Gen. Math. Notes, Vol. 16, No. 2, June, 2013, pp.14-31
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

$g^\star b$-Separation Axioms

Alias B. Khalaf[1] **and Suzan N. Dawod**[2]

[1]Department of Mathematics, Faculty of Science
University of Duhok, Kurdistan-Region, Iraq
E-mail: aliasbkhalaf@gmail.com
[2]Department of Mathematics, Faculty of Science
University of Zakho, Kurdistan-Region, Iraq
E-mail: suzan.dawod@yahoo.com

(Received: 18-2-13 / Accepted: 1-4-13)

Abstract

In this paper, we define some new types of separation axioms in topological spaces by using $g^\star b$-open set also the concept of $g^\star b$-R_0 and $g^\star b$-R_1 are introduced. Several properties of these spaces are investigated.

Keywords: *$g^\star b$-open set, $g^\star b$-R_0, $g^\star b$-R_1, $g^\star b$-T_i(i=0,1,2).*

1 Introduction

Mashhour et al [12] introduced and investigated the notion of preopen sets and precontinuity in topological spaces. Since then many separation axioms and mappings have been studied using preopen sets. In [[1], [9]], weak pre-separation axioms, namely, pre-T_0, pre-T_1 and pre-T_2 are introduced and studied. Further, the notion of preopen sets are used to introduce some more pre-separation axioms called pre-R_0, pre-R_1 spaces. Caldas and Jafari [3], introduced and studied $b-T_0$, $b-T_1$, $b-T_2$, $b-D_0$, $b-D_1$ and $b-D_2$ via b-open sets after that Keskin and Noiri [10], introduced the notion of $b-T_{\frac{1}{2}}$. The aim of this paper is to introduce new types of separation axiom via $g^\star b$-open sets, and investigate the relations among these concepts.
Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, $Cl(A)$ and $Int(A)$ represents the closure of A and Interior of A respectively. A subset A is said

to be preopen set [12] if $A \subseteq IntCl(A)$, b-open [2] or(γ-open) [6] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$. The family of all b-open sets in (X, τ) is denoted by $bO(X, \tau)$.

2 Preliminaries

Definition 2.1 *A subset A of a topological space (X, τ) is called:*

1. *generalized closed set (briefly g-closed) [11], if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*

2. *$g^{\star}b$-closed [**?**], if $bCl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.*

Definition 2.2 *[3] A subset A of a topological space X is called a bdifference set (briefly, bD-set) if there are $U, V \in bO(X, \tau)$ such that $U \neq X$ and $A = U \setminus V$.*

Definition 2.3 *[3] A space X is said to be:*

1. *$b - T_0$ if for each pair of distinct points x and y in X, there exists a b-open set A containing x but not y or a b-open set B containing y but not x.*

2. *$b - T_1$ if for each pair x; y in X, $x \neq y$, there exists a b-open set G containing x but not y and a b-open set B containing y but not x.*

3. *b-D_0 (resp., b-D_1) if for any pair of distinct points x and y of X there exists a bD-set of X containing x but not y or (resp., and) a bD-set of X containing y but not x.*

4. *$b - D_2$ if for any pair of distinct points x and y of X, there exist disjoint bD-sets G and H of X containing x and y, respectively.*

Definition 2.4 *[13] A space X is said to be $b - T_2$ if for any pair of distinct points x and y in X, there exist $U \in BO(X, x)$ and $V \in BO(X, y)$ such that $U \cap V = \phi$.*

Definition 2.5 *[10] A topological space X is called $b - T_{\frac{1}{2}}$ if every gb-closed set is b-closed.*

Definition 2.6 *[8] Let X be a topological space. A subset $S \subseteq X$ is called a pre-difference set (briefly pD-set), if there are two preopen sets A_1, A_2 in X such that $A_1 \neq X$ and $B = A_1 \setminus A_2$.*

Definition 2.7 ([1], [9]) *A space X is said to be:*

1. *pre-T_0 if for each pair of distinct points x, y of X, there exists a preopen set containing one but not the other.*

2. *pre-T_1 if for each pair of distinct points x, y of X, there exist a pair of preopen sets, one containing x but not y, and the other containing y but not x.*

3. *pre-T_2 if for each pair of distinct points x, y of X, there exist a pair of disjoint preopen sets, one containing x and the other containing y.*

Definition 2.8 *[8] A topological space X is said to be pre-D_0 (resp., pre-D_1) if for $x, y \in X$ with $x \neq y$, there exists an pD-set of X containing x but not y or (resp., and) an pD-set containing y but not x.*

Definition 2.9 *[8] A topological space X is said to be pre-D_2 if for each $x, y \in X$ and $x \neq y$, there exist disjoint pD-sets S_1 and S_2 such that $x \in S_1$ and $y \in S_2$.*

Definition 2.10 *[7] A space X is said to be:*

1. *pre-R_0 if for each preopen set G and $x \in G$ implies $Clx \subseteq G$.*

2. *pre-R_1 if for $x, y \in X$ with $Clx \neq Cly$, there exist disjoint preopen sets U and V such that $Clx \subseteq U$ and $Cly \subseteq V$.*

Definition 2.11 ([4], [5]) 1. *A topological space (X, τ) is called b-R_0 (or $\gamma - R_0$) if every b-open set contains the b-closure of each of its singletons.*

2. *A topological space (X, τ) is called b-R_1 (or $\gamma - R_1$) if for every x and y in X with $bCl(\{x\}) \neq bCl(\{y\})$, there exist disjoint b-open sets U and V such that $bCl(\{x\}) \subseteq U$ and $bCl(\{y\}) \subseteq V$.*

3 $g^\star b$-T_k Space $(k = 0, \frac{1}{2}, 1, 2)$

In this section, some new types of separation axioms are defined and studied in topological spaces called $g^\star b$-T_k for $k = 0, \frac{1}{2}, 1, 2$ and $g^\star b$-D_k for $k = 0, 1, 2$, and also some properties of these spaces are explained.

The following definitions are introduced via $g^\star b$-open sets.

Definition 3.1 *A topological space (X, τ) is said to be:*

1. *$g^\star b$-T_0 if for each pair of distinct points x, y in X, there exists a $g^\star b$-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.*

2. *$g^\star b$-T_1 if for each pair of distinct points x, y in X, there exist two $g^\star b$-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*

3. $g^\star b$-T_2 *if for each distinct points* x, y *in* X*, there exist two disjoint* $g^\star b$-*open sets* U *and* V *containing* x *and* y *respectively.*

4. $g^\star b$-$T_{\frac{1}{2}}$ *if every* $g^\star b$*-closed set is* g*-closed.*

5. $g^\star b$*-space if every* $g^\star b$*-open set of* X *is open in* X*.*

The following result can be simply obtained from the definitions.

Proposition 3.2 *For a topological space* (X, τ)*, the following properties hold:*

1. *If* (X, τ) *is* b-T_k*, then it is* $g^\star b$-T_k*, for* $k = 0, \frac{1}{2}, 1, 2$*.*

2. *If* (X, τ) *is Pre-*T_k*, then it is* $g^\star b$-T_k*, for* $k = 0, 1, 2$*.*

The converse of Proposition 3.2 is not true in general as it is shown in the following examples.

Example 3.3 *Consider* $X = \{a, b, c\}$ *with the topology* $\tau = \{\phi, \{a\}, X\}$*. Then the space* X *is* $g^\star b$-T_k *but it is not pre-*T_k *for* $k = 1, 2$*.*

Example 3.4 *Consider* $X = \{a, b, c\}$ *with the topology* $\tau = \{\phi, \{a\}, \{a, c\}, X\}$*. Then the space* X *is* $g^\star b$-T_k *but it is not* b-T_k *for* $k = 1, 2$*.*

Proposition 3.5 *A topological space* (X, τ) *is* $g^\star b$-T_0 *if and only if for each pair of distinct points* x, y *of* X*,* $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$*.*

Proof. **Necessity.** Let (X, τ) be a $g^\star b$-T_0 space and x, y be any two distinct points of X. There exists a $g^\star b$-open set U containing x or y, say x but not y. Then $X \setminus U$ is a $g^\star b$-closed set which does not contain x but contains y. Since $g^\star bCl(\{y\})$ is the smallest $g^\star b$-closed set containing y, $g^\star bCl(\{y\}) \subseteq X \setminus U$ and therefore $x \notin g^\star bCl(\{y\})$. Consequently $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$.
Sufficiency. Suppose that $x, y \in X$, $x \neq y$ and $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$. Let z be a point of X such that $z \in g^\star bCl(\{x\})$ but $z \notin g^\star bCl(\{y\})$. We claim that $x \notin g^\star bCl(\{y\})$. For, if $x \in g^\star bCl(\{y\})$ then $g^\star bCl(\{x\}) \subseteq g^\star bCl(\{y\})$. This contradicts the fact that $z \notin g^\star bCl(\{y\})$. Consequently x belongs to the $g^\star b$-open set $X \setminus g^\star bCl(\{y\})$ to which y does not belong.

Proposition 3.6 *A topological space* (X, τ) *is* $g^\star b$-T_1 *if and only if the singletons are* $g^\star b$*-closed sets.*

Proof. Let (X, τ) be $g^\star b$-T_1 and x any point of X. Suppose $y \in X \setminus \{x\}$, then $x \neq y$ and so there exists a $g^\star b$-open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subseteq X \setminus \{x\}$, that is $X \setminus \{x\} = \cup\{U : y \in X \setminus \{x\}\}$ which is $g^\star b$-open.
Conversely, suppose $\{p\}$ is $g^\star b$-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X \setminus \{x\}$. Hence $X \setminus \{x\}$ is a $g^\star b$-open set contains y but not x. Similarly $X \setminus \{y\}$ is a $g^\star b$-open set contains x but not y. Accordingly X is a $g^\star b$-T_1 space.

Proposition 3.7 *A topological space* (X,τ) *is* $g^{\star}b$-$T_{\frac{1}{2}}$ *if each singleton* $\{x\}$ *of* X *is either* g*-open or* g*-closed.*

Proof. Suppose $\{x\}$ is not g-closed, then it is obvious that $(X\setminus\{x\})$ is $g^{\star}b$-closed. Since (X,τ) is $g^{\star}b$-$T_{\frac{1}{2}}$, so $(X\setminus\{x\})$ is g-closed, that is $\{x\}$ is g-open.

Proposition 3.8 *The following statements are equivalent for a topological space* (X,τ)*:*

1. X *is* $g^{\star}b$-T_2.
2. *Let* $x\in X$*. For each* $y\neq x$*, there exists a* $g^{\star}b$*-open set* U *containing* x *such that* $y\notin g^{\star}bCl(U)$.
3. *For each* $x\in X$, $\cap\{g^{\star}bCl(U):U\in g^{\star}bO(X)$ *and* $x\in U\}=\{x\}$.

Proof. (1) $\Rightarrow$ (2). Since X is $g^{\star}b$-T_2, there exist disjoint $g^{\star}b$-open sets U and V containing x and y respectively. So, $U\subseteq X\setminus V$. Therefore, $g^{\star}bCl(U)\subseteq X\setminus V$. So $y\notin g^{\star}bCl(U)$.
(2) $\Rightarrow$ (3). If possible for some $y\neq x$, we have $y\in g^{\star}bCl(U)$ for every $g^{\star}b$-open set U containing x, which contradicts (2).
(3) $\Rightarrow$ (1). Let $x,y\in X$ and $x\neq y$. Then there exists a $g^{\star}b$-open set U containing x such that $y\notin g^{\star}bCl(U)$. Let $V=X\setminus g^{\star}bCl(U)$, then $y\in V$ and $x\in U$ and also $U\cap V=\phi$.

Proposition 3.9 *Let* (X,τ) *be a topological space, then the following statements are true:*

1. *Every* $g^{\star}b$-T_2 *space is* $g^{\star}b$-T_1.
2. *Every* $g^{\star}b$*-space is* $g^{\star}b$-$T_{\frac{1}{2}}$.
3. *Every* $g^{\star}b$-T_1 *space is* $g^{\star}b$-$T_{\frac{1}{2}}$.

Proof. The proof is straightforward from the definitions and proposition 3.6.

Definition 3.10 *A subset* A *of a topological space* X *is called a* $g^{\star}b$ *difference set (briefly,* $g^{*}bD$*-set) if there are* $U,V\in g^{*}bO(X,\tau)$ *such that* $U\neq X$ *and* $A=U\setminus V$.

It is true that every $g^{\star}b$-open set U different from X is a $g^{*}bD$-set if $A=U$ and $V=\phi$. So, we can observe the following.

Remark 3.11 *Every proper* $g^{\star}b$*-open set is a* $g^{*}bD$*-set. But, the converse is not true in general as the next example shows.*

Example 3.12 *Consider* $X = \{a, b, c, d\}$ *with the topology* $\tau = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. *So,* $g^*bO(X, \tau) = \{\phi, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{c, d\}, \{b, d\}, \{a, d\}, \{d\}, X\}$, *then* $U = \{a, b, d\} \neq X$ *and* $V = \{a, c, d\}$ *are* $g^\star b$*-open sets in* X *and* $A = U \setminus V = \{a, b, d\} \setminus \{a, c, d\} = \{b\}$, *then we have* $A = \{b\}$ *is a* g^*bD*-set but it is not* $g^\star b$*-open.*

Now we define another set of separation axioms called $g^\star b$-D_k, for $k = 0, 1, 2$, by using the g^*bD-sets.

Definition 3.13 *A topological space* (X, τ) *is said to be:*

1. $g^\star b$-D_0 *if for any pair of distinct points* x *and* y *of* X *there exists a* g^*bD*-set of* X *containing* x *but not* y *or a* g^*bD*-set of* X *containing* y *but not* x.

2. $g^\star b$-D_1 *if for any pair of distinct points* x *and* y *of* X *there exists a* g^*bD*-set of* X *containing* x *but not* y *and a* g^*bD*-set of* X *containing* y *but not* x.

3. $g^\star b$-D_2 *if for any pair of distinct points* x *and* y *of* X *there exist disjoint* g^*bD*-set* G *and* E *of* X *containing* x *and* y*, respectively.*

Remark 3.14 *For a topological space* (X, τ)*, the following properties hold:*

1. *If* (X, τ) *is* $g^\star b$-T_k*, then it is* $g^\star b$-D_k*, for* $k = 0, 1, 2$.

2. *If* (X, τ) *is* $g^\star b$-D_k*, then it is* $g^\star b$-D_{k-1}*, for* $k = 1, 2$.

3. *If* (X, τ) *is Pre-*D_k*, then it is* $g^\star b$-D_k*, for* $k = 0, 1, 2$.

Proof. Obvious.

Proposition 3.15 *A space* X *is* $g^\star b$-D_0 *if and only if it is* $g^\star b$-T_0.

Proof. Suppose that X is $g^\star b$-D_0. Then for each distinct pair $x, y \in X$, at least one of x, y, say x, belongs to a g^*bD-set G but $y \notin G$. Let $G = U_1 \setminus U_2$ where $U_1 \neq X$ and $U_1, U_2 \in g^*bO(X, \tau)$. Then $x \in U_1$, and for $y \notin G$ we have two cases: (a) $y \notin U_1$, (b) $y \in U_1$ and $y \in U_2$.
In case (a), $x \in U_1$ but $y \notin U_1$.
In case (b), $y \in U_2$ but $x \notin U_2$.
Thus in both the cases, we obtain that X is $g^\star b$-T_0.
Conversely, if X is $g^\star b$-T_0, by Remark 3.14 (1), X is $g^\star b$-D_0.

Proposition 3.16 *A space* X *is* $g^\star b$-D_1 *if and only if it is* $g^\star b$-D_2.

Proof. **Necessity.** Let $x, y \in X$, $x \neq y$. Then there exist g^*bD-sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where U_1, U_2, U_3 and U_4 are $g^\star b$-open sets in X. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.
(i) $x \notin U_3$. By $y \notin G_1$ we have two sub-cases:
(a) $y \notin U_1$. Since $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$, and since $y \in U_3 \setminus U_4$ we have $y \in U_3 \setminus (U_1 \cup U_4)$. Therefore $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \phi$.
(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2$, and $y \in U_2$. Therefore $(U_1 \setminus U_2) \cap U_2 = \phi$.
(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4$ and $x \in U_4$. Hence $(U_3 \setminus U_4) \cap U_4 = \phi$. Therefore X is $g^\star b$-D_2.
sufficiency. Follows from Remark 3.14 (2).

Corollary 3.17 *If (X, τ) is $g^\star b$-D_1, then it is $g^\star b$-T_0.*

Proof. Follows from Remark 3.14 (2) and Proposition 3.15.

Here is an example which shows that the converse of Corollary 3.17 is not true in general.

Example 3.18 *Consider $X = \{a, b\}$ with the topology $\tau = \{\phi, \{a\}, X\}$. Then (X, τ) is $g^\star b$-T_0, but not $g^\star b$-D_1, since there is no g^*bD-set containing b but not a.*

From Proposition 3.9, Remark 3.14, and Proposition 3.2 we obtain the following diagram of implications:

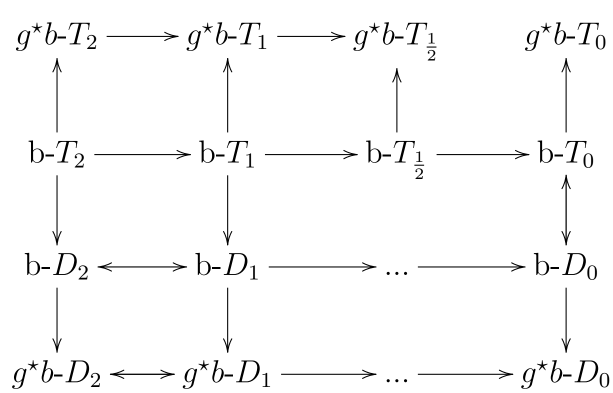

Diagram 3

The following examples show that implications in Diagram 3, are not reversible.

Example 3.19 *Consider* $X = \{a, b, c, d\}$ *with the topology* $\tau = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. *Then* (X, τ) *is* $g^{\star}b$-T_0 *but not* $g^{\star}b$-$T_{\frac{1}{2}}$.

Example 3.20 *Consider* $X = \{a, b, c\}$ *with the topology* $\tau = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. *Then* (X, τ) *is* $g^{\star}b$-$T_{\frac{1}{2}}$ *but not* $g^{\star}b$-T_1.

Remark 3.21 *From Example 3.18, it is clear that* X *is* $g^{\star}b$-D_0 *but not* $g^{\star}b$-D_1. *And from Example 3.4, the space* X *is* $g^{\star}b$-D_k *but it is not* b-D_k *for* $k = 0, 1, 2$. *In Example 3.3, the space* X *is* $g^{\star}b$-D_k *but it is not pre-*D_k *for* $k = 0, 1, 2$.

Definition 3.22 *A point* $x \in X$ *which has only* X *as the* $g^{\star}b$*-neighbourhood is called a* $g^{\star}b$*-neat point.*

Proposition 3.23 *For a* $g^{\star}b$-T_0 *topological space* (X, τ) *the following are equivalent:*

1. (X, τ) *is* $g^{\star}b$-D_1.
2. (X, τ) *has no* $g^{\star}b$*-neat point.*

Proof. (1) $\Rightarrow$ (2). Since (X, τ) is $g^{\star}b$-D_1, then each point x of X is contained in a g^*bD-set $A = U \setminus V$ and thus in U. By definition $U \neq X$. This implies that x is not a $g^{\star}b$-neat point.
(2) $\Rightarrow$ (1). If X is $g^{\star}b$-T_0, then for each distinct pair of points $x, y \in X$, at least one of them, x (say) has a $g^{\star}b$-neighbourhood U containing x and not y. Thus U which is different from X is a g^*bD-set. If X has no $g^{\star}b$-neat point, then y is not a $g^{\star}b$-neat point. This means that there exists a $g^{\star}b$-neighbourhood V of y such that $V \neq X$. Thus $y \in V \setminus U$ but not x and $V \setminus U$ is a g^*bD-set. Hence X is $g^{\star}b$-D_1.

Corollary 3.24 *A* $g^{\star}b$-T_0 *space* X *is not* $g^{\star}b$-D_1 *if and only if there is a unique* $g^{\star}b$*-neat point in* X.

Proof. We only prove the uniqueness of the $g^{\star}b$-neat point. If x and y are two $g^{\star}b$-neat points in X, then since X is $g^{\star}b$-T_0, at least one of x and y, say x, has a $g^{\star}b$-neighbourhood U containing x but not y. Hence $U \neq X$. Therefore x is not a $g^{\star}b$-neat point which is a contradiction.

Definition 3.25 *A topological space* (X, τ) *is said to be* $g^{\star}b$*-symmetric if for* x *and* y *in* X, $x \in g^{\star}bCl(\{y\})$ *implies* $y \in g^{\star}bCl(\{x\})$.

Proposition 3.26 *If* (X, τ) *is a topological space, then the following are equivalent:*

1. (X,τ) *is a* $g^{\star}b$*-symmetric space.*
2. $\{x\}$ *is* $g^{\star}b$*-closed, for each* $x \in X$.

Proof. (1) $\Rightarrow$ (2). Assume that $\{x\} \subseteq U \in g^{\star}bO(X)$, but $g^{\star}bCl(\{x\}) \nsubseteq U$. Then $g^{\star}bCl(\{x\}) \cap X \setminus U \neq \phi$. Now, we take $y \in g^{\star}bCl(\{x\}) \cap X \setminus U$, then by hypothesis $x \in g^{\star}bCl(\{y\}) \subseteq X \setminus U$ and $x \notin U$, which is a contradiction. Therefore $\{x\}$ is $g^{\star}b$-closed, for each $x \in X$.
(2) $\Rightarrow$ (1). Assume that $x \in g^{\star}bCl(\{y\})$, but $y \notin g^{\star}bCl(\{x\})$. Then $\{y\} \subseteq X \setminus g^{\star}bCl(\{x\})$ and hence $g^{\star}bCl(\{y\}) \subseteq X \setminus g^{\star}bCl(\{x\})$. Therefore $x \in X \setminus g^{\star}bCl(\{x\})$, which is a contradiction and hence $y \in g^{\star}bCl(\{x\})$.

Corollary 3.27 *If a topological space* (X,τ) *is a* $g^{\star}b$-T_1 *space, then it is* $g^{\star}b$*-symmetric.*

Proof. In a $g^{\star}b$-T_1 space, every singleton is $g^{\star}b$-closed (Proposition 3.6) and therefore is by Proposition 3.26, (X,τ) is $g^{\star}b$-symmetric.

Corollary 3.28 *If a topological space* (X,τ) *is* $g^{\star}b$*-symmetric and* $g^{\star}b$-T_0*, then* (X,τ) *is* $g^{\star}b$-T_1.

Proof. Let $x \neq y$ and as (X,τ) is $g^{\star}b$-T_0, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in g^{\star}bO(X)$. Then $x \notin g^{\star}bCl(\{y\})$ and hence $y \notin g^{\star}bCl(\{x\})$. There exists a $g^{\star}b$-open set V such that $y \in V \subseteq X \setminus \{x\}$ and thus (X,τ) is a $g^{\star}b$-T_1 space.

Corollary 3.29 *If a topological space* (X,τ) *is* $g^{\star}b$-T_1*, then* (X,τ) *is* $g^{\star}b$*-symmetric and* $g^{\star}b$-$T_{\frac{1}{2}}$.

Proof. By Corollary 3.27 and Proposition 3.9, it is true.

Corollary 3.30 *For a* $g^{\star}b$*-symmetric space* (X,τ)*, the following are equivalent:*

1. (X,τ) *is* $g^{\star}b$-T_0.
2. (X,τ) *is* $g^{\star}b$-D_1.
3. (X,τ) *is* $g^{\star}b$-T_1.

Proof. (1) $\Rightarrow$ (3). Follows from Corollary 3.28.
(3) $\Rightarrow$ (2) $\Rightarrow$ (1). Follows from Remark 3.14 and Corollary 3.17.

Definition 3.31 *Let* A *be a subset of a topological space* (X,τ)*. The* $g^{\star}b$*-kernel of* A*, denoted by* $g^{\star}bker(A)$ *is defined to be the set*

$$g^\star bker(A) = \cap\{U \in g^\star bO(X) \colon A \subseteq U\}.$$

Proposition 3.32 *Let (X, τ) be a topological space and $x \in X$. Then $y \in g^\star bker(\{x\})$ if and only if $x \in g^\star bCl(\{y\})$.*

Proof. Suppose that $y \notin g^\star bker(\{x\})$. Then there exists a $g^\star b$-open set V containing x such that $y \notin V$. Therefore, we have $x \notin g^\star bCl(\{y\})$. The proof of the converse case can be done similarly.

Proposition 3.33 *Let (X, τ) be a topological space and A be a subset of X. Then, $g^\star bker(A) = \{x \in X \colon g^\star bCl(\{x\}) \cap A \neq \phi\}$.*

Proof. Let $x \in g^\star bker(A)$ and suppose $g^\star bCl(\{x\}) \cap A = \phi$. Hence $x \notin X \setminus g^\star bCl(\{x\})$ which is a $g^\star b$-open set containing A. This is impossible, since $x \in g^\star bker(A)$. Consequently, $g^\star bCl(\{x\}) \cap A \neq \phi$. Next, let $x \in X$ such that $g^\star bCl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin g^\star bker(A)$. Then, there exists a $g^\star b$-open set V containing A and $x \notin V$. Let $y \in g^\star bCl(\{x\}) \cap A$. Hence, V is a $g^\star b$-neighbourhood of y which does not contain x. By this contradiction $x \in g^\star bker(A)$ and the claim.

Proposition 3.34 *The following properties hold for the subsets A, B of a topological space (X, τ):*

1. *$A \subseteq g^\star bker(A)$.*

2. *$A \subseteq B$ implies that $g^\star bker(A) \subseteq g^\star bker(B)$.*

3. *If A is $g^\star b$-open in (X, τ), then $A = g^\star bker(A)$.*

4. *$g^\star bker(g^\star bker(A)) = g^\star bker(A)$.*

Proof. (1), (2) and (3) are immediate consequences of Definition 3.31. To prove (4), first observe that by (1) and (2), we have $g^\star bker(A) \subseteq g^\star bker(g^\star bker(A))$. If $x \notin g^\star bker(A)$, then there exists $U \in g^* bO(X, \tau)$ such that $A \subseteq U$ and $x \notin U$. Hence $g^\star bker(A) \subseteq U$, and so we have $x \notin g^\star bker(g^\star bker(A))$. Thus $g^\star bker(g^\star bker(A)) = g^\star bker(A)$.

Proposition 3.35 *If a singleton $\{x\}$ is a $g^* bD$-set of (X, τ), then $g^\star bker(\{x\}) \neq X$.*

Proof. Since $\{x\}$ is a $g^* bD$-set of (X, τ), then there exist two subsets $U_1, U_2 \in g^* bO(X, \tau)$ such that $\{x\} = U_1 \setminus U_2$, $\{x\} \subseteq U_1$ and $U_1 \neq X$. Thus, we have that $g^\star bker(\{x\}) \subseteq U_1 \neq X$ and so $g^\star bker(\{x\}) \neq X$.

4 $g^{\star}b$-R_k Space $(k = 0, 1)$

In this section, new classes of topological spaces called $g^{\star}b$-R_0 and $g^{\star}b$-R_1 spaces are introduced.

Definition 4.1 *A topological space (X, τ) is said to be $g^{\star}b$-R_0 if U is a $g^{\star}b$-open set and $x \in U$ then $g^{\star}bCl(\{x\}) \subseteq U$.*

Proposition 4.2 *For a topological space (X, τ) the following properties are equivalent:*

1. *(X, τ) is $g^{\star}b$-R_0.*
2. *For any $F \in g^{\star}bC(X)$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in g^{\star}bO(X)$.*
3. *For any $F \in g^{\star}bC(X)$, $x \notin F$ implies $F \cap g^{\star}bCl(\{x\}) = \phi$.*
4. *For any distinct points x and y of X, either $g^{\star}bCl(\{x\}) = g^{\star}bCl(\{y\})$ or $g^{\star}bCl(\{x\}) \cap g^{\star}bCl(\{y\}) = \phi$.*

Proof. (1) $\Rightarrow$ (2). Let $F \in g^{\star}bC(X)$ and $x \notin F$. Then by (1), $g^{\star}bCl(\{x\}) \subseteq X \setminus F$. Set $U = X \setminus g^{\star}bCl(\{x\})$, then U is a $g^{\star}b$-open set such that $F \subseteq U$ and $x \notin U$.
(2) $\Rightarrow$ (3). Let $F \in g^{\star}bC(X)$ and $x \notin F$. There exists $U \in g^{\star}bO(X)$ such that $F \subseteq U$ and $x \notin U$. Since $U \in g^{\star}bO(X)$, $U \cap g^{\star}bCl(\{x\}) = \phi$ and $F \cap g^{\star}bCl(\{x\}) = \phi$.
(3) $\Rightarrow$ (4). Suppose that $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$ for distinct points $x, y \in X$. There exists $z \in g^{\star}bCl(\{x\})$ such that $z \notin g^{\star}bCl(\{y\})$ (or $z \in g^{\star}bCl(\{y\})$ such that $z \notin g^{\star}bCl(\{x\})$). There exists $V \in g^{\star}bO(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin g^{\star}bCl(\{y\})$. By (3), we obtain $g^{\star}bCl(\{x\}) \cap g^{\star}bCl(\{y\}) = \phi$.
(4) $\Rightarrow$ (1). let $V \in g^{\star}bO(X)$ and $x \in V$. For each $y \notin V$, $x \neq y$ and $x \notin g^{\star}bCl(\{y\})$. This shows that $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$. By (4), $g^{\star}bCl(\{x\}) \cap g^{\star}bCl(\{y\}) = \phi$ for each $y \in X \setminus V$ and hence $g^{\star}bCl(\{x\}) \cap (\bigcup_{y \in X \setminus V} g^{\star}bCl(\{y\})) = \phi$. On other hand, since $V \in g^{\star}bO(X)$ and $y \in X \setminus V$, we have $g^{\star}bCl(\{y\}) \subseteq X \setminus V$ and hence $X \setminus V = \bigcup_{y \in X \setminus V} g^{\star}bCl(\{y\})$. Therefore, we obtain $(X \setminus V) \cap g^{\star}bCl(\{x\}) = \phi$ and $g^{\star}bCl(\{x\}) \subseteq V$. This shows that (X, τ) is a $g^{\star}b$-R_0 space.

Remark 4.3 *Every pre-R_0 and b-R_0 spaces is $g^{\star}b$-R_0 space but converse is not true in general.*

Example 4.4 $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$, *is $g^{\star}b$-R_0 but not pre-R_0 and b-R_0, since for preopen (b-open) set $\{a\}$, $a \in \{a\}$, then $Cl\{a\}(bCl\{a\}) = X \not\subseteq \{a\}$*

Proposition 4.5 *If a topological space* (X, τ) *is* $g^{\star}b$-T_0 *and a* $g^{\star}b$-R_0 *space then it is* $g^{\star}b$-T_1.

Proof. Let x and y be any distinct points of X. Since X is $g^{\star}b$-T_0, there exists a $g^{\star}b$-open set U such that $x \in U$ and $y \notin U$. As $x \in U$ implies that $g^{\star}bCl(\{x\}) \subseteq U$. Since $y \notin U$, so $y \notin g^{\star}bCl(\{x\})$. Hence $y \in V = X \setminus g^{\star}bCl(\{x\})$ and it is clear that $x \notin V$. Hence it follows that there exist $g^{\star}b$-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$. This implies that X is $g^{\star}b$-T_1.

Proposition 4.6 *For a topological space* (X, τ) *the following properties are equivalent:*

1. (X, τ) *is* $g^{\star}b$-R_0.
2. $x \in g^{\star}bCl(\{y\})$ *if and only if* $y \in g^{\star}bCl(\{x\})$, *for any points* x *and* y *in* X.

Proof. (1) $\Rightarrow$ (2). Assume that X is $g^{\star}b$-R_0. Let $x \in g^{\star}bCl(\{y\})$ and V be any $g^{\star}b$-open set such that $y \in V$. Now by hypothesis, $x \in V$. Therefore, every $g^{\star}b$-open set which contain y contains x. Hence $y \in g^{\star}bCl(\{x\})$.
(2) $\Rightarrow$ (1). Let U be a $g^{\star}b$-open set and $x \in U$. If $y \notin U$, then $x \notin g^{\star}bCl(\{y\})$ and hence $y \notin g^{\star}bCl(\{x\})$. This implies that $g^{\star}bCl(\{x\}) \subseteq U$. Hence (X, τ) is $g^{\star}b$-R_0.

From Definition 3.25 and Proposition 4.6, the notions of $g^{\star}b$-symmetric and $g^{\star}b$-R_0 are equivalent.

Proposition 4.7 *The following statements are equivalent for any points* x *and* y *in a topological space* (X, τ):

1. $g^{\star}bker(\{x\}) \neq g^{\star}bker(\{y\})$.
2. $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$.

Proof. (1) $\Rightarrow$ (2). Suppose that $g^{\star}bker(\{x\}) \neq g^{\star}bker(\{y\})$, then there exists a point z in X such that $z \in g^{\star}bker(\{x\})$ and $z \notin g^{\star}bker(\{y\})$. From $z \in g^{\star}bker(\{x\})$ it follows that $\{x\} \cap g^{\star}bCl(\{z\}) \neq \phi$ which implies $x \in g^{\star}bCl(\{z\})$. By $z \notin g^{\star}bker(\{y\})$, we have $\{y\} \cap g^{\star}bCl(\{z\}) = \phi$. Since $x \in g^{\star}bCl(\{z\})$, $g^{\star}bCl(\{x\}) \subseteq g^{\star}bCl(\{z\})$ and $\{y\} \cap g^{\star}bCl(\{x\}) = \phi$. Therefore, it follows that $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$. Now $g^{\star}bker(\{x\}) \neq g^{\star}bker(\{y\})$ implies that $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$.
(2) $\Rightarrow$ (1). Suppose that $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$. Then there exists a point z in X such that $z \in g^{\star}bCl(\{x\})$ and $z \notin g^{\star}bCl(\{y\})$. Then, there exists a $g^{\star}b$-open set containing z and therefore x but not y, namely, $y \notin g^{\star}bker(\{x\})$ and thus $g^{\star}bker(\{x\}) \neq g^{\star}bker(\{y\})$.

Proposition 4.8 *Let (X, τ) be a topological space. Then $\cap\{g^\star bCl(\{x\}) : x \in X\} = \phi$ if and only if $g^\star bker(\{x\}) \neq X$ for every $x \in X$.*

Proof. **Necessity.** Suppose that $\cap\{g^\star bCl(\{x\}) : x \in X\} = \phi$. Assume that there is a point y in X such that $g^\star bker(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every $g^\star b$-open set V containing y and hence $y \in g^\star bCl(\{x\})$ for any $x \in X$. This implies that $y \in \cap\{g^\star bCl(\{x\}) : x \in X\}$. But this is a contradiction.
Sufficiency. Assume that $g^\star bker(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap\{g^\star bCl(\{x\}) : x \in X\}$, then every $g^\star b$-open set containing y must contain every point of X. This implies that the space X is the unique $g^\star b$-open set containing y. Hence $g^\star bker(\{y\}) = X$ which is a contradiction. Therefore, $\cap\{g^\star bCl(\{x\}) : x \in X\} = \phi$.

Proposition 4.9 *A topological space (X, τ) is $g^\star b$-R_0 if and only if for every x and y in X, $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$ implies $g^\star bCl(\{x\}) \cap g^\star bCl(\{y\}) = \phi$.*

Proof. **Necessity.** Suppose that (X, τ) is $g^\star b$-R_0 and $x, y \in X$ such that $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$. Then, there exists $z \in g^\star bCl(\{x\})$ such that $z \notin g^\star bCl(\{y\})$ (or $z \in g^\star bCl(\{y\})$ such that $z \notin g^\star bCl(\{x\})$). There exists $V \in g^\star bO(X)$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin g^\star bCl(\{y\})$. Thus $x \in [X \setminus g^\star bCl(\{y\})] \in g^\star bO(X)$, which implies $g^\star bCl(\{x\}) \subseteq [X \setminus g^\star bCl(\{y\})]$ and $g^\star bCl(\{x\}) \cap g^\star bCl(\{y\}) = \phi$.
Sufficiency. Let $V \in g^\star bO(X)$ and let $x \in V$. We still show that $g^\star bCl(\{x\}) \subseteq V$. Let $y \notin V$, that is $y \in X \setminus V$. Then $x \neq y$ and $x \notin g^\star bCl(\{y\})$. This shows that $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$. By assumption, $g^\star bCl(\{x\}) \cap g^\star bCl(\{y\}) = \phi$. Hence $y \notin g^\star bCl(\{x\})$ and therefore $g^\star bCl(\{x\}) \subseteq V$.

Proposition 4.10 *A topological space (X, τ) is $g^\star b$-R_0 if and only if for any points x and y in X, $g^\star bker(\{x\}) \neq g^\star bker(\{y\})$ implies $g^\star bker(\{x\}) \cap g^\star bker(\{y\}) = \phi$.*

Proof. Suppose that (X, τ) is a $g^\star b$-R_0 space. Thus by Proposition 4.7, for any points x and y in X if $g^\star bker(\{x\}) \neq g^\star bker(\{y\})$ then $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$. Now we prove that $g^\star bker(\{x\}) \cap g^\star bker(\{y\}) = \phi$. Assume that $z \in g^\star bker(\{x\}) \cap g^\star bker(\{y\})$. By $z \in g^\star bker(\{x\})$ and Proposition 3.32, it follows that $x \in g^\star bCl(\{z\})$. Since $x \in g^\star bCl(\{x\})$, by Proposition 4.2, $g^\star bCl(\{x\}) = g^\star bCl(\{z\})$. Similarly, we have $g^\star bCl(\{y\}) = g^\star bCl(\{z\}) = g^\star bCl(\{x\})$. This is a contradiction. Therefore, we have $g^\star bker(\{x\}) \cap g^\star bker(\{y\}) = \phi$.
Conversely, let (X, τ) be a topological space such that for any points x and y in X, $g^\star bker(\{x\}) \neq g^\star bker(\{y\})$ implies $g^\star bker(\{x\}) \cap g^\star bker(\{y\}) = \phi$. If $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$, then by Proposition 4.7, $g^\star bker(\{x\}) \neq$

$g^{\star}bker(\{y\})$. Hence, $g^{\star}bker(\{x\}) \cap g^{\star}bker(\{y\}) = \phi$ which implies $g^{\star}bCl(\{x\}) \cap g^{\star}bCl(\{y\}) = \phi$. Because $z \in g^{\star}bCl(\{x\})$ implies that $x \in g^{\star}bker(\{z\})$ and therefore $g^{\star}bker(\{x\}) \cap g^{\star}bker(\{z\}) \neq \phi$. By hypothesis, we have $g^{\star}bker(\{x\}) = g^{\star}bker(\{z\})$. Then $z \in g^{\star}bCl(\{x\}) \cap g^{\star}bCl(\{y\})$ implies that $g^{\star}bker(\{x\}) = g^{\star}bker(\{z\}) = g^{\star}bker(\{y\})$. This is a contradiction. Therefore, $g^{\star}bCl(\{x\}) \cap g^{\star}bCl(\{y\}) = \phi$ and by Proposition 4.2, (X, τ) is a $g^{\star}b$-R_0 space.

Proposition 4.11 *For a topological space (X, τ) the following properties are equivalent:*

1. *(X, τ) is a $g^{\star}b$-R_0 space.*
2. *For any non-empty set A and $G \in g^{\star}bO(X)$ such that $A \cap G \neq \phi$, there exists $F \in g^{\star}bC(X)$ such that $A \cap F \neq \phi$ and $F \subseteq G$.*
3. *For any $G \in g^{\star}bO(X)$, we have $G = \cup\{F \in g^{\star}bC(X)\colon F \subseteq G\}$.*
4. *For any $F \in g^{\star}bC(X)$, we have $F = \cap\{G \in g^{\star}bO(X)\colon F \subseteq G\}$.*
5. *For every $x \in X$, $g^{\star}bCl(\{x\}) \subseteq g^{\star}bker(\{x\})$.*

Proof. (1) $\Rightarrow$ (2). Let A be a non-empty subset of X and $G \in g^{\star}bO(X)$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in g^{\star}bO(X)$, $g^{\star}bCl(\{x\}) \subseteq G$. Set $F = g^{\star}bCl(\{x\})$, then $F \in g^{\star}bC(X)$, $F \subseteq G$ and $A \cap F \neq \phi$.
(2) $\Rightarrow$ (3). Let $G \in g^{\star}bO(X)$, then $G \supseteq \cup\{F \in g^{\star}bC(X)\colon F \subseteq G\}$. Let x be any point of G. There exists $F \in g^{\star}bC(X)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in g^{\star}bC(X)\colon F \subseteq G\}$ and hence $G = \cup\{F \in g^{\star}bC(X)\colon F \subseteq G\}$.
(3) $\Rightarrow$ (4). Obvious.
(4) $\Rightarrow$ (5). Let x be any point of X and $y \notin g^{\star}bker(\{x\})$. There exists $V \in g^{\star}bO(X)$ such that $x \in V$ and $y \notin V$, hence $g^{\star}bCl(\{y\}) \cap V = \phi$. By (4), $(\cap\{G \in g^{\star}bO(X)\colon g^{\star}bCl(\{y\}) \subseteq G\}) \cap V = \phi$ and there exists $G \in g^{\star}bO(X)$ such that $x \notin G$ and $g^{\star}bCl(\{y\}) \subseteq G$. Therefore $g^{\star}bCl(\{x\}) \cap G = \phi$ and $y \notin g^{\star}bCl(\{x\})$. Consequently, we obtain $g^{\star}bCl(\{x\}) \subseteq g^{\star}bker(\{x\})$.
(5) $\Rightarrow$ (1). Let $G \in g^{\star}bO(X)$ and $x \in G$. Let $y \in g^{\star}bker(\{x\})$, then $x \in g^{\star}bCl(\{y\})$ and $y \in G$. This implies that $g^{\star}bker(\{x\}) \subseteq G$. Therefore, we obtain $x \in g^{\star}bCl(\{x\}) \subseteq g^{\star}bker(\{x\}) \subseteq G$. This shows that (X, τ) is a $g^{\star}b$-R_0 space.

Corollary 4.12 *For a topological space (X, τ) the following properties are equivalent:*

1. *(X, τ) is a $g^{\star}b$-R_0 space.*

2. $g^\star bCl(\{x\}) = g^\star bker(\{x\})$ *for all* $x \in X$.

Proof. (1) $\Rightarrow$ (2). Suppose that (X,τ) is a $g^\star b$-R_0 space. By Proposition 4.11, $g^\star bCl(\{x\}) \subseteq g^\star bker(\{x\})$ for each $x \in X$. Let $y \in g^\star bker(\{x\})$, then $x \in g^\star bCl(\{y\})$ and by Proposition 4.2, $g^\star bCl(\{x\}) = g^\star bCl(\{y\})$. Therefore, $y \in g^\star bCl(\{x\})$ and hence $g^\star bker(\{x\}) \subseteq g^\star bCl(\{x\})$. This shows that $g^\star bCl(\{x\}) = g^\star bker(\{x\})$.
(2) $\Rightarrow$ (1). Follows from Proposition 4.11.

Proposition 4.13 *For a topological space* (X,τ) *the following properties are equivalent:*

1. (X,τ) *is a* $g^\star b$-R_0 *space.*

2. *If* F *is* $g^\star b$*-closed, then* $F = g^\star bker(F)$.

3. *If* F *is* $g^\star b$*-closed and* $x \in F$, *then* $g^\star bker(\{x\}) \subseteq F$.

4. *If* $x \in X$, *then* $g^\star bker(\{x\}) \subseteq g^\star bCl(\{x\})$.

Proof. (1) $\Rightarrow$ (2). Let F be a $g^\star b$-closed and $x \notin F$. Thus $(X \setminus F)$ is a $g^\star b$-open set containing x. Since (X,τ) is $g^\star b$-R_0, $g^\star bCl(\{x\}) \subseteq (X \setminus F)$. Thus $g^\star bCl(\{x\}) \cap F = \phi$ and by Proposition 3.33, $x \notin g^\star bker(F)$. Therefore $g^\star bker(F) = F$.
(2) $\Rightarrow$ (3). In general, $A \subseteq B$ implies $g^\star bker(A) \subseteq g^\star bker(B)$. Therefore, it follows from (2), that $g^\star bker(\{x\}) \subseteq g^\star bker(F) = F$.
(3) $\Rightarrow$ (4). Since $x \in g^\star bCl(\{x\})$ and $g^\star bCl(\{x\})$ is $g^\star b$-closed, by (3), $g^\star bker(\{x\}) \subseteq g^\star bCl(\{x\})$.
(4) $\Rightarrow$ (1). We show the implication by using Proposition 4.6. Let $x \in g^\star bCl(\{y\})$. Then by Proposition 3.32, $y \in g^\star bker(\{x\})$. Since $x \in g^\star bCl(\{x\})$ and $g^\star bCl(\{x\})$ is $g^\star b$-closed, by (4), we obtain $y \in g^\star bker(\{x\}) \subseteq g^\star bCl(\{x\})$. Therefore $x \in g^\star bCl(\{y\})$ implies $y \in g^\star bCl(\{x\})$. The converse is obvious and (X,τ) is $g^\star b$-R_0.

Definition 4.14 *A topological space* (X,τ) *is said to be* $g^\star b$-R_1 *if for* x, y *in* X *with* $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$, *there exist disjoint* $g^\star b$*-open sets* U *and* V *such that* $g^\star bCl(\{x\}) \subseteq U$ *and* $g^\star bCl(\{y\}) \subseteq V$.

Remark 4.15 *Every pre-*R_1 *and b-*R_1 *space is* $g^\star b$-R_1 *space but converse is not true in general.*

Example 4.16 $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$, *is* $g^\star b$-R_1 *but not pre-*R_1 *and b-*R_1, *since for* $b, c \in X$, $pCl\{b\} = bCl\{b\} = \{b\} \neq \{c\} = pCl\{c\} = bCl\{c\}$, *there do not exist disjoint preopen (resp. b-open) sets containing* $pCl\{b\}, bCl\{b\}$ *and* $pCl\{c\}, bCl\{c\}$ *resp.*

Proposition 4.17 *A topological space* (X, τ) *is* $g^\star b$-R_1 *if it is* $g^\star b$-T_2.

Proof. Let x and y be any points of X such that $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$. By Proposition 3.9 (1), every $g^\star b$-T_2 space is $g^\star b$-T_1. Therefore, by Proposition 3.6, $g^\star bCl(\{x\}) = \{x\}$, $g^\star bCl(\{y\}) = \{y\}$ and hence $\{x\} \neq \{y\}$. Since (X, τ) is $g^\star b$-T_2, there exist disjoint $g^\star b$-open sets U and V such that $g^\star bCl(\{x\}) = \{x\} \subseteq U$ and $g^\star bCl(\{y\}) = \{y\} \subseteq V$. This shows that (X, τ) is $g^\star b$-R_1.

Proposition 4.18 *If a topological space* (X, τ) *is* $g^\star b$*-symmetric, then the following are equivalent:*

1. (X, τ) *is* $g^\star b$-T_2.
2. (X, τ) *is* $g^\star b$-R_1 *and* $g^\star b$-T_1.
3. (X, τ) *is* $g^\star b$-R_1 *and* $g^\star b$-T_0.

Proof. Straightforward.

Proposition 4.19 *For a topological space* (X, τ) *the following statements are equivalent:*

1. (X, τ) *is* $g^\star b$-R_1.
2. *If* $x, y \in X$ *such that* $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$, *then there exist* $g^\star b$*-closed sets* F_1 *and* F_2 *such that* $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ *and* $X = F_1 \cup F_2$.

Proof. Obvious.

Proposition 4.20 *If* (X, τ) *is* $g^\star b$-R_1, *then* (X, τ) *is* $g^\star b$-R_0.

Proof. Let U be $g^\star b$-open such that $x \in U$. If $y \notin U$, since $x \notin g^\star bCl(\{y\})$, we have $g^\star bCl(\{x\}) \neq g^\star bCl(\{y\})$. So, there exists a $g^\star b$-open set V such that $g^\star bCl(\{y\}) \subseteq V$ and $x \notin V$, which implies $y \notin g^\star bCl(\{x\})$. Hence $g^\star bCl(\{x\}) \subseteq U$. Therefore, (X, τ) is $g^\star b$-R_0.

Corollary 4.21 *A topological space* (X, τ) *is* $g^\star b$-R_1 *if and only if for* $x, y \in X$, $g^\star bker(\{x\}) \neq g^\star bker(\{y\})$, *there exist disjoint* $g^\star b$*-open sets* U *and* V *such that* $g^\star bCl(\{x\}) \subseteq U$ *and* $g^\star bCl(\{y\}) \subseteq V$.

Proof. Follows from Proposition 4.7.

Proposition 4.22 *A topological space* (X, τ) *is* $g^\star b$-R_1 *if and only if* $x \in X \setminus g^\star bCl(\{y\})$ *implies that* x *and* y *have disjoint* $g^\star b$*-open neighbourhoods.*

Proof. **Necessity.** Let $x \in X \setminus g^{\star}bCl(\{y\})$. Then $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$, so, x and y have disjoint $g^{\star}b$-open neighbourhoods.
Sufficiency. First, we show that (X,τ) is $g^{\star}b$-R_0. Let U be a $g^{\star}b$-open set and $x \in U$. Suppose that $y \notin U$. Then, $g^{\star}bCl(\{y\}) \cap U = \phi$ and $x \notin g^{\star}bCl(\{y\})$. There exist $g^{\star}b$-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \phi$. Hence, $g^{\star}bCl(\{x\}) \subseteq g^{\star}bCl(U_x)$ and $g^{\star}bCl(\{x\}) \cap U_y \subseteq g^{\star}bCl(U_x) \cap U_y = \phi$. Therefore, $y \notin g^{\star}bCl(\{x\})$. Consequently, $g^{\star}bCl(\{x\}) \subseteq U$ and (X,τ) is $g^{\star}b$-R_0. Next, we show that (X,τ) is $g^{\star}b$-R_1. Suppose that $g^{\star}bCl(\{x\}) \neq g^{\star}bCl(\{y\})$. Then, we can assume that there exists $z \in g^{\star}bCl(\{x\})$ such that $z \notin g^{\star}bCl(\{y\})$. There exist $g^{\star}b$-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \phi$. Since $z \in g^{\star}bCl(\{x\})$, $x \in V_z$. Since (X,τ) is $g^{\star}b$-R_0, we obtain $g^{\star}bCl(\{x\}) \subseteq V_z$, $g^{\star}bCl(\{y\}) \subseteq V_y$ and $V_z \cap V_y = \phi$. This shows that (X,τ) is $g^{\star}b$-R_1.

References

[1] A. Chattopadhyay, Pre-T_0 and pre-T_1 topological spaces, *J. Indian Acad.Math.*, 17(2) (1995), 156-159.

[2] D. Andrijevic, On b-open set, *Mat. Vesink*, 48(1996), 59-64.

[3] M. Caldas and S. Jafari, On some applications of b-open sets in topological spaces, *Kochi J. Math.*, 2(2007), 11-19.

[4] E. Ekici, On γ-US spaces, *Indian J. Math.*, 47(2-3) (2005), 131-138.

[5] E. Ekici, On R spaces, *Int. J. Pure Appl. Math.*, 25(2) (2005), 163-172.

[6] A.A. El-Atik, A study on some types of mappings on topological spaces, *Masters Thesis*, Tanta University, Egypt, (1997).

[7] G. Navalagi, Further properties of pre-T_0, pre-T_1 and pre-T_2 spaces, *International Journal of Mathematics and Computing Applications*, 3(1-2) (2011), 67-75.

[8] S. Jafari, On a weak separation axiom, *Far East J. Math. Sci.*, 3(5) (2001), 779-787.

[9] A. Kar and P. Bhattacharyya, Some weak separation axioms, *Bull. Cal. Math. Soc.*, 82(1990), 415-422.

[10] A. Keskin and T. Noiri, On bD-sets and associated separation axioms, *Bulletin of the Iranian Math. Soc.*, 35(1) (2009), 179-198.

[11] N. Levine, Strong continuity in topology, *Amer. Math. Monthly*, 67(1960), 269.

[12] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous and week precontinuous mappings, *Proc. Math. Phys. So.*, Egypt, 53(1982), 47-53.

[13] J.H. Park, Strongly θ-b-continuous functions, *Acta Math. Hungar*, 110(4) (2006), 347-359.

[14] D. Vidhya and R. Parimelazhagan, $g^{\star}b$-closed sets in topological spaces, *Int. J. Contemp. Math. Sciences*, 7(2012), 1305 -1312.