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Orders of Generalized Hypersubstitutions of Type $\tau = (3)$

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Abstract

The concept of generalized hypersubstitutions was introduced by S. Leeratanavalee and K. Denecke in 2000. We used it as the tool to study strong hyperidentities and strongly solid varieties. In this paper we characterize all idempotent generalized hypersubstitutions of type $\tau = (3)$ and determine the order of each generalized hypersubstitution of this type. It turns out that the order is 1, 2, 3 or infinite.

Keywords: *generalized superposition, generalized hypersubstitution, idempotent element, cyclic subsemigroup, the order of generalized hypersubstitutions.*

1 Introduction

The order of hypersubstitutions, all idempotent elements on the monoid of all hypersubstitutions of type $\tau = (2)$ were studied by K. Denecke and Sh.L. Wismath [5] and the order of hypersubstitutions of type $\tau = (3)$ was studied by Th. Changphas [1]. In [10], W. Puninagool and S. Leeratanavalee studied similar problems for the monoid of all generalized hypersubstitutions of type $\tau = (2)$. In this paper we characterize all idempotent generalized hypersub-

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stitutions of type $\tau = (3)$ and then determine the order of each generalized hypersubstitution of type $\tau = (3)$. At first, we will give briefly the concept of generalized hypersubstitutions which was introduced by S. Leeratanavalee and K. Denecke [8]. A generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$, for simply, a generalized hypersubstitution is a mapping σ which maps each n_i-ary operation symbol of type τ to the set $W_\tau(X)$ of all terms of type τ built up by operation symbols from $\{f_i \mid i \in I\}$ where f_i is n_i-ary and variables from a countably infinite alphabet $X := \{x_1, x_2, x_3, \ldots\}$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we define at first the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, j \in I\!N, m < j$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

We extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i supposed that $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and σ_{id} is the identity element.

For more details on generalized hypersubstitutions see [8].

2 Idempotent Elements in $Hyp_G(3)$

In this section we characterize idempotent generalized hypersubstitutions of type $\tau = (3)$. We have only one ternary operation symbol, say f. The generalized hypersubstitution σ which maps f to the term t is denoted by σ_t. For any term $t \in W_{(3)}(X)$, the set of all variables occurring in t is denoted by $var(t)$. Firstly, we will recall the definition of an idempotent element.

Definition 2.1 ([6]) *For any semigroup S, an element $e \in S$ is called* idempotent *if $ee = e$. In general, by $E(S)$ we denote the set of all idempotent elements of S.*

Proposition 2.2 *An element $\sigma_t \in Hyp_G(3)$ is idempotent if and only if $\hat{\sigma}_t[t] = t$.*

Proof. Assume that σ_t is idempotent, i.e. $\sigma_t^2 = \sigma_t$. Then

$$\hat{\sigma}_t[t] = \hat{\sigma}_t[\sigma_t(f)] = \sigma_t^2(f) = \sigma_t(f) = t.$$

Conversely, let $\hat{\sigma}_t[t] = t$. We have $(\sigma_t \circ_G \sigma_t)(f) = \hat{\sigma}_t[\sigma_t(f)] = \hat{\sigma}_t[t] = t = \sigma_t(f)$. Thus $\sigma_t^2 = \sigma_t$, i.e. σ_t is idempotent. ■

Proposition 2.3 *For every $x_i \in X$, σ_{x_i} and σ_{id} are idempotent.*

Proof. Since for every $x_i \in X$, $\hat{\sigma}_{x_i}[x_i] = x_i$. By Proposition 2.2 we have σ_{x_i} is idempotent. σ_{id} is idempotent because it is a neutral element. ■

Note that for any $t \in W_{(3)}(X) \setminus X$ and $x_1, x_2, x_3 \notin var(t), \sigma_t$ is idempotent. Because there has nothing to substitute in the term $\hat{\sigma}_t[t]$. Thus $\hat{\sigma}_t[t] = t$.

Theorem 2.4 *Let $t = f(t_1, t_2, t_3) \in W_{(3)}(X)$ and $var(t) \cap X_3 \neq \emptyset$. Then σ_t is idempotent if and only if $t_i = x_i$ for all $x_i \in var(t) \cap X_3$.*

Proof. Assume that σ_t is idempotent. Then
$S^3(f(t_1,t_2,t_3), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3]) = \sigma^2_{f(t_1,t_2,t_3)}(f)$
$= \sigma_{f(t_1,t_2,t_3)}(f) = f(t_1,t_2,t_3)$. Suppose that there exists $x_i \in var(t) \cap X_3$ such that $t_i \neq x_i$. If $t_i \in X$, then $\hat{\sigma}_{f(t_1,t_2,t_3)}[t_i] = t_i \neq x_i$.
So $S^3(f(t_1,t_2,t_3), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3]) \neq f(t_1,t_2,t_3)$ and it is a contradiction. If $t_i \notin X$, then $\hat{\sigma}_{f(t_1,t_2,t_3)}[t_i] \notin X$. We obtain $op(t) = op(S^3(f(t_1,t_2,t_3), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3])) > op(t)$ where $op(t)$ denotes the number of all operation symbols occurring in t. This is a contradiction. For the converse direction, consider

$$\begin{aligned} \hat{\sigma}_t[t] &= \hat{\sigma}_{f(t_1,t_2,t_3)}[f(t_1,t_2,t_3)] \\ &= S^3(\sigma_{f(t_1,t_2,t_3)}(f), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3]). \end{aligned}$$

Since $var(t) \cap X_3 \neq \emptyset$ and $t_i = x_i$ for all $x_i \in var(t) \cap X_3$. Then after substitution in the term t we get the term t again. Thus σ_t is idempotent. ■

Let $i, j, k \in \mathbb{N}$. For convenience, we denote:

$E_0 := \{\sigma_t \mid t \in X\} \cup \{\sigma_t \mid t \in W_{(3)}(X) \setminus X \text{ and } x_1, x_2, x_3 \notin var(t)\}$,
$E_1 := \{\sigma_{f(x_1,x_2,x_2)}, \sigma_{f(x_i,x_3,x_3)}, \sigma_{f(x_3,x_j,x_3)}, \sigma_{f(x_2,x_2,x_k)} \mid i \neq 2, j, k \neq 1\}$,
$E_2 := \{\sigma_{f(x_1,x_j,x_k)} \mid j \neq 3, k \neq 2\}$,
$E_3 := \{\sigma_{f(x_i,x_2,x_k)} \mid i > 3, k \neq 1\}$,

$E_4 := \{\sigma_{f(x_i,x_j,x_k)} \mid i,j > 3, k \geq 3\}$,

$E_5 := \{\sigma_{f(x_1,x_j,t)} \mid j \notin \{2,3\}, t \notin X$ and $x_2, x_3 \notin var(t)\} \cup \{\sigma_{f(x_1,x_2,t)} \mid t \notin X$ and $x_3 \notin var(t)\} \cup \{\sigma_{f(x_i,x_2,t)} \mid i \notin \{1,3\}, t \notin X$ and $x_1, x_3 \notin var(t)\}$,

$E_6 := \{\sigma_{f(x_1,t,x_k)} \mid t \notin X, x_2, x_3 \notin var(t)$ and $k \notin \{2,3\}\} \cup \{\sigma_{f(x_1,t,x_3)} \mid t \notin X$ and $x_2 \notin var(t)\} \cup \{\sigma_{f(x_i,t,x_3)} \mid i \notin \{1,2\}, t \notin X$ and $x_1, x_2 \notin var(t)\}$,

$E_7 := \{\sigma_{f(t,x_2,x_k)} \mid t \notin X, x_1, x_3 \notin var(t)$ and $k \notin \{1,3\}\} \cup \{\sigma_{f(t,x_2,x_3)} \mid t \notin X$ and $x_1 \notin var(t)\} \cup \{\sigma_{f(t,x_j,x_3)} \mid t \notin X, x_1, x_2 \notin var(t)$ and $j \notin \{1,2\}\}$,

$E_8 := \{\sigma_{f(x_1,t_1,t_2)} \mid t_1, t_2 \notin X$ and $x_2, x_3 \notin var(t_1) \cup var(t_2)\} \cup \{\sigma_{f(t_1,x_2,t_2)} \mid t_1, t_2 \notin X$ and $x_1, x_3 \notin var(t_1) \cup var(t_2)\} \cup \{\sigma_{f(t_1,t_2,x_3)} \mid t_1, t_2 \notin X$ and $x_1, x_2 \notin var(t_1) \cup var(t_2)\}$.

By Theorem 2.4, we have

Corollary 2.5 $E(Hyp_G(3)) = E_0 \cup E_1 \cup E_2 \cup \ldots \cup E_8$.

3 Orders of Generalized Hypersubstitutions of Type $\tau = (3)$

The order of the element a in a semigroup S is defined as the order of the cyclic subsemigroup $\langle a \rangle$. The order of any generalized hypersubstitution of type $\tau = (2)$ was determined in [10]. In this section, we characterize the order of generalized hypersubstitutions of type $\tau = (3)$.

It is clearly that an element a in a semigroup S is idempotent if and only if the order of a is 1. Then we consider only the order of generalized hypersubstitutions of type $\tau = (3)$ which are not idempotent. We consider the generalized hypersubstitutions σ_t where $t = f(t_1, t_2, t_3) \in W_{(3)}(X)$ into four cases.

Case 1: t_1, t_2, t_3 are variables.

Case 2: There exists a unique $i \in \{1, 2, 3\}$ such that t_i is not a variable.

Case 3: There exists a unique $i \in \{1, 2, 3\}$ such that t_i is a variable.

Case 4: t_1, t_2, t_3 are not variables.

To determine the orders of generalized hypersubstitutions in Case 1 to Case 4 we need the definition of $vb_k(t)$, the x_k- variable count of the term t and the following proposition.

Definition 3.1 ([3]) *Let $t \in W_\tau(X_n)$ be an n-ary term. For each variable x_k, the x_k- variable count of t denoted by $vb_k(t)$ is defined inductively as follows :*

(i) $vb_k(x_k) = 1$;

(ii) if $x_k \notin var(t)$, *then* $vb_k(t) = 0$;

(iii) *if* $t = f_i(t_1, ..., t_{n_i})$ *and* $x_k \in var(t)$, *then* $vb_k(t) = \sum_{j=1}^{n_i} vb_k(t_j)$.

Proposition 3.2 ([9]) *Let* $s, t_1, ..., t_m \in W_\tau(X)$. *Then*

$$op(S^m(s, t_1, ..., t_m)) = \sum_{j=1}^{m} vb_j(s)op(t_j) + op(s).$$

We have the following propositions.

Proposition 3.3 *Let* $t = f(t_1, t_2, t_3)$ *where* $t_1, t_2, t_3 \in X$. *Let* $i, j, k \in \{1, 2, 3\}$ *and all are distinct. Then* σ_t *has order 2 if one of the following statements is satisfied.*

(i) $t_i = x_i, t_j = x_k$ *and* $t_k \in X \setminus \{x_k\}$.

(ii) $t_i = x_j, t_j = x_i$ *and* $t_k \neq x_k$.

(iii) $t_i = t_j = x_k$ *and* $t_k = x_m$ *where* $m > 3$.

(iv) $t_i = x_m, t_j = x_n$ *where* $m, n > 3$ *and* $t_k = x_l$ *for some* $l \in \{i, j\}$.

Proof. (i) Assume that $t_i = x_i, t_j = x_k$ and $t_k \in X \setminus \{x_k\}$. So $t_k \in \{x_i, x_j, x_n \mid n > 3\}$. We consider into two cases.

(a) $t_k \in \{x_i, x_n \mid n > 3\}$.

(b) $t_k = x_j$.

Case (a): Assume that $t_k \in \{x_i, x_n \mid n > 3\}$.
Since $\sigma_t^2(f) = S^3(f(t_1, t_2, t_3), \hat{\sigma}_t[t_1], \hat{\sigma}_t[t_2], \hat{\sigma}_t[t_3])$ and $\hat{\sigma}_t[t_i] = x_i, \hat{\sigma}_t[t_k] = t_k$, after substitution for the term $f(t_1, t_2, t_3)$ we obtain a new term by replacing each of the occurrences t_i, t_j and t_k by x_i, t_k and t_k, respectively. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing each of the occurrences t_i, t_j and t_k by x_i, t_k and t_k respectively. Thus $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

The proof of Case (b) is the same manner as the proof of Case (a).

(ii) Assume that $t_i = x_j, t_j = x_i$ and $t_k \neq x_k$, i.e. $t_k \in \{x_i, x_j, x_n \mid n > 3\}$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing t_i by x_i, t_j by x_j and t_k by x_j, x_i or x_n. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing t_i by x_j, t_j by x_i and t_k by x_i, x_j or x_n. So $\sigma_t^3(f) = \sigma_t(f)$. Hence σ_t has order 2.

The proofs of (iii), (iv) are the same manner as the proof of (ii). ∎

Proposition 3.4 *Let $t = f(t_1, t_2, t_3)$ where $t_1, t_2, t_3 \in X$ and $t_1 \neq t_2 \neq t_3$. Then σ_t has order 3 if there exist at least two elements $i, j \in \{1, 2, 3\}$ such that $t_i, t_j \in \{x_1, x_2, x_3\}, t_i \neq x_j, t_j \neq x_i$ and $t_k \neq x_k$ for all $k \in \{1, 2, 3\}$.*

Proof. Assume that there exist at least two elements $i, j \in \{1, 2, 3\}$ such that $t_i, t_j \in \{x_1, x_2, x_3\}, t_i \neq x_j, t_j \neq x_i$ and $t_k \neq x_k$ for all $k \in \{1, 2, 3\}$. Then $t_k \in \{x_i, x_j, x_m \mid m > 3\}$. We consider into two cases.

Case (a): $t_k \neq x_m$ where $m > 3$ and $k \in \{1, 2, 3\}$. We have t is either $f(x_2, x_3, x_1)$ or $f(x_3, x_1, x_2)$. If $t = f(x_2, x_3, x_1)$, then $\sigma_t^2(f) = S^3(f(x_2, x_3, x_1), x_2, x_3, x_1) = f(x_3, x_1, x_2), \sigma_t^3(f) = S^3(f(x_2, x_3, x_1), x_3, x_1, x_2) = f(x_1, x_2, x_3)$ and $\sigma_t^4(f) = S^3(f(x_2, x_3, x_1), x_1, x_2, x_3) = f(x_2, x_3, x_1)$. For $t = f(x_3, x_1, x_2)$, we can show in the same manner. Hence σ_t has order 3.

Case (b): There exists a unique $k \in \{1, 2, 3\}$ such that $t_k = x_m$ where $m > 3$. Then t must be one of the following forms $f(x_m, x_i, x_j), f(x_i, x_m, x_j)$ or $f(x_i, x_j, x_m)$. If $t = f(x_m, x_i, x_j)$, then

$$\sigma_t^2(f) = S^3(f(x_m, x_i, x_j), x_m, x_i, x_j) = \begin{cases} f(x_m, x_m, x_i) & ; \quad i = 1, j = 2 \\ f(x_m, x_j, x_m) & ; \quad i = 3, j = 1, \end{cases}$$

$$\begin{aligned} \sigma_t^3(f) &= \begin{cases} S^3(f(x_m, x_i, x_j), x_m, x_m, x_i) & ; \quad i = 1, j = 2 \\ S^3(f(x_m, x_i, x_j), x_m, x_j, x_m) & ; \quad i = 3, j = 1 \end{cases} \\ &= f(x_m, x_m, x_m) \end{aligned}$$

and $\sigma_t^4(f) = S^3(f(x_m, x_i, x_j), x_m, x_m, x_m) = f(x_m, x_m, x_m) = \sigma_t^3(f)$. For the other forms, we can show in the same manner. Hence σ_t has order 3. ■

Proposition 3.5 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $i \in \{1, 2, 3\}$ such that $t_i \notin X$. Let $j, k \in \{1, 2, 3\}$ and i, j, k are distinct. Then σ_t has order 2 if one of the following statements is satisfied.*

(i) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$ *and one of the following conditions is satisfied.*

(a) $t_j = x_m$ *and* $t_k \neq x_i$.

(b) $t_j = t_k = x_k$.

(c) $t_j = x_k$ *and* $t_k = x_j$.

(d) $t_j = x_j$ *and* $t_k = x_i$.

(ii) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$ *and one of the following conditions is satisfied.*

(a) $t_j = x_j$ *and* $t_k = x_j$ *or* x_m.

(b) $t_j = x_k$ *and* $t_k = x_j$.

(c) $t_j, t_k \notin X_3$.

(iii) $\emptyset \neq var(t_i) \subseteq \{x_m \mid m > 3\}$ *and one of the following conditions is satisfied.*

(a) $t_j = x_i$ *and* $t_k \neq x_j$.

(b) $t_j = x_k$ *and* $t_k = x_j$.

(c) $t_j = x_k$ *and* $t_k = x_m$.

Proof. (i) Let $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$.

(a) Assume that $t_j = x_m$ and $t_k \neq x_i$. So $t_k \in \{x_j, x_k, x_n \mid n > 3\}$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by x_m, t_j by x_m and t_k by $x_m, \hat{\sigma}_t[t_k]$ or x_n. Notice that each of variable which occurs in the term which obtained from the term t_i after substitution is only x_m. So after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by x_m, t_j by x_m and t_k by $x_m, \hat{\sigma}_t[t_k]$ or x_n. Hence $\sigma_t^3(f) = \sigma_t^2(f)$. Therefore σ_t has order 2.

The proofs of (b),(c) and (d) are the same manner as the proof of (a).

The proof of (ii) is also the same manner as the proof of (i).

(iii) Let $\emptyset \neq var(t_i) \subseteq \{x_m | m > 3\}$.

(a) Assume that $t_j = x_i$ and $t_k \neq x_j$, i.e. $t_k \in \{x_i, x_k, x_n | n > 3\}$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing t_j by $\hat{\sigma}_t[t_i], t_i$ is untouched, t_k by

$$\begin{cases} \hat{\sigma}_t[t_i] & if \quad t_k = x_i \quad , \\ \hat{\sigma}_t[t_k] & if \quad t_k = x_k \quad , \end{cases}$$

and if $t_k = x_n$ where $n > 3$, t_k is untouched. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing t_j by $\hat{\sigma}_t[t_i], t_i$ is untouched, and for t_k we have the same conclusion as in $\sigma_t^2(f)$. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

(b) Assume that $t_j = x_k$ and $t_k = x_j$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing t_j by x_j, t_i is untouched , and t_k by x_k. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing t_j by x_k, t_i is untouched, and t_k by x_j. So $\sigma_t^3(f) = \sigma_t(f)$. Hence σ_t has order 2.

The proof of (c) is the same manner as the proof of (b). ∎

Proposition 3.6 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $i \in \{1, 2, 3\}$ such that $t_i \notin X$. Let $j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. Then σ_t has order 3 if one of the following statements is satisfied.*

(i) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$ *and either*

(a) $t_j = x_k$ *and* $t_k = x_m$*, or*

(b) $t_j = x_m$ *and* $t_k = x_i$*.*

(ii) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$ *and* $t_j = x_k, t_k = x_m$*.*

(iii) $\emptyset \neq var(t_i) \subseteq \{x_m \mid m > 3\}$ *and* $t_j = x_i, t_k = x_j$*.*

Proof. (i) Let $x_j \in var(t_i) \subseteq \{x_j, x_m | m > 3\}$.

(a) Assume that $t_j = x_k$ and $t_k = x_m$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $x_k, x_m \in var(t_i)$ is untouched, t_j by x_m and t_k is untouched. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $x_m, x_m \in var(t_i)$ is untouched, t_j by x_m and t_k is untouched. Since $\sigma_t^4(f) = \hat{\sigma}_t[\sigma_t^3(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^4(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $x_m, x_m \in var(t_i)$ is untouched, t_j by x_m and t_k is untouched. So $\sigma_t^4(f) = \sigma_t^3(f)$. Hence σ_t has order 3.

The proof of (b) is the same manner as the proof of (a).

The proofs of (ii) and (iii) are the same manner as the proof of (i). ■

Proposition 3.7 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $i \in \{1, 2, 3\}$ such that $t_i \notin X$. Let $j, k \in \{1, 2, 3\}$ and i, j, k are distinct. Let $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $t_j = x_i$ and $t_k \in X$. Then σ_t has order 3 if t satisfies each of the following*

(i) x_m *where* $m > 3$ *is in the* i^{th}, j^{th} *coordinates for all subterms of the term* t_i*.*

(ii) $t_k \neq x_k$*.*

Otherwise, σ_t has infinite order.

Proof. Assume that x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_i and $t_k \neq x_k$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by x_i and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s where $var(s) = \{x_i, x_m | m > 3\}$. t_j is substituted by $\hat{\sigma}_t[t_i]$

and $var(\hat{\sigma}_t[t_i]) = \{x_m | m > 3\}$. Since $t_k \neq x_k$, i.e. $t_k \in \{x_i, x_j, x_n | n > 3\}$, t_k is substituted by

$$\begin{cases} \hat{\sigma}_t[t_i] & if \quad t_k = x_i \quad , \\ \hat{\sigma}_t[t_j] = x_i & if \quad t_k = x_j \quad , \end{cases}$$

and if $t_k = x_n$ where $n > 3$, t_k is untouched. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[\hat{\sigma}_t[t_i]]$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s' where $var(s') = \{x_m | m > 3\}$. t_j is substituted by $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s and t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$) and t_k is substituted by

$$\begin{cases} \hat{\sigma}_t[s] = \hat{\sigma}_t[t_i] & if \quad t_k = x_i \quad , \\ \hat{\sigma}_t[\hat{\sigma}_t[t_i]] & if \quad t_k = x_j \quad , \end{cases}$$

and if $t_k = x_n$ where $n > 3$, t_k is untouched. Since $\sigma_t^4(f) = \hat{\sigma}_t[\sigma_t^3(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^4(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[\hat{\sigma}_t[t_i]]$ and $x_m \in var(t_i)$ is untouched, i.e. t_i is substituted by the term s'. t_j is substituted by $\hat{\sigma}_t[s'] = \hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s', s, t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s'] = \hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$) and t_k is substituted by

$$\begin{cases} \hat{\sigma}_t[s'] = \hat{\sigma}_t[s] = \hat{\sigma}_t[t_i] & if \quad t_k = x_i \quad , \\ \hat{\sigma}_t[\hat{\sigma}_t[t_i]] & if \quad t_k = x_j \quad , \end{cases}$$

and if $t_k = x_n$ where $n > 3$, t_k is untouched. So $\sigma_t^4(f) = \sigma_t^3(f)$. Hence σ_t has order 3.

Now, let $a \in \mathbb{N}$. Since $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}, t_j = x_i$ and $t_k = x_k$. So $vb_i(t) \geq 1, vb_j(t) \geq 1, vb_k(t) = 1$. Consider

$$\begin{aligned} op(\sigma_t^{a+1}(f)) &= op(\hat{\sigma}_t[\sigma_t^a(f)]) \quad where \quad \sigma_t^a(f) = f(s_i, s_j, s_k) \\ &= \Big(vb_i(t)\ op(\hat{\sigma}_t[s_i])\Big) + \Big(vb_j(t)\ op(\hat{\sigma}_t[s_j])\Big) + \Big(vb_k(t)\ op(\hat{\sigma}_t[s_k])\Big) \\ &\quad + op(t) \\ &\geq op(\hat{\sigma}_t[s_i]) + op(\hat{\sigma}_t[s_j]) + op(\hat{\sigma}_t[s_k]) + op(t) \\ &\geq op(s_i) + op(s_j) + op(s_k) + op(t) \\ &> op(s_i) + op(s_j) + op(s_k) + 1 \qquad (since \quad op(t) > 1) \\ &= op(\sigma_t^a(f)). \end{aligned}$$

So $op(\sigma_t^{a+1}(f)) > op(\sigma_t^a(f))$ for all $a \in \mathbb{N}$. Hence σ_t has infinite order. ∎

Proposition 3.8 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $i \in \{1, 2, 3\}$ such that $t_i \notin X$. Let $j, k \in \{1, 2, 3\}$ and i, j, k are distinct. Then σ_t has infinite order if one of the following statements is satisfied.*

(i) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}, t_j = x_k$ *and* $t_k = x_i$.

(ii) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$ *and* $t_j = x_i, t_k \in X$.

The proofs of (i), (ii) are the same manner as the proof of Proposition 3.7 in case of infinite order.

Proposition 3.9 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_k$ and $i, j \in \{1, 2, 3\}$ where i, j, k are distinct. Then*

(i) σ_t *has order 2 if* $\emptyset \neq var(t_i) \subseteq \{x_k, x_m \mid m > 3\}$ *and* $x_i \in var(t_j) \subseteq \{x_i, x_k, x_m \mid m > 3\}$,

(ii) σ_t *has infinite order if* $x_j \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$ *and* $x_i \in var(t_j) \subseteq \{x_i, x_k, x_m \mid m > 3\}$.

Proof. (i) Assume that $\emptyset \neq var(t_i) \subseteq \{x_k, x_m | m > 3\}$ and $x_i \in var(t_j) \subseteq \{x_i, x_k, x_m | m > 3\}$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_k \in var(t_i)$ by x_k and $x_m \in var(t_i)$ is untouched. $x_i, x_k \in var(t_j)$ are substituted by $\hat{\sigma}_t[t_i], x_k$ and $x_m \in var(t_i)$ is untouched. t_k is substituted by x_k. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_k \in var(t_i)$ by x_k and $x_m \in var(t_i)$ is untouched. $x_i, x_k \in var(t_j)$ are substituted by $\hat{\sigma}_t[t_i], x_k$ and $x_m \in var(t_i)$ is untouched. t_k is substituted by x_k. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

The proof of (ii) is the same manner as the proof of Proposition 3.7 in case of infinite order. ■

Proposition 3.10 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_m$ where $m > 3$ and $i, j \in \{1, 2, 3\}$ where i, j, k are distinct. Then σ_t has order 2 if one of the following statements is satisfied.*

(i) $\emptyset \neq var(t_i) \subseteq \{x_m \mid m > 3\}$ *and*

- *(a)* $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ *or*
- *(b)* $x_k \in var(t_j) \subseteq \{x_k, x_m \mid m > 3\}$.

(ii) $x_k \in var(t_i) \subseteq \{x_k, x_m \mid m > 3\}$ *and* $x_k \in var(t_j) \subseteq \{x_k, x_m \mid m > 3\}$.

Proposition 3.11 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_m$ where $m > 3$ and $i, j \in \{1, 2, 3\}$ where i, j, k are distinct. Then σ_t has order 3 if $x_k \in var(t_j) \subseteq \{x_k, x_m \mid m > 3\}$ and either*

(i) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, or

(ii) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$.

The proofs of Proposition 3.10 and Proposition 3.11 are the same manner as the proof of Proposition 3.9(i).

Proposition 3.12 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ and $t_k = x_m$ where $m > 3$ and $i, j \in \{1, 2, 3\}$ where i, j, k are distinct. Then σ_t has order 2 if x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the terms t_i and t_j. And σ_t has order 3 if one of the following statements is satisfied:*

(i) x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_i.

(ii) $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ and x_m where $m > 3$ is not in the i^{th}, j^{th} coordinates for all subterms of the term t_j.

Otherwise, σ_t has infinite order.

Proof. Assume that x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the terms t_i and t_j. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ where $var(\hat{\sigma}_t[t_j]) = \{x_m | m > 3\}$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s where $var(s) = \{x_m | m > 3\}$. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[t_i]$ where $var(\hat{\sigma}_t[t_i]) = \{x_m | m > 3\}$ and $x_m \in var(t_j)$ is untouched. This means after substitution for the term t_j we obtain a term, say s' where $var(s') = \{x_m | m > 3\}$ and t_k is substituted by $\hat{\sigma}_t[t_k] = \hat{\sigma}_t[x_m] = x_m$. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[s'] = \hat{\sigma}_t[t_j]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s' and t_j are x_m where $m > 3$, so $\hat{\sigma}_t[s'] = \hat{\sigma}_t[t_j]$) and $x_m \in var(t_i)$ is untouched, i.e. t_i is substituted by the term s. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s and t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$) and $x_m \in var(t_j)$ is untouched, i.e. t_j is substituted by the term s' and t_k is substituted by $\hat{\sigma}_t[t_k] = \hat{\sigma}_t[x_m] = x_m$. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

Now, suppose that x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_i, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ and x_m where $m > 3$ is not in the i^{th}, j^{th} coordinates for all subterms of the term t_j. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[t_i]$ where $var(\hat{\sigma}_t[t_i]) = \{x_m | m > 3\}$ and $x_m \in var(t_j)$ is untouched. This means after substitution for the term t_j we obtain a term, say s' where $var(s') = \{x_m | m > 3\}$ and t_k is substituted by $\hat{\sigma}_t[t_k] = \hat{\sigma}_t[x_m] = x_m$. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[s']$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s_1 where $var(s_1) = \{x_m | m > 3\}$. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[s]$=$\hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s and t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$) and $x_m \in var(t_j)$ is untouched, i.e. t_j is substituted by the term s' and t_k is substituted by $\hat{\sigma}_t[t_k] = \hat{\sigma}_t[x_m] = x_m$. Since $\sigma_t^4(f) = \hat{\sigma}_t[\sigma_t^3(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^4(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[s']$ and $x_m \in var(t_i)$ is untouched, i.e. t_i is substituted by the term s_1. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[s_1]$=$\hat{\sigma}_t[s]$=$\hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s_1, s and t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s_1]$=$\hat{\sigma}_t[s]$=$\hat{\sigma}_t[t_i]$) and $x_m \in var(t_j)$ is untouched, i.e. t_j is substituted by the term s' and t_k is substituted by $\hat{\sigma}_t[t_k] = \hat{\sigma}_t[x_m] = x_m$. So $\sigma_t^4(f) = \sigma_t^3(f)$. Hence σ_t has order 3.

Now, let $a \in I\!N$. Since $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ and $t_k = x_m$ where $m > 3$. So $vb_i(t) \geq 1, vb_j(t) \geq 1, vb_k(t) = 0$. Consider

$$\begin{aligned}
op(\sigma_t^{a+1}(f)) &= op(\hat{\sigma}_t[\sigma_t^a(f)]) \;\; where \;\; \sigma_t^a(f) = f(s_i, s_j, s_k) \\
&= \Big(vb_i(t)\ op(\hat{\sigma}_t[s_i])\Big) + \Big(vb_j(t)\ op(\hat{\sigma}_t[s_j])\Big) + \Big(vb_k(t)\ op(\hat{\sigma}_t[s_k])\Big) \\
&\quad +op(t) \\
&= \Big(vb_i(t)\ op(\hat{\sigma}_t[s_i])\Big) + \Big(vb_j(t)\ op(\hat{\sigma}_t[s_j])\Big) + op(t) \\
&\quad (since \;\; vb_k(t) = 0) \\
&\geq op(\hat{\sigma}_t[s_i]) + op(\hat{\sigma}_t[s_j]) + op(t) \;\; (since \;\; vb_i(t) \geq 1, vb_j(t) \geq 1) \\
&\geq op(s_i) + op(s_j) + op(t) \\
&= op(s_i) + op(s_j) + op(s_k) + op(t) \\
&\quad (since \;\; s_k = x_m, so \;\; op(s_k) = 0) \\
&> op(s_i) + op(s_j) + op(s_k) + 1 \qquad (since \;\; op(t) > 1) \\
&= op(\sigma_t^a(f)).
\end{aligned}$$

So $op(\sigma_t^{a+1}(f)) > op(\sigma_t^a(f))$ for all $a \in I\!N$. Hence σ_t has infinite order. ∎

Proposition 3.13 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_m$ where $m > 3$ and $i, j \in \{1, 2, 3\}$ where i, j, k are distinct. Then σ_t has infinite order if one of the following statements is satisfied.*

(i) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$ *and* $x_i, x_k \in var(t_j) \subseteq \{x_i, x_k, x_m \mid m > 3\}$.

(ii) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$ *and* $x_i, x_k \in var(t_j) \subseteq \{x_i, x_k, x_m \mid m > 3\}$.

The proofs of (i), (ii) are the same manner as the proof of Proposition 3.12 in case of infinite order.

Proposition 3.14 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_i$ for some $i \in \{1, 2, 3\} \setminus \{k\}$ and let $j \in \{1, 2, 3\} \setminus \{i, k\}$. Then σ_t has order 2 if $\emptyset \neq var(t_i) \subseteq \{x_m \mid m > 3\}$ and one of the following conditions is satisfied.*

(i) $\emptyset \neq var(t_j) \subseteq \{x_m \mid m > 3\}$.

(ii) $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$.

Proof. (i) Assume that $\emptyset \neq var(t_j) \subseteq \{x_m | m > 3\}$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing t_k by $\hat{\sigma}_t[t_i], t_i$ and t_j are untouched. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing t_k by $\hat{\sigma}_t[t_i], t_i$ and t_j are untouched. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

The proof of (ii) is the same manner as the proof of (i). ∎

Proposition 3.15 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_i$ for some $i \in \{1, 2, 3\} \setminus \{k\}$ and let $j \in \{1, 2, 3\} \setminus \{i, k\}$. Then σ_t has order 3 if one of the following statements is satisfied.*

(i) $\emptyset \neq var(t_i) \subseteq \{x_m \mid m > 3\}$ *and either* $x_k \in var(t_j) \subseteq \{x_k, x_m \mid m > 3\}$ *or* $x_k, x_i \in var(t_j) \subseteq \{x_k, x_i, x_m \mid m > 3\}$.

(ii) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$ *and* $\emptyset \neq var(t_j) \subseteq \{x_m \mid m > 3\}$.

Proof. (i) Let $\emptyset \neq var(t_i) \subseteq \{x_m | m > 3\}$. We consider into two cases.

Case (a): $x_k \in var(t_j) \subseteq \{x_k, x_m | m > 3\}$. We have that after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_k \in var(t_j)$ by $\hat{\sigma}_t[t_k] = x_i, t_i$ and $x_m \in var(t_j)$ are untouched. This means after substitution for the term t_j we get a new term, say s. t_k is substituted by $\hat{\sigma}_t[t_i]$. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_k \in var(t_j)$ by $\hat{\sigma}_t[\hat{\sigma}_t[t_i]], t_i$ and $x_m \in var(t_j)$ are untouched. This means after substitution for the term t_j we obtain a term, say s'. t_k is substituted by $\hat{\sigma}_t[t_i]$. Since $\sigma_t^4(f) = \hat{\sigma}_t[\sigma_t^3(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^4(f)$ we obtain a new term by replacing $x_k \in var(t_j)$ by $\hat{\sigma}_t[\hat{\sigma}_t[t_i]], t_i$ and $x_m \in var(t_j)$ are untouched, i.e. t_j is substituted by term s' and t_k is substituted by $\hat{\sigma}_t[t_i]$. So $\sigma_t^4(f) = \sigma_t^3(f)$. Hence σ_t has order 3.

Case (b): $x_k, x_i \in var(t_j) \subseteq \{x_k, x_i, x_m | m > 3\}$. We can prove the same as (a) that σ_t has order 3.

The proof of (ii) is the same manner as the proof of (i) (a). ■

Proposition 3.16 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_i$ for some $i \in \{1, 2, 3\} \setminus \{k\}$ and let $j \in \{1, 2, 3\} \setminus \{i, k\}$. Let $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$. Then σ_t has order 2 if x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_i and x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_j. And σ_t has order 3 if one of the following statements is satisfied.*

(i) x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_i and x_m where $m > 3$ is not in the i^{th}, j^{th} coordinates for all subterms of the term t_j.

(ii) x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_j and x_m where $m > 3$ is not in the i^{th}, j^{th} coordinates for all subterms of the term t_i.

Otherwise, σ_t has infinite order.

Proposition 3.17 *Let $t = f(t_1, t_2, t_3)$ and there exists a unique $k \in \{1, 2, 3\}$ such that $t_k \in X$. Let $t_k = x_i$ for some $i \in \{1, 2, 3\} \setminus \{k\}$ and let $j \in \{1, 2, 3\} \setminus \{i, k\}$. Let $x_k \in var(t_i) \subseteq \{x_k, x_m \mid m > 3\}$. Then σ_t has order 3 if one of the following statements is satisfied.*

(i) x_m where $m > 3$ is in the i^{th}, k^{th} coordinates for all subterms of the term t_i, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$.

(ii) x_m *where* $m > 3$ *is in the* i^{th}, k^{th} *coordinates for all subterms of the term* t_i, $x_k \in var(t_j) \subseteq \{x_k, x_m \mid m > 3\}$.

(iii) x_m *where* $m > 3$ *is in the* i^{th}, k^{th} *coordinates for all subterms of the term* t_i, $x_i, x_k \in var(t_j) \subseteq \{x_i, x_k, x_m \mid m > 3\}$.

Otherwise, σ_t *has infinite order.*

The proofs of Proposition 3.16 and Proposition 3.17 are the same manner as the proof of Proposition 3.12.

Proposition 3.18 *Let* $t = f(t_1, t_2, t_3)$ *and there exists a unique* $k \in \{1, 2, 3\}$ *such that* $t_k \in X$. *Let* $t_k = x_i$ *for some* $i \in \{1, 2, 3\} \setminus \{k\}$ *and let* $j \in \{1, 2, 3\} \setminus \{i, k\}$. *Then* σ_t *has infinite order if one of the following statements is satisfied.*

(i) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $x_k \in var(t_j) \subseteq \{x_k, x_m \mid m > 3\}$.

(ii) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $x_i, x_k \in var(t_j) \subseteq \{x_i, x_k, x_m \mid m > 3\}$.

(iii) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$, $x_k \in var(t_j) \subseteq \{x_k, x_m \mid m > 3\}$.

(iv) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$, $x_i, x_k \in var(t_j) \subseteq \{x_i, x_k, x_m \mid m > 3\}$.

(v) $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$.

The proofs of (i)-(v) are the same manner as the proof of Proposition 3.12 in case of infinite order.

Proposition 3.19 *Let* $t = f(t_1, t_2, t_3)$ *and there exists a unique* $k \in \{1, 2, 3\}$ *such that* $t_k \in X$. *Let* $x_k \in var(t_i) \subseteq \{x_k, x_m \mid m > 3\}$, $\emptyset \neq var(t_j) \subseteq \{x_m \mid m > 3\}$, $t_k = x_i$ *for some* $i \in \{1, 2, 3\} \setminus \{k\}$ *and let* $j \in \{1, 2, 3\} \setminus \{i, k\}$. *Then* σ_t *has order 3 if* x_m *where* $m > 3$ *is in the* i^{th}, k^{th} *coordinates for all subterms of the term* t_i. *Otherwise,* σ_t *has infinite order.*

The proof of Proposition 3.19 is the same manner as the proof of Proposition 3.12.

Proposition 3.20 *Let* $t = f(t_1, t_2, t_3)$ *and there exists a unique* $k \in \{1, 2, 3\}$ *such that* $t_k \in X$. *Let* $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$, $\emptyset \neq var(t_j) \subseteq \{x_m \mid m > 3\}$, $t_k = x_i$ *for some* $i \in \{1, 2, 3\} \setminus \{k\}$ *and let* $j \in \{1, 2, 3\} \setminus \{i, k\}$. *Then* σ_t *has infinite order.*

The proof of Proposition 3.20 is the same manner as the proof of Proposition 3.12 in case of infinite order.

Proposition 3.21 *Let* $t = f(t_1, t_2, t_3)$ *where* $t_1, t_2, t_3 \notin X$. *Let* $i, j, k \in \{1, 2, 3\}$ *and all are distinct. Then* σ_t *has order 2 if one of the following statements is satisfied.*

(i) $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}, \emptyset \neq var(t_j) \subseteq \{x_m \mid m > 3\}$ *and either* $x_j \in var(t_k) \subseteq \{x_j, x_m \mid m > 3\}$ *or* $\emptyset \neq var(t_k) \subseteq \{x_m \mid m > 3\}$.

(ii) $\emptyset \neq var(t_i) \subseteq \{x_m \mid m > 3\}, \emptyset \neq var(t_j) \subseteq \{x_m \mid m > 3\}$ *and* $x_i, x_j \in var(t_k) \subseteq \{x_i, x_j, x_m \mid m > 3\}$.

Proof. (i) Let $x_j \in var(t_i) \subseteq \{x_j, x_m | m > 3\}$, $\emptyset \neq var(t_j) \subseteq \{x_m | m > 3\}$. We consider into two cases.

Case (a): $x_j \in var(t_k) \subseteq \{x_j, x_m | m > 3\}$. We have that after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s_1. t_j of the term t is untouched. $x_j \in var(t_k)$ is substituted by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_k)$ is untouched. This means after substitution for the term t_k we obtain a term, say s_2. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_i)$ is untouched, i.e. t_i is substituted by the term s_1 and t_j is untouched. $x_j \in var(t_k)$ is substituted by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_k)$ is untouched, i.e. t_k is substituted by the term s_2. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

Case (b): $\emptyset \neq var(t_k) \subseteq \{x_m | m > 3\}$. We have that after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s. t_j and t_k are untouched. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_i)$ is untouched, i.e. t_i is substituted by the term s and t_j, t_k are untouched. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

(ii) Assume that $\emptyset \neq var(t_i) \subseteq \{x_m | m > 3\}$, $\emptyset \neq var(t_j) \subseteq \{x_m | m > 3\}$ and $x_i, x_j \in var(t_k) \subseteq \{x_i, x_j, x_m | m > 3\}$. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_i, x_j \in var(t_k)$ by $\hat{\sigma}_t[t_i]$, $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_k)$ is untouched. t_i, t_j are untouched. This means after substitution for the term t_k we obtain a term, say s. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_i, x_j \in var(t_k)$ by $\hat{\sigma}_t[t_i]$, $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_k)$ is untouched, i.e. t_k is substituted by the term s. t_i, t_j are untouched. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2. ∎

Proposition 3.22 *Let $t = f(t_1, t_2, t_3)$ where $t_1, t_2, t_3 \notin X$. Let $i, j, k \in \{1, 2, 3\}$ and all are distinct. Then σ_t has order 3 if $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$ and $\emptyset \neq var(t_j) \subseteq \{x_m \mid m > 3\}$ and either $x_i \in var(t_k) \subseteq \{x_i, x_m \mid m > 3\}$ or $x_i, x_j \in var(t_k) \subseteq \{x_i, x_j, x_m \mid m > 3\}$.*

The proof of Proposition 3.22 is the same manner as the proof of Proposition 3.21.

Proposition 3.23 *Let $t = f(t_1, t_2, t_3)$ where $t_1, t_2, t_3 \notin X$. Let $i, j, k \in \{1, 2, 3\}$ and all are distinct. If $x_j, x_k \in var(t_i) \subseteq \{x_j, x_k, x_m \mid m > 3\}$ and $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ and $\emptyset \neq var(t_k) \subseteq \{x_i, x_j, x_m \mid m > 3\}$, then σ_t has infinite order.*

The proof of Proposition 3.23 is the same manner as the proof of Proposition 3.12 in case of infinite order.

Proposition 3.24 *Let $t = f(t_1, t_2, t_3)$ where $t_1, t_2, t_3 \notin X$. Let $i, j, k \in \{1, 2, 3\}$ and all are distinct. Let $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ and $\emptyset \neq var(t_k) \subseteq \{x_i, x_j, x_m \mid m > 3\}$. Then σ_t has order 2 if x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the terms t_i and t_j. And σ_t has order 3 if one of the following statements is satisfied.*

(i) x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_i and x_m where $m > 3$ is not in the i^{th}, j^{th} coordinates for all subterms of the term t_j.

(ii) x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_j and x_m where $m > 3$ is not in the i^{th}, j^{th} coordinates for all subterms of the term t_i.

Otherwise, σ_t has infinite order.

Proof. Assume that x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the terms t_i and t_j. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ where $var(\hat{\sigma}_t[t_j]) = \{x_m | m > 3\}$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s where $var(s) = \{x_m | m > 3\}$. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[t_i]$ where $var(\hat{\sigma}_t[t_i]) = \{x_m | m > 3\}$ and $x_m \in var(t_j)$ is untouched. This means after substitution for the term t_j we obtain a term, say s' where $var(s') = \{x_m | m > 3\}$ and $x_i, x_j \in var(t_k)$ are substituted by $\hat{\sigma}_t[t_i], \hat{\sigma}_t[t_j]$ where $var(\hat{\sigma}_t[t_i]) = var(\hat{\sigma}_t[t_j]) = \{x_m | m > 3\}$ and $x_m \in var(t_k)$ is untouched. This means after substitution for the term t_k we obtain a term, say s'' where $var(s'') = \{x_m | m > 3\}$. Since $\sigma_t^3(f) =$

$\hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[s']$=$\hat{\sigma}_t[t_j]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s' and t_j are x_m where $m > 3$, so $\hat{\sigma}_t[s'] = \hat{\sigma}_t[t_j]$) and $x_m \in var(t_i)$ is untouched, i.e. t_i is substituted by the term s. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[s]$=$\hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s and t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$) and $x_m \in var(t_j)$ is untouched, i.e. t_j is substituted by the term s' and $x_i, x_j \in var(t_k)$ are substituted by $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i], \hat{\sigma}_t[s'] = \hat{\sigma}_t[t_j]$ and $x_m \in var(t_k)$ is untouched, i.e. t_k is substituted by the term s''. So $\sigma_t^3(f) = \sigma_t^2(f)$. Hence σ_t has order 2.

(i) Assume that x_m where $m > 3$ is in the i^{th}, j^{th} coordinates for all subterms of the term t_i and x_m where $m > 3$ is not in the i^{th}, j^{th} coordinate for all subterms of the term t_j. Then after substitution for the term $f(t_1, t_2, t_3)$ in $\sigma_t^2(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[t_j]$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[t_i]$ where $var(\hat{\sigma}_t[t_i]) = \{x_m | m > 3\}$ and $x_m \in var(t_j)$ is untouched. This means after substitution for the term t_j we obtain a term, say s' where $var(s') = \{x_m | m > 3\}$ and $x_i, x_j \in var(t_k)$ are substituted by $\hat{\sigma}_t[t_i], \hat{\sigma}_t[t_j]$ and $x_m \in var(t_k)$ is untouched. This means after substitution for the term t_k we obtain a term, say s''. Since $\sigma_t^3(f) = \hat{\sigma}_t[\sigma_t^2(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^3(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[s']$ and $x_m \in var(t_i)$ is untouched. This means after substitution for the term t_i we obtain a term, say s_1. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[s]$=$\hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s and t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i]$) and $x_m \in var(t_j)$ is untouched, i.e. t_j is substituted by the term s' and $x_i, x_j \in var(t_k)$ are substituted by $\hat{\sigma}_t[s] = \hat{\sigma}_t[t_i], \hat{\sigma}_t[s']$ and $x_m \in var(t_k)$ is untouched. This means after substitution for the term t_k we obtain a term, say s_1''. Since $\sigma_t^4(f) = \hat{\sigma}_t[\sigma_t^3(f)]$ and from the previous substitution, after substitution for the term $t = f(t_1, t_2, t_3)$ in $\sigma_t^4(f)$ we obtain a new term by replacing $x_j \in var(t_i)$ by $\hat{\sigma}_t[s']$ and $x_m \in var(t_i)$ is untouched, i.e. t_i is substituted by the term s_1. $x_i \in var(t_j)$ is substituted by $\hat{\sigma}_t[s_1]$=$\hat{\sigma}_t[s]$=$\hat{\sigma}_t[t_i]$ (since $var(t) = \{x_i, x_j, x_m | m > 3\}$ and the i^{th}, j^{th} coordinates of the subterms of the terms s_1, s and t_i are x_m where $m > 3$, so $\hat{\sigma}_t[s_1]$=$\hat{\sigma}_t[s]$=$\hat{\sigma}_t[t_i]$) and $x_m \in var(t_j)$ is untouched, i.e. t_j is substituted by the term s' and $x_i, x_j \in var(t_k)$ are substituted by $\hat{\sigma}_t[s_1] = \hat{\sigma}_t[s] = \hat{\sigma}_t[t_i], \hat{\sigma}_t[s']$ and $x_m \in var(t_k)$ is untouched. This means after substitution for the term t_k we obtain a term, say s_1''. So $\sigma_t^4(f) = \sigma_t^3(f)$. Hence σ_t has order 3.

The proof of (ii) is the same manner as the proof of (i).

Now, let $a \in I\!N$. Since $x_j \in var(t_i) \subseteq \{x_j, x_m \mid m > 3\}$, $x_i \in var(t_j) \subseteq \{x_i, x_m \mid m > 3\}$ and $\emptyset \neq var(t_k) \subseteq \{x_i, x_j, x_m \mid m > 3\}$. So $vb_i(t) \geq$

$1, vb_j(t) \geq 1, vb_k(t) = 0$. Consider

$$\begin{aligned}
op(\sigma_t^{a+1}(f)) &= op(\hat{\sigma}_t[\sigma_t^a(f)]) \ \ where \ \ \sigma_t^a(f) = f(s_i, s_j, s_k) \\
&= \Big(vb_i(t)\ op(\hat{\sigma}_t[s_i])\Big) + \Big(vb_j(t)\ op(\hat{\sigma}_t[s_j])\Big) + \Big(vb_k(t)\ op(\hat{\sigma}_t[s_k])\Big) \\
&\quad + op(t) \\
&= \Big(vb_i(t)\ op(\hat{\sigma}_t[s_i])\Big) + \Big(vb_j(t)\ op(\hat{\sigma}_t[s_j])\Big) + op(t) \\
&\quad (since \ \ vb_k(t) = 0) \\
&\geq op(\hat{\sigma}_t[s_i]) + op(\hat{\sigma}_t[s_j]) + op(t) \ \ (since \ \ vb_i(t) \geq 1, vb_j(t) \geq 1) \\
&= 1 + \Big(op(t_i) + op(\hat{\sigma}_t[s_j])\Big) + \Big(op(t_j) + op(\hat{\sigma}_t[s_i])\Big) + op(t_k) \\
&\quad \Big(since\ op(t) = 1 + op(t_i) + op(t_j) + op(t_k)\Big) \\
&> 1 + op(s_i) + op(s_j) + op(s_k) \\
&= op(\sigma_t^a(f)).
\end{aligned}$$

So $op(\sigma_t^{a+1}(f)) > op(\sigma_t^a(f))$ for all $a \in I\!N$. Hence σ_t has infinite order. ■

Proposition 3.25 *Let $t = f(t_1, t_2, t_3)$ where $t_1, t_2, t_3 \notin X$. If for all $i \in \{1, 2, 3\}$ there exists at least x_j or x_k is in $var(t_i)$ where $j, k \in \{1, 2, 3\}, i, j, k$ are distinct and $x_i \notin var(t_i)$, then σ_t has infinite order.*

Proposition 3.26 *Let $t = f(t_1, t_2, t_3)$ where $t_1, t_2, t_3 \notin X$ and there exists a unique $i \in \{1, 2, 3\}$ such that $\emptyset \neq var(t_i) \subseteq \{x_m \mid m > 3\}$. Let $j, k \in \{1, 2, 3\}$ and i, j, k are distinct. If $x_i, x_j \in var(t_k), x_i, x_k \in var(t_j)$ and $x_j \notin var(t_j), x_k \notin var(t_k)$, then σ_t has infinite order.*

Proposition 3.27 *Let $t = f(t_1, t_2, t_3)$ and there exists at least one element $i \in \{1, 2, 3\}$ such that $t_i \notin X$ and $x_i \in var(t_i)$. Then σ_t has infinite order.*

The proofs of Proposition 3.25, 3.26 and 3.27 are the same manner as the proof of Proposition 3.24 in case of infinite order.

Theorem 3.28 *The order of any generalized hypersubstitutions of type $\tau = (3)$ is 1, 2, 3 or infinite.*

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References

[1] Th. Changphas, The order of hypersubstitutions of type $\tau = (3)$, *Algebra Colloquium*, 13(2) (2006), 307-313.

[2] K. Denecke, J. Koppitz and Sh.L. Wismath, Solid varieties of arbitrary type, *Algebra Universalis*, 48 (2002), 357-378.

[3] Denecke and Sh.L. Wismath, Complexity of terms, composition and hypersubstitution, *International Journal of Mathematics and Mathematical Sciences*, 15 (2003), 950-969.

[4] K. Denecke and Sh.L. Wismath, *Hyperidentities and Clones*, Gordon and Breach Scientific Publishers, Singapore, (2000).

[5] K. Denecke and Sh.L. Wismath, The Monoid of Hypersubstitutions of type $\tau = (2)$, *Contributions to General Algebra 10*, Klagenfurt, (1998), 109-127.

[6] J.M. Howie, *An Introduction to Semigroup Theory*, Academic Press Inc., London, (1976).

[7] J. Koppitz and K. Denecke, *M-Solid Varieties of Algebras*, Springer Science+Business Media, Inc., New York, (2006).

[8] S. Leeratanavalee and K. Denecke, Generalized hypersubstitutions and strongly solid varieties, *General Algebra and Applications, Proc. of the 59th Workshop on General Algebra, 15 th Conference for Young Algebraists Potsdam 2000*, Shaker Verlag, (2000), 135-145.

[9] W. Puninagool and S. Leeratanavalee, Complexity of term, superpositions and generalized hypersubstitutions, *Computers and Mathematics with Applications*, 59 (2010), 1038-1045.

[10] W. Puninagool and S. Leeratanavalee, The order of generalized hypersubstitutions of type $\tau = (2)$, *International Journal of Mathematics and Mathematical Sciences*, 2008 (2008), Article ID 263541, 8 pages.