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A Study of a New Family of Functions on the Space of Analytic Functions

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Abstract

By making use of a linear differential operator, we give some applications of the new families of analytic functions on the same space associated with quasi-Hadamard product in the unit disk U.

Keywords: *Analytic functions, Differential operator, Quasi-Hadamard product.*

1 Introduction

Let H be the class of functions analytic in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of H consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ...$ Let $A_p \subseteq H[a, n]$ denote the class of all functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, p \in \mathbb{N} = \{1, 2, ...\}$.

Let A denote the class of functions of the form $f(z) = z + a_2z^2 + a_3z^3 + ...$ or $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$.

For functions $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z+\sum_{n=2}^{\infty} b_n z^n$ the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

see [7], for $f \in A_p$ we define the operator as follows:

$$\Theta_p^0(\beta,\gamma)f(z) = f(z);$$

$$(p(\gamma+1)+\beta)\Theta_p^1(\beta,\gamma)f(z) = \beta f(z) + p(\gamma+1)(\frac{zf'(z)}{p});$$

$$\vdots$$

$$\Theta_p^m(\beta,\gamma)f(z) = D(\Theta_p^{m-1}(\beta,\gamma)).$$

This gives rise to

$$\Theta_p^m(\beta,\gamma)f(z) = z^p + \sum_{k=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^m a_k z^k, \beta,\gamma \geq 0, p \in \mathbb{N}, \quad (1)$$

which was given for $k = p+1$ in [4]. This operator generalize certain differential operators which already exist in literature as under.

- $\beta = \lambda, \gamma = 0$ we get $\Theta_p^m(m,\lambda,0)$ of Aghalary et al. differential operator [1].
- $\beta = \lambda, \gamma = 0$ and $p = 1$ we get Cho-Kim [2] and Cho-Srivastava [3] differential operator.
- $\beta = 1, \gamma = 0$ and $p = 1$ we get Uralegaddi and Somanatha differential operator [9].
- $\beta = 0, \gamma = 0$ and $p = 1$ we get Salagean differential operator [6].
- $\beta = l, \gamma = 0$ and $p = 1$ we get Kumar et al. differential operator [5] and Srivastava et al. differential operator [8].

Note that

$$(\gamma+1)z(\Theta_p^m(\beta,\gamma)f(z))' = (p(\gamma+1)+\beta)\Theta_p^{m+1}(\beta,\gamma)f(z) - \beta\Theta_p^m(\beta,\gamma)f(z).$$

Throughout this paper, we consider the functions of the form as follow

$$f(z) = a_1 z + \sum_{n=2}^{\infty} a_n z^n, (a_1 > 0, a_n \geq 0), \quad (2)$$

$$f_i(z) = a_{1,i}z + \sum_{n=2}^{\infty} a_{n,i}z^n, (a_{1,i} > 0, a_{n,i} \geq 0), \tag{3}$$

$$g(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, (b_1 > 0, b_n \geq 0), \tag{4}$$

$$g_j(z) = b_{1,j}z + \sum_{n=2}^{\infty} b_{n,j}z^n, (b_{1,j} > 0, b_{n,j} \geq 0), \tag{5}$$

be regular and univalent in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
For $0 \leq \rho < 1, 0 \leq \delta < 1$ and $\eta \geq 0$, we let $\mathfrak{U}(k, \rho, \delta, \eta)$ denote the class of functions f defined by (2) and satisfying the analytic criterion

$$\Re\{\frac{z(\Theta_p^m(\beta,\gamma))'}{(1-\rho)(\Theta_p^m(\beta,\gamma)) + \rho z(\Theta_p^m(\beta,\gamma))'} - \delta\} > \eta\{\frac{z(\Theta_p^m(\beta,\gamma))'}{(1-\rho)(\Theta_p^m(\beta,\gamma)) + \rho z(\Theta_p^m(\beta,\gamma))'} - 1\}.$$

Also let $\mathfrak{E}(k, \rho, \delta, \eta)$ denote the class of functions f defined by (2) and satisfying the analytic criterion

$$\Re\{\frac{(\Theta_p^m(\beta,\gamma))' + z(\Theta_p^m(\beta,\gamma))''}{(\Theta_p^m(\beta,\gamma))' + \rho z(\Theta_p^m(\beta,\gamma))''} - \delta\} > \eta\{\frac{(\Theta_p^m(\beta,\gamma))' + z(\Theta_p^m(\beta,\gamma))''}{(\Theta_p^m(\beta,\gamma))' + \rho z(\Theta_p^m(\beta,\gamma))''} - 1\}.$$

A function $f \in \mathfrak{U}(k, \rho, \delta, \eta)(0 \leq \rho < 1, 0 \leq \delta < 1, \eta \geq 0)$ if and only if

$$\sum_{n=2}^{\infty} (\frac{\beta + (p+n-1)(\gamma+1)}{\beta + p(\gamma+1)})^k [n(1+\eta) - (\delta+\eta)(1+n\rho-\rho)]a_{n,i}| \leq (1-\delta)|a_{1,i}|,$$

and $f \in \mathfrak{E}(k, \rho, \delta, \eta)(0 \leq \rho < 1, 0 \leq \delta < 1, \eta \geq 0)$ if and only if

$$\sum_{n=2}^{\infty} (\frac{\beta + (p+n-1)(\gamma+1)}{\beta + p(\gamma+1)})^{k+1} [n(1+\eta) - (\delta+\eta)(1+n\rho-\rho)]a_{n,i}| \leq (1-\delta)|a_{1,i}|.$$

A function f which is analytic in U belonging to the class $\mathfrak{M}_s(k, \rho, \delta, \eta)$ if and only if

$$\sum_{n=2}^{\infty} (\frac{\beta + (p+n-1)(\gamma+1)}{\beta + p(\gamma+1)})^s [n(1+\eta) - (\delta+\eta)(1+n\rho-\rho)]a_{n,i}| \leq (1-\delta)|a_{1,i}|, \tag{6}$$

where $0 \leq \rho < 1, 0 \leq \delta < 1, \eta \geq 0$ and s is any fixed nonnegative real number. For $s = k$ and $s = k+1$, it is identical to the family of functions denoted by $\mathfrak{U}(k, \rho, \delta, \eta)$ and $\mathfrak{E}(k, \rho, \delta, \eta)$ respectively. Further, for any positive integer $s > h > h-1 > ... > k+1 > k$, we have the inclusion relation

$$\mathfrak{U}(k, \rho, \delta, \eta) \subseteq \mathfrak{E}(k, \rho, \delta, \eta) \subseteq ... \subseteq \mathfrak{M}_h(k, \rho, \delta, \eta) \subseteq \mathfrak{M}_s(k, \rho, \delta, \eta).$$

The class $\mathfrak{M}_s(k, \rho, \delta, \eta)$ is nonempty for any nonnegative real number s as the functions of the form

$$f(z) = a_1 z + \sum_{n=2}^{\infty} \left(\frac{(\beta + p(\gamma+1))^s (1-\delta)}{(\beta + (p+n-1)(\gamma+1))^s [n(1+\eta) - (\delta+\eta)(1+n\rho-\rho)]} \lambda_n z^n, \right.$$

where $a_1 > 0, \lambda_n \geq 0$ and $\sum_{n=2}^{\infty} \lambda_n \leq 1$; satisfy the inequality (6).

2 Main Results

Theorem 2.1: *Let the functions f_i defined by (3) belonging to the family of functions $\mathfrak{E}(k, \rho, \delta, \eta)$ defined on space of analytic functions for all $i = 1, 2, ..., r$. Then quasi-Hadamard product of $f_1 * f_2 * ... * f_r$ belongs to the family $\mathfrak{M}_{r(k+2)-1}(n, \rho, \delta, \eta)$ on same space of analytic functions.*

Proof: Since $f_i \in \mathfrak{E}(k, \rho, \delta, \eta)$, implies

$$\sum_{n=2}^{\infty} \left(\frac{\beta + (p+n-1)(\gamma+1)}{\beta + p(\gamma+1)} \right)^{k+1} [n(1+\eta) - (\delta+\eta)(1+n\rho-\rho)] |a_{n,i}| \leq (1-\delta)|a_{1,i}|, \tag{7}$$

implies

$$|a_{n,i}| \leq \left(\frac{\beta + (p+n-1)(\gamma+1)}{\beta + p(\gamma+1)} \right)^{-k-2} |a_{1,i}|, \forall i = 1, 2, ..., r. \tag{8}$$

Using (7) as well as (8) for $i = 1, 2, ..., r-1$, implies

$$\sum_{n=2}^{\infty} \left(\frac{\beta + (p+n-1)(\gamma+1)}{\beta + p(\gamma+1)} \right)^{r(k+2)-1} [n(1+\eta) - (\delta+\eta)(1+n\rho-\rho)] \prod_{i=1}^{r-1} |a_{n,i}| \leq$$

$$\sum_{n=2}^{\infty} \left(\frac{\beta + (p+n-1)(\gamma+1)}{\beta + p(\gamma+1)} \right)^{k+1} [n(1+\eta) - (\delta+\eta)(1+n\rho-\rho)] |a_{n,r}| \prod_{i=1}^{r} |a_{1,i}| =$$

$$(1-\delta) \prod_{i=1}^{r} |a_{1,i}|.$$

Thus

$$f_1 * f_2 * ... * f_r \in \mathfrak{M}_{r(k+2)-1}(k, \rho, \delta, \eta).$$

Hence the proof is complete.

Theorem 2.2: *Let the functions f_i defined by (3) belonging to the family of functions $\mathfrak{U}(k, \rho, \delta, \eta)$ defined on space of analytic functions for all $i =$*

$1,2,...,r$. *Then quasi-Hadamard product of* $f_1 * f_2 * ... * f_r$ *belongs to the family* $\mathfrak{M}_{r(k+1)-1}(n,\rho,\delta,\eta)$ *on the same space of analytic functions.*

Proof: Using the same techniques of the proof of Theorem 2.1, we proved that

$$\sum_{n=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^{r(k+1)-1}[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)]\prod_{n=1}^{r}|a_{n,i}| \leq$$

$$\sum_{n=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^{k+1}[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)]|a_{n,r}|\prod_{i=1}^{r-1}|a_{1,i}| =$$

$$(1-\delta)\prod_{i=1}^{r}|a_{1,i}|.$$

Thus

$$f_1 * f_2 * ... * f_r \in \mathfrak{M}_{r(k+1)-1}(k,\rho,\delta,\eta).$$

Hence the proof is complete.

Theorem 2.3: *Let the functions* f_i *defined by (3) belonging to the family* $\mathfrak{E}(k,\rho,\delta,\eta)$ *of functions on space of analytic functions for all* $i = 1,2,...,r$ *and let* g_i *defined by (5) belonging to family* $\mathfrak{U}(k,\rho,\delta,\eta)$ *of functions on space of analytic functions for all* $j = 1,2,...,q$. *Then quasi-Hadamard product of* $f_1 * f_2 * ... * f_r * g_1 * g_2 * ... * g_q$ *belongs to the class* $\mathfrak{M}_{r(k+2)+q(k+1)-1}(n,\rho,\delta,\eta)$ *on the same space of analytic functions.*

Proof: Let us denote $f_1 * f_2 * ... * f_r * g_1 * g_2 * ... * g_q$ by H. Then

$$H(z) = [\prod_{i=1}^{r}|a_{1,i}|][\prod_{j=1}^{q}|b_{1,j}|]z + \sum_{n=2}^{\infty}[\prod_{i=1}^{r}|a_{n,i}|][\prod_{j=1}^{q}|b_{n,j}|]z^n.$$

Since $f_i \in \mathfrak{E}(k,\rho,\delta,\eta)$ and $g_j \in \mathfrak{U}(k,\rho,\delta,\eta)$, implies

$$\sum_{n=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^{k+1}[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)]a_{n,i}| \leq (1-\delta)|a_{1,i}|, \forall i = 1,2,...,r.$$

$$|a_{n,i}| \leq \frac{(\beta+p(\gamma+1))^{k+1}(1-\delta)|a_{1,i}|}{(\beta+(p+n-1)(\gamma+1))^{k+1}[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)]},$$

$$|a_{n,i}| \leq (\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^{-k-2}|a_{1,i}|, \forall i = 1,2,...,r. \quad (9)$$

$$|b_{n,i}| \leq (\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^{-k-1}|a_{b,i}|, \forall i = 1,2,...,q. \quad (10)$$

Also

$$\sum_{n=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^k[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)]|b_{n,j}| \leq (1-\delta)|b_{1,j}|. \tag{11}$$

Using (9), (11) and (10) for $i = 1, 2, ..., r, j = q$ and $j = 1, 2, ..., q-1$ respectively. We have (consider $t = r(k+2) + q(k+1) - 1$)

$$\sum_{n=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^t[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)][\prod_{i=1}^{r}|a_{n,i}|][\prod_{j=1}^{q}|b_{n,j}|] \leq$$

$$\sum_{n=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^k[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)]|b_{n,q}|[\prod_{i=1}^{r}|a_{1,i}|][\prod_{j=1}^{q-1}|b_{1,j}|] =$$

$$\sum_{n=2}^{\infty}(\frac{\beta+(p+n-1)(\gamma+1)}{\beta+p(\gamma+1)})^k[n(1+\eta)-(\delta+\eta)(1+n\rho-\rho)]|b_{n,q}|[\prod_{i=1}^{r}|a_{1,i}|][\prod_{j=1}^{q-1}|b_{1,j}|] \leq$$

$$(1-\delta)[\prod_{i=1}^{r}|a_{1,i}|][\prod_{j=1}^{q}|b_{1,j}|].$$

Thus

$$f_1 * f_2 * ... * f_r * g_1 * g_2 * ... * g_q \in \mathfrak{M}_{r(k+2)+q(k+1)-1}(n, \rho, \delta, \eta).$$

Hence the proof is complete.

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