A) \subseteq U \subseteq \beta O(X)_\gamma$-$Cl(U) \subseteq B$.*

7. *For each β-γ-g.closed set A and any β-γ-open set B containing A, there exists a β-γ-g.open set G such that $\beta O(X)_\gamma$-$Cl(A) \subseteq G \subseteq \beta O(X)_\gamma$-$Cl(G) \subseteq B$.*

Proof. (1) $\Rightarrow$ (2). Follows from the fact that every β-γ-open set is β-γ-g.open.
(2) $\Rightarrow$ (3). Let A be a β-γ-closed set and V any β-γ-open set containing A. Since A and $(X \setminus V)$ are disjoint β-γ-closed sets, there exist β-γ-g.open sets U and W such that $A \subseteq U$, $(X \setminus V) \subseteq W$ and $U \cap W = \phi$. By Proposition 3.16, we get $(X \setminus V) \subseteq \beta O(X)_\gamma$-$Int(W)$. Since $U \cap \beta O(X)_\gamma$-$Int(W) = \phi$, we have $\beta O(X)_\gamma$-$Cl(U) \cap \beta O(X)_\gamma$-$Int(W) = \phi$, and hence $\beta O(X)_\gamma$-$Cl(U) \subseteq X \setminus \beta O(X)_\gamma$-$Int(W) \subseteq V$. Therefore $A \subseteq U \subseteq \beta O(X)_\gamma$-$Cl(U) \subseteq V$.
(3) $\Rightarrow$ (1). Let A and B be any two disjoint β-γ-closed sets of X. Since $(X \setminus B)$ is a β-γ-open set containing A, there exists a β-γ-g.open set G such that $A \subseteq G \subseteq \beta O(X)_\gamma$-$Cl(G) \subseteq (X \setminus B)$. Since G is a β-γ-g.open set, using Proposition 3.16, we have $A \subseteq \beta O(X)_\gamma$-$Int(G)$. Taking $U = \beta O(X)_\gamma$-$Int(G)$ and $V = X \setminus \beta O(X)_\gamma$-$Cl(G)$, we have two disjoint β-γ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Hence X is β-γ-normal.
(5) $\Rightarrow$ (4). Obvious.
(4) $\Rightarrow$ (3). Obvious.
(5) $\Rightarrow$ (3). Let A be any β-γ-closed set and V any β-γ-open set containing

A. Since every β-γ-open set is β-γ-g.open, there exists a β-γ-open set G such that $A \subseteq G \subseteq \beta O(X)_\gamma\text{-}Cl(G) \subseteq \beta O(X)_\gamma\text{-}Int(V)$. Also we have a β-γ-g.open set G such that $A \subseteq G \subseteq \beta O(X)_\gamma\text{-}Cl(G) \subseteq \beta O(X)_\gamma\text{-}Int(V) \subseteq V$.
(6) $\Rightarrow$ (7). Obvious.
(7) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (5). Let A be a β-γ-closed set and B any β-γ-g.open set containing A. Using Proposition 3.16, of a β-γ-g.open set we get $A \subseteq \beta O(X)_\gamma\text{-}Int(B) = V$, say. Then applying (3), we get a β-γ-g.open set U such that $A = \beta O(X)_\gamma\text{-}Cl(A) \subseteq U \subseteq \beta O(X)_\gamma\text{-}Cl(U) \subseteq V$. Again, using the same Proposition 3.16, we get $A \subseteq \beta O(X)_\gamma\text{-}Int(U)$, and hence $A \subseteq \beta O(X)_\gamma\text{-}Int(U) \subseteq U \subseteq \beta O(X)_\gamma\text{-}Cl(U) \subseteq V$, which implies $A \subseteq \beta O(X)_\gamma\text{-}Int(U) \subseteq \beta O(X)_\gamma\text{-}Cl(\beta O(X)_\gamma\text{-}Int(U)) \subseteq \beta O(X)_\gamma\text{-}Cl(U) \subseteq V$, i.e.
$A \subseteq G \subseteq \beta O(X)_\gamma\text{-}Cl(G) \subseteq \beta O(X)_\gamma\text{-}Int(B)$, where $G = \beta O(X)_\gamma\text{-}Int(U)$.
(3) $\Rightarrow$ (7). Let A be a β-γ-g.closed set and B any β-γ-open set containing A. Since A is a β-γ-g.closed set, we have $\beta O(X)_\gamma\text{-}Cl(A) \subseteq B$, therefore by (3) we can find a β-γ-g.open set U such that $\beta O(X)_\gamma\text{-}Cl(A) \subseteq U \subseteq \beta O(X)_\gamma\text{-}Cl(U) \subseteq B$.
(7) $\Rightarrow$ (6). Let A be a β-γ-g.closed set and B any β-γ-open set containing A, then by (7) there exists a β-γ-g.open set G such that $\beta O(X)_\gamma\text{-}Cl(A) \subseteq G \subseteq \beta O(X)_\gamma\text{-}Cl(G) \subseteq B$. Since G is a β-γ-g.open set, then by Proposition 3.16, we get $\beta O(X)_\gamma\text{-}Cl(A) \subseteq \beta O(X)_\gamma\text{-}Int(G)$. If we take $U = \beta O(X)_\gamma\text{-}Int(G)$, the proof follows.

Every β-γ-T_4' space is clearly a β-γ-T_3' space, but it should not be surprising that β-γ-normal spaces need not be β-γ-regular.

Example 4.6 *Consider $X = \{a, b, c\}$ with the discrete topology on X. Define an operation γ on $\beta O(X)$ by*

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a\} \\ A \cup \{c\} & \text{if } A \neq \{a\} \end{cases}$$

Then X is β-γ-normal but not β-γ-regular.

Definition 4.7 *A topological space (X, τ) with an operation γ on $\beta O(X)$, is said to be β-γ-symmetric if for x and y in X, $x \in \beta O(X)_\gamma\text{-}Cl(\{y\})$ implies $y \in \beta O(X)_\gamma\text{-}Cl(\{x\})$.*

Proposition 4.8 *If (X, τ) is a topological space with an operation γ on $\beta O(X)$, then the following are equivalent:*

1. *(X, τ) is β-γ-symmetric space.*

2. *Every singleton is β-γ-g.closed, for each $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Assume that $\{x\} \subseteq U \in \beta O(X)_\gamma$, but $\beta O(X)_\gamma$-$Cl(\{x\}) \nsubseteq U$. Then $\beta O(X)_\gamma$-$Cl(\{x\}) \cap X \setminus U \neq \phi$. Now, we take $y \in \beta O(X)_\gamma$-$Cl(\{x\}) \cap X \setminus U$, then by hypothesis $x \in \beta O(X)_\gamma$-$Cl(\{y\}) \subseteq X \setminus U$ and $x \notin U$, which is a contradiction. Therefore $\{x\}$ is β-γ-g.closed, for each $x \in X$.
(2) $\Rightarrow$ (1). Assume that $x \in \beta O(X)_\gamma$-$Cl(\{y\})$, but $y \notin \beta O(X)_\gamma$-$Cl(\{x\})$. Then $\{y\} \subseteq X \setminus \beta O(X)_\gamma$-$Cl(\{x\})$ and hence $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq X \setminus \beta O(X)_\gamma$-$Cl(\{x\})$. Therefore $x \in X \setminus \beta O(X)_\gamma$-$Cl(\{x\})$, which is a contradiction and hence $y \in \beta O(X)_\gamma$-$Cl(\{x\})$.

Corollary 4.9 *If a topological space (X, τ) with an operation γ on $\beta O(X)$ is a β-γ-T_1' space, then it is β-γ-symmetric.*

Proof. In a β-γ-T_1' space, every singleton is β-γ-closed (Theorem 2.8) and therefore is β-γ-g.closed. Then by Proposition 4.8, (X, τ) is β-γ-symmetric.

Corollary 4.10 *For a topological space (X, τ) with an operation γ on $\beta O(X)$, the following statements are equivalent:*

1. *(X, τ) is β-γ-symmetric and β-γ-T_0'.*
2. *(X, τ) is β-γ-T_1'.*

Proof. (1) $\Rightarrow$ (2). Let $x \neq y$ and by β-γ-T_0', we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in \beta O(X)_\gamma$. Then $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$ and hence $y \notin \beta O(X)_\gamma$-$Cl(\{x\})$. There exists a β-γ-open set V such that $y \in V \subseteq X \setminus \{x\}$ and thus (X, τ) is a β-γ-T_1' space.
(2) $\Rightarrow$ (1). Follows from Corollary 4.9.

Proposition 4.11 *If (X, τ) is a β-γ-symmetric space with an operation γ on $\beta O(X)$, then the following statements are equivalent:*

1. *(X, τ) is a β-γ-T_0' space.*
2. *(X, τ) is a β-γ-$T_{\frac{1}{2}}'$ space.*
3. *(X, τ) is a β-γ-T_1' space.*

Proof. (1) $\Leftrightarrow$ (3). Follows from Corollary 4.10.
(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious.

5 β-γ-R_0 and β-γ-R_1 Spaces

Definition 5.1 *[4] Let A be a subset of a topological space (X,τ) and γ be an operation on $\beta O(X)$. The β-γ-kernel of A, denoted by $\beta O(X)_\gamma$-$Ker(A)$ is defined to be the set*

$$\beta O(X)_\gamma\text{-}Ker(A) = \cap\{U \in \beta O(X)_\gamma\text{: } A \subseteq U\}.$$

Lemma 5.2 *Let (X,τ) be a topological space with an operation γ on τ and $x \in X$. Then $y \in \beta O(X)_\gamma$-$Ker(\{x\})$ if and only if $x \in \beta O(X)_\gamma$-$Cl(\{y\})$.*

Proof. Suppose that $y \notin \beta O(X)_\gamma$-$Ker(\{x\})$. Then there exists a β-γ-open set V containing x such that $y \notin V$. Therefore, we have $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$. The proof of the converse case can be done similarly.

Theorem 5.3 *Let (X,τ) be a topological space with an operation γ on τ and A be a subset of X. Then, $\beta O(X)_\gamma$-$Ker(A) = \{x \in X\text{: } \beta O(X)_\gamma\text{-}Cl(\{x\}) \cap A \neq \phi\}$.*

Proof. Let $x \in \beta O(X)_\gamma$-$Ker(A)$ and suppose $\beta O(X)_\gamma$-$Cl(\{x\}) \cap A = \phi$. Hence $x \notin X \setminus \beta O(X)_\gamma$-$Cl(\{x\})$ which is a β-γ-open set containing A. This is impossible, since $x \in \beta O(X)_\gamma$-$Ker(A)$. Consequently, $\beta O(X)_\gamma$-$Cl(\{x\}) \cap A \neq \phi$. Next, let $x \in X$ such that $\beta O(X)_\gamma$-$Cl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin \beta O(X)_\gamma$-$Ker(A)$. Then, there exists a β-γ-open set V containing A and $x \notin V$. Let $y \in \beta O(X)_\gamma$-$Cl(\{x\}) \cap A$. Hence, V is a β-γ-nbd of y which does not contain x. By this contradiction $x \in \beta O(X)_\gamma$-$Ker(A)$ and the claim.

Theorem 5.4 *The following properties hold for the subsets A, B of a topological space (X,τ) with an operation γ on τ:*

1. *$A \subseteq \beta O(X)_\gamma$-$Ker(A)$.*
2. *$A \subseteq B$ implies that $\beta O(X)_\gamma$-$Ker(A) \subseteq \beta O(X)_\gamma$-$Ker(B)$.*
3. *If A is β-γ-open in (X,τ), then $A = \beta O(X)_\gamma$-$Ker(A)$.*
4. *$\beta O(X)_\gamma$-$Ker(\beta O(X)_\gamma$-$Ker(A)) = \beta O(X)_\gamma$-$Ker(A)$.*

Proof. (1), (2) and (3) are immediate consequences of Definition 5.1. To prove (4), first observe that by (1) and (2), we have $\beta O(X)_\gamma$-$Ker(A) \subseteq \beta O(X)_\gamma$-$Ker(\beta O(X)_\gamma$-$Ker(A))$. If $x \notin \beta O(X)_\gamma$-$Ker(A)$, then there exists $U \in \beta O(X)_\gamma$ such that $A \subseteq U$ and $x \notin U$. Hence $\beta O(X)_\gamma$-$Ker(A) \subseteq U$, and so we have $x \notin \beta O(X)_\gamma$-$Ker(\beta O(X)_\gamma$-$Ker(A))$. Thus $\beta O(X)_\gamma$-$Ker(\beta O(X)_\gamma$-$Ker(A)) = \beta O(X)_\gamma$-$Ker(A)$.

Definition 5.5 *A topological space (X,τ) with an operation γ on τ, is said to be β-γ-R_0 if U is a β-γ-open set and $x \in U$ then $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq U$.*

Theorem 5.6 *For a topological space (X,τ) with an operation γ on τ, the following properties are equivalent:*

1. *(X,τ) is β-γ-R_0.*
2. *For any $F \in \beta C(X)_\gamma$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in \beta O(X)_\gamma$.*
3. *For any $F \in \beta C(X)_\gamma$, $x \notin F$ implies $F \cap \beta O(X)_\gamma$-$Cl(\{x\}) = \phi$.*
4. *For any distinct points x and y of X, either $\beta O(X)_\gamma$-$Cl(\{x\}) = \beta O(X)_\gamma$-$Cl(\{y\})$ or $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$.*

Proof. (1) $\Rightarrow$ (2). Let $F \in \beta C(X)_\gamma$ and $x \notin F$. Then by (1), $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq X \setminus F$. Set $U = X \setminus \beta O(X)_\gamma$-$Cl(\{x\})$, then U is a β-γ-open set such that $F \subseteq U$ and $x \notin U$.
(2) $\Rightarrow$ (3). Let $F \in \beta C(X)_\gamma$ and $x \notin F$. There exists $U \in \beta O(X)_\gamma$ such that $F \subseteq U$ and $x \notin U$. Since $U \in \beta O(X)_\gamma$, $U \cap \beta O(X)_\gamma$-$Cl(\{x\}) = \phi$ and $F \cap \beta O(X)_\gamma$-$Cl(\{x\}) = \phi$.
(3) $\Rightarrow$ (4). Suppose that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$ for distinct points $x, y \in X$. There exists $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ such that $z \notin \beta O(X)_\gamma$-$Cl(\{y\})$ (or $z \in \beta O(X)_\gamma$-$Cl(\{y\})$ such that $z \notin \beta O(X)_\gamma$-$Cl(\{x\})$). There exists $V \in \beta O(X)_\gamma$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$. By (3), we obtain $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$.
(4) $\Rightarrow$ (1). let $V \in \beta O(X)_\gamma$ and $x \in V$. For each $y \notin V$, $x \neq y$ and $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$. This shows that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. By (4), $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$ for each $y \in X \setminus V$ and hence $\beta O(X)_\gamma$-$Cl(\{x\}) \cap (\bigcup_{y \in X\setminus V} \beta O(X)_\gamma$-$Cl(\{y\})) = \phi$. On other hand, since $V \in \beta O(X)_\gamma$ and $y \in X \setminus V$, we have $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq X \setminus V$ and hence $X \setminus V = \bigcup_{y \in X\setminus V} \beta O(X)_\gamma$-$Cl(\{y\})$. Therefore, we obtain $(X \setminus V) \cap \beta O(X)_\gamma$-$Cl(\{x\}) = \phi$ and $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq V$. This shows that (X,τ) is a β-γ-R_0 space.

Theorem 5.7 *For a topological space (X,τ) with an operation γ on τ, the following properties are equivalent:*

1. *(X,τ) is β-γ-R_0.*
2. *$x \in \beta O(X)_\gamma$-$Cl(\{y\})$ if and only if $y \in \beta O(X)_\gamma$-$Cl(\{x\})$, for any points x and y in X.*

Proof. (1) $\Rightarrow$ (2). Assume that X is β-γ-R_0. Let $x \in \beta O(X)_\gamma$-$Cl(\{y\})$ and V be any β-γ-open set such that $y \in V$. Now by hypothesis, $x \in V$. Therefore, every β-γ-open set which contain y contains x. Hence $y \in \beta O(X)_\gamma$-$Cl(\{x\})$.
(2) $\Rightarrow$ (1). Let U be a β-γ-open set and $x \in U$. If $y \notin U$, then $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$ and hence $y \notin \beta O(X)_\gamma$-$Cl(\{x\})$. This implies that $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq U$. Hence (X, τ) is β-γ-R_0.

Theorem 5.8 *The following statements are equivalent for any points x and y in a topological space (X, τ) with an operation γ on τ:*

1. $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$.

2. $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$, then there exists a point z in X such that $z \in \beta O(X)_\gamma$-$Ker(\{x\})$ and $z \notin \beta O(X)_\gamma$-$Ker(\{y\})$. From $z \in \beta O(X)_\gamma$-$Ker(\{x\})$ it follows that $\{x\} \cap \beta O(X)_\gamma$-$Cl(\{z\}) \neq \phi$ which implies $x \in \beta O(X)_\gamma$-$Cl(\{z\})$. By $z \notin \beta O(X)_\gamma$-$Ker(\{y\})$, we have $\{y\} \cap \beta O(X)_\gamma$-$Cl(\{z\}) = \phi$. Since $x \in \beta O(X)_\gamma$-$Cl(\{z\})$, $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq \beta O(X)_\gamma$-$Cl(\{z\})$ and $\{y\} \cap \beta O(X)_\gamma$-$Cl(\{x\}) = \phi$. Therefore, it follows that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. Now $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$ implies that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$.
(2) $\Rightarrow$ (1). Suppose that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. Then there exists a point z in X such that $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ and $z \notin \beta O(X)_\gamma$-$Cl(\{y\})$. Then, there exists a β-γ-open set containing z and therefore x but not y, namely, $y \notin \beta O(X)_\gamma$-$Ker(\{x\})$ and thus $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$.

Theorem 5.9 *Let (X, τ) be a topological space and γ be an operation on $\beta O(X)$. Then $\cap\{\beta O(X)_\gamma$-$Cl(\{x\}) : x \in X\} = \phi$ if and only if $\beta O(X)_\gamma$-$Ker(\{x\}) \neq X$ for every $x \in X$.*

Proof. Necessity. Suppose that $\cap\{\beta O(X)_\gamma$-$Cl(\{x\}) : x \in X\} = \phi$. Assume that there is a point y in X such that $\beta O(X)_\gamma$-$Ker(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every β-γ-open set V containing y and hence $y \in \beta O(X)_\gamma$-$Cl(\{x\})$ for any $x \in X$. This implies that $y \in \cap\{\beta O(X)_\gamma$-$Cl(\{x\}) : x \in X\}$. But this is a contradiction.
Sufficiency. Assume that $\beta O(X)_\gamma$-$Ker(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap\{\beta O(X)_\gamma$-$Cl(\{x\}) : x \in X\}$, then every β-γ-open set containing y must contain every point of X. This implies that the space X is the unique β-γ-open set containing y. Hence $\beta O(X)_\gamma$-$Ker(\{y\}) = X$ which is a contradiction. Therefore, $\cap\{\beta O(X)_\gamma$-$Cl(\{x\}) : x \in X\} = \phi$.

Theorem 5.10 *A topological space (X, τ) with an operation γ on τ is β-γ-R_0 if and only if for every x and y in X, $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$ implies $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$.*

Proof. Necessity. Suppose that (X, τ) is β-γ-R_0 and $x, y \in X$ such that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. Then, there exists $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ such that $z \notin \beta O(X)_\gamma$-$Cl(\{y\})$ (or $z \in \beta O(X)_\gamma$-$Cl(\{y\})$ such that $z \notin \beta O(X)_\gamma$-$Cl(\{x\})$). There exists $V \in \beta O(X)_\gamma$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$. Thus $x \in [X \backslash \beta O(X)_\gamma$-$Cl(\{y\})] \in \beta O(X)_\gamma$, which implies $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq [X \setminus \beta O(X)_\gamma$-$Cl(\{y\})]$ and $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$.
Sufficiency. Let $V \in \beta O(X)_\gamma$ and let $x \in V$. We still show that $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq V$. Let $y \notin V$, that is $y \in X \setminus V$. Then $x \neq y$ and $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$. This shows that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. By assumption, $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$. Hence $y \notin \beta O(X)_\gamma$-$Cl(\{x\})$ and therefore $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq V$.

Theorem 5.11 *A topological space (X, τ) with an operation γ on τ is β-γ-R_0 if and only if for any points x and y in X, $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$ implies $\beta O(X)_\gamma$-$Ker(\{x\}) \cap \beta O(X)_\gamma$-$Ker(\{y\}) = \phi$.*

Proof. Suppose that (X, τ) is a β-γ-R_0 space. Thus by Theorem 5.8, for any points x and y in X if $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$ then $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. Now we prove that $\beta O(X)_\gamma$-$Ker(\{x\}) \cap \beta O(X)_\gamma$-$Ker(\{y\}) = \phi$. Assume that $z \in \beta O(X)_\gamma$-$Ker(\{x\}) \cap \beta O(X)_\gamma$-$Ker(\{y\})$. By $z \in \beta O(X)_\gamma$-$Ker(\{x\})$ and Lemma 5.2, it follows that $x \in \beta O(X)_\gamma$-$Cl(\{z\})$. Since $x \in \beta O(X)_\gamma$-$Cl(\{x\})$, by Theorem 5.6, $\beta O(X)_\gamma$-$Cl(\{x\}) = \beta O(X)_\gamma$-$Cl(\{z\})$. Similarly, we have $\beta O(X)_\gamma$-$Cl(\{y\}) = \beta O(X)_\gamma$-$Cl(\{z\}) = \beta O(X)_\gamma$-$Cl(\{x\})$. This is a contradiction. Therefore, we have $\beta O(X)_\gamma$-$Ker(\{x\}) \cap \beta O(X)_\gamma$-$Ker(\{y\}) = \phi$.
Conversely, let (X, τ) be a topological space such that for any points x and y in X, $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$ implies $\beta O(X)_\gamma$-$Ker(\{x\}) \cap \beta O(X)_\gamma$-$Ker(\{y\}) = \phi$. If $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$, then by Theorem 5.8, $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$. Hence, $\beta O(X)_\gamma$-$Ker(\{x\}) \cap \beta O(X)_\gamma$-$Ker(\{y\}) = \phi$ which implies $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$. Because $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ implies that $x \in \beta O(X)_\gamma$-$Ker(\{z\})$ and therefore $\beta O(X)_\gamma$-$Ker(\{x\}) \cap \beta O(X)_\gamma$-$Ker(\{z\}) \neq \phi$. By hypothesis, we have $\beta O(X)_\gamma$-$Ker(\{x\}) = \beta O(X)_\gamma$-$Ker(\{z\})$. Then $z \in \beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\})$ implies that $\beta O(X)_\gamma$-$Ker(\{x\}) = \beta O(X)_\gamma$-$Ker(\{z\}) = \beta O(X)_\gamma$-$Ker(\{y\})$. This is a contradiction. Therefore, $\beta O(X)_\gamma$-$Cl(\{x\}) \cap \beta O(X)_\gamma$-$Cl(\{y\}) = \phi$ and by Theorem 5.6, (X, τ) is a β-γ-R_0 space.

Theorem 5.12 *For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:*

1. *(X, τ) is a β-γ-R_0 space.*

2. *For any non-empty set A and $G \in \beta O(X)_\gamma$ such that $A \cap G \neq \phi$, there exists $F \in \beta C(X)_\gamma$ such that $A \cap F \neq \phi$ and $F \subseteq G$.*

3. *For any $G \in \beta O(X)_\gamma$, we have $G = \cup\{F \in \beta C(X)_\gamma$: $F \subseteq G\}$.*

4. *For any $F \in \beta C(X)_\gamma$, we have $F = \cap\{G \in \beta O(X)_\gamma$: $F \subseteq G\}$.*

5. *For every $x \in X$, $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq \beta O(X)_\gamma$-$Ker(\{x\})$.*

Proof. (1) $\Rightarrow$ (2). Let A be a non-empty subset of X and $G \in \beta O(X)_\gamma$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \beta O(X)_\gamma$, $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq G$. Set $F = \beta O(X)_\gamma$-$Cl(\{x\})$, then $F \in \beta C(X)_\gamma$, $F \subseteq G$ and $A \cap F \neq \phi$.
(2) $\Rightarrow$ (3). Let $G \in \beta O(X)_\gamma$, then $G \supseteq \cup\{F \in \beta C(X)_\gamma$: $F \subseteq G\}$. Let x be any point of G. There exists $F \in \beta C(X)_\gamma$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in \beta C(X)_\gamma$: $F \subseteq G\}$ and hence $G = \cup\{F \in \beta C(X)_\gamma$: $F \subseteq G\}$.
(3) $\Rightarrow$ (4). Obvious.
(4) $\Rightarrow$ (5). Let x be any point of X and $y \notin \beta O(X)_\gamma$-$Ker(\{x\})$. There exists $V \in \beta O(X)_\gamma$ such that $x \in V$ and $y \notin V$, hence $\beta O(X)_\gamma$-$Cl(\{y\}) \cap V = \phi$. By (4), $(\cap\{G \in \beta O(X)_\gamma$: $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq G\}) \cap V = \phi$ and there exists $G \in \beta O(X)_\gamma$ such that $x \notin G$ and $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq G$. Therefore $\beta O(X)_\gamma$-$Cl(\{x\}) \cap G = \phi$ and $y \notin \beta O(X)_\gamma$-$Cl(\{x\})$. Consequently, we obtain $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq \beta O(X)_\gamma$-$Ker(\{x\})$.
(5) $\Rightarrow$ (1). Let $G \in \beta O(X)_\gamma$ and $x \in G$. Let $y \in \beta O(X)_\gamma$-$Ker(\{x\})$, then $x \in \beta O(X)_\gamma$-$Cl(\{y\})$ and $y \in G$. This implies that $\beta O(X)_\gamma$-$Ker(\{x\}) \subseteq G$. Therefore, we obtain $x \in \beta O(X)_\gamma$-$Cl(\{x\}) \subseteq \beta O(X)_\gamma$-$Ker(\{x\}) \subseteq G$. This shows that (X, τ) is a β-γ-R_0 space.

Corollary 5.13 *For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:*

1. *(X, τ) is a β-γ-R_0 space.*

2. *$\beta O(X)_\gamma$-$Cl(\{x\}) = \beta O(X)_\gamma$-$Ker(\{x\})$ for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Suppose that (X, τ) is a β-γ-R_0 space. By Theorem 5.12, $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq \beta O(X)_\gamma$-$Ker(\{x\})$ for each $x \in X$. Let $y \in \beta O(X)_\gamma$-$Ker(\{x\})$, then $x \in \beta O(X)_\gamma$-$Cl(\{y\})$ and by Theorem 5.6, $\beta O(X)_\gamma$-$Cl(\{x\}) = \beta O(X)_\gamma$-$Cl(\{y\})$. Therefore, $y \in \beta O(X)_\gamma$-$Cl(\{x\})$ and hence $\beta O(X)_\gamma$-$Ker(\{x\}) \subseteq \beta O(X)_\gamma$-$Cl(\{x\})$. This shows that $\beta O(X)_\gamma$-$Cl(\{x\}) = \beta O(X)_\gamma$-$Ker(\{x\})$.
(2) $\Rightarrow$ (1). Follows from Theorem 5.12.

Theorem 5.14 *For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:*

1. (X, τ) *is a* β-γ-R_0 *space.*

2. *If* F *is* β-γ-*closed, then* $F = \beta O(X)_\gamma$-$Ker(F)$.

3. *If* F *is* β-γ-*closed and* $x \in F$, *then* $\beta O(X)_\gamma$-$Ker(\{x\}) \subseteq F$.

4. *If* $x \in X$, *then* $\beta O(X)_\gamma$-$Ker(\{x\}) \subseteq \beta O(X)_\gamma$-$Cl(\{x\})$.

Proof. (1) $\Rightarrow$ (2). Let F be a β-γ-closed and $x \notin F$. Thus $(X \setminus F)$ is a β-γ-open set containing x. Since (X, τ) is β-γ-R_0, $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq (X \setminus F)$. Thus $\beta O(X)_\gamma$-$Cl(\{x\}) \cap F = \phi$ and by Theorem 5.3, $x \notin \beta O(X)_\gamma$-$Ker(F)$. Therefore $\beta O(X)_\gamma$-$Ker(F) = F$.
(2) $\Rightarrow$ (3). In general, $A \subseteq B$ implies $\beta O(X)_\gamma$-$Ker(A) \subseteq \beta O(X)_\gamma$-$Ker(B)$. Therefore, it follows from (2), that $\beta O(X)_\gamma$-$Ker(\{x\}) \subseteq \beta O(X)_\gamma$-$Ker(F) = F$.
(3) $\Rightarrow$ (4). Since $x \in \beta O(X)_\gamma$-$Cl(\{x\})$ and $\beta O(X)_\gamma$-$Cl(\{x\})$ is β-γ-closed, by (3), $\beta O(X)_\gamma$-$Ker(\{x\})$
$\subseteq \beta O(X)_\gamma$-$Cl(\{x\})$.
(4) $\Rightarrow$ (1). We show the implication by using Theorem 5.7. Let $x \in \beta O(X)_\gamma$-$Cl(\{y\})$. Then by Lemma 5.2, $y \in \beta O(X)_\gamma$-$Ker(\{x\})$. Since $x \in \beta O(X)_\gamma$-$Cl(\{x\})$ and $\beta O(X)_\gamma$-$Cl(\{x\})$ is β-γ-closed, by (4), we obtain $y \in \beta O(X)_\gamma$-$Ker(\{x\}) \subseteq \beta O(X)_\gamma$-$Cl(\{x\})$. Therefore $x \in \beta O(X)_\gamma$-$Cl(\{y\})$ implies $y \in \beta O(X)_\gamma$-$Cl(\{x\})$. The converse is obvious and (X, τ) is β-γ-R_0.

Definition 5.15 *A topological space (X, τ) with an operation γ on τ, is said to be β-γ-R_1 if for x, y in X with $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$, there exist disjoint β-γ-open sets U and V such that $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq U$ and $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq V$.*

Theorem 5.16 *For a topological space (X, τ) with an operation γ on τ, the following statements are equivalent:*

1. (X, τ) *is* β-γ-R_1.

2. *If* $x, y \in X$ *such that* $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$, *then there exist* β-γ-*closed sets* F_1 *and* F_2 *such that* $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ *and* $X = F_1 \cup F_2$.

Proof. Obvious.

Theorem 5.17 *If (X, τ) is β-γ-R_1, then (X, τ) is β-γ-R_0.*

Proof. Let U be β-γ-open such that $x \in U$. If $y \notin U$, since $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$, we have $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. So, there exists a β-γ-open set V such that $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq V$ and $x \notin V$, which implies $y \notin \beta O(X)_\gamma$-$Cl(\{x\})$. Hence $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq U$. Therefore, (X, τ) is β-γ-R_0.

The converse of the above Theorem need not be ture in general as shown in the following example.

Example 5.18 *Consider* $X = \{a, b, c\}$ *with the discrete topology on* X. *Define an operation* γ *on* $\beta O(X)$ *by*

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \\ X & \text{otherwise} \end{cases}$$

Then X *is a* β-γ-R_0 *space but not a* β-γ-R_1 *space.*

Corollary 5.19 *A topological space* (X, τ) *with an operation* γ *on* τ *is* β-γ-R_1 *if and only if for* $x, y \in X$, $\beta O(X)_\gamma$-$Ker(\{x\}) \neq \beta O(X)_\gamma$-$Ker(\{y\})$, *there exist disjoint* β-γ-*open sets* U *and* V *such that* $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq U$ *and* $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq V$.

Proof. Follows from Theorem 5.8.

Theorem 5.20 *A topological space* (X, τ) *is* β-γ-R_1 *if and only if* $x \in X \setminus \beta O(X)_\gamma$-$Cl(\{y\})$ *implies that* x *and* y *have disjoint* β-γ-*nbds.*

Proof. Necessity. Let $x \in X \setminus \beta O(X)_\gamma$-$Cl(\{y\})$. Then $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$, so, x and y have disjoint β-γ-nbds.

Sufficiency. First, we show that (X, τ) is β-γ-R_0. Let U be a β-γ-open set and $x \in U$. Suppose that $y \notin U$. Then, $\beta O(X)_\gamma$-$Cl(\{y\}) \cap U = \phi$ and $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$. There exist β-γ-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \phi$. Hence, $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq \beta O(X)_\gamma$-$Cl(U_x)$ and $\beta O(X)_\gamma$-$Cl(\{x\}) \cap U_y \subseteq \beta O(X)_\gamma$-$Cl(U_x) \cap U_y = \phi$. Therefore, $y \notin \beta O(X)_\gamma$-$Cl(\{x\})$. Consequently, $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq U$ and (X, τ) is β-γ-R_0. Next, we show that (X, τ) is β-γ-R_1. Suppose that $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. Then, we can assume that there exists $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ such that $z \notin \beta O(X)_\gamma$-$Cl(\{y\})$. There exist β-γ-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \phi$. Since $z \in \beta O(X)_\gamma$-$Cl(\{x\})$, $x \in V_z$. Since (X, τ) is β-γ-R_0, we obtain $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq V_z$, $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq V_y$ and $V_z \cap V_y = \phi$. This shows that (X, τ) is β-γ-R_1.

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