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Lightlike Submanifolds of a Para-Sasakian Manifold

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Abstract

In the present paper we study lightlike submanifolds of almost paracontact metric manifolds. We define invariant lightlike submanifolds. We study radical transversal lightlike submanifolds of para-Sasakian manifolds and investigate the geometry of distributions. Also we introduce a general notion of paracontact Cauchy-Riemann (CR) lightlike submanifolds and we derive some necessary and sufficient conditions for integrability of various distributions of paracontact CR-lightlike submanifold of a para-Sasakian manifold.

Keywords: *Lightlike Submanifolds, Para-Sasakian Manifolds.*

1 Introduction

Given a semi-Riemannian manifold, one can consider its lightlike submanifold whose study is important from application point of view and difficult in the sense that the intersection of normal vector bundle and tangent bundle of these submanifolds is nonempty. This unique feature makes the study of lightlike submanifolds different from the study of non-degenerate submanifolds. The general theory of lightlike submanifolds was developed by D. N. Küpeli [2], K. L. Duggal and A. Bejancu [1]. Since then many authors have studied lightlike submanifolds of semi-Riemannian manifolds and especially indefinite Sasakian manifolds ([4], [5], [6], [7]). For differential geometry of lightlike submanifolds we refer the book [3].

The study of paracontact geometry was initiated by S. Kaneyuki and M. Konzai in [8]. The authors defined almost paracontact structure on a pseudo-Riemannian manifold M of dimension $(2n+1)$ and constructed the almost paracomplex structure on $M^{2n+1} \times R$. Recently, S. Zamkovoy [9] studied para-contact metric manifolds and some remarkable subclasses like para-Sasakian manifolds. Especially, in the recent years, many authors ([10], [11], [12], [13], [14]) have pointed out the importance of paracontact geometry, and in particular of para-Sasakian geometry, by several papers giving the relationships with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics.

These circumstances motivated us to initiate the study of lightlike geometry of almost paracontact metric manifolds. In the present paper we study the lightlike submanifolds of para-Sasakian manifolds and obtain several geometric results. The paper is organized as follows. Section 1 is devoted to some basic definitions for almost paracontact metric manifolds and lightlike submanifolds, respectively. In section 2, we define invariant lightlike submanifolds of a para-Sasakian manifold and prove that if a lightlike submanifold of a para-Sasakian manifold is totally umbilical, then it is totally geodesic and invariant. In section 3, we introduce paracontact CR-lightlike submanifolds of a para-Sasakian manifold and investigate integrability conditions for certain natural distributions arising on paracontact CR-lightlike submanifolds. Section 4 contains definition of radical transversal lightlike submanifolds of para-Sasakian manifolds and an example. It is proved that there exist no isotropic or totally lightlike radical transversal lightlike submanifold of a para-Sasakian manifold. Moreover, we obtain some necessary and sufficient conditions for the radical distrubiton and screen distrubiton of a radical transversal lightlike submanifold to be integrable.

2 Preliminaries

2.1 Almost Paracontact Metric Manifolds

A differentiable manifold $\bar{M}$ of dimension $(2n+1)$ is called almost paracontact manifold with the almost paracontact structure $(\bar{\phi}, \xi, \eta)$ if it admits a tensor field $\bar{\phi}$ of type $(1,1)$, a vector field ξ, a $1-$form η satisfying the following conditions [8]:

$$\bar{\phi}^2 = I - \eta \otimes \xi, \tag{1}$$

$$\eta(\xi) = 1, \tag{2}$$

$$\bar{\phi}\xi = 0, \tag{3}$$

$$\eta \circ \bar{\phi} = 0, \tag{4}$$

where I denotes the identity transformation. Moreover, the tensor field $\bar{\phi}$ induces an almost paracomplex structure on the paracontact distribution $D = \ker\eta$, i.e. the eigendistributions $D^{\pm}$ corresponding to the eigenvalues ± 1 of $\bar{\phi}$ are both $n-$dimensional.

If a $(2n+1)-$dimensional almost paracontact manifold $\bar{M}$ with an almost paracontact structure $(\bar{\phi}, \xi, \eta)$ admits a pseudo-Riemannian metric $\bar{g}$ such that [9]

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X, Y) + \eta(X)\eta(Y), \quad X,\, Y \in \Gamma(T\bar{M}), \tag{5}$$

then we say that $\bar{M}$ is an *almost paracontact metric manifold* with an *almost paracontact metric structure* $(\bar{\phi}, \xi, \eta, \bar{g})$ and such metric $\bar{g}$ is called *compatible metric*. Any compatible metric $\bar{g}$ is necessarily of signature $(n+1, n)$.

From (5) it can be easily seen that [9]

$$\bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y), \tag{6}$$

$$\bar{g}(X, \xi) = \eta(X), \tag{7}$$

for any $X,\, Y \in \Gamma(T\bar{M})$. The *fundamental* 2*-form* of $\bar{M}$ is defined by

$$\Phi(X, Y) = \bar{g}(X, \bar{\phi}Y).$$

An almost paracontact metric structure becomes a *paracontact metric structure* if $\bar{g}(X, \bar{\phi}Y) = d\eta(X, Y)$, for all $X,\, Y \in \Gamma(T\bar{M})$, where $d\eta(X, Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X, Y])\}$.

For a $(2n+1)-$dimensional manifold $\bar{M}$ with an almost paracontact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$, one can also construct a local orthonormal basis which is called $\bar{\phi}-$*basis* $(X_i, \bar{\phi}X_i, \xi)$ $(i = 1, 2, ..., n)$ [9].

An almost paracontact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is a *para-Sasakian manifold* if and only if [9]

$$(\bar{\nabla}_X \bar{\phi})Y = -\bar{g}(X, Y)\xi + \eta(Y)X, \quad X,\, Y \in \Gamma(T\bar{M}), \tag{8}$$

where $X,\, Y \in \Gamma(T\bar{M})$ and $\bar{\nabla}$ is a Levi-Civita connection on $\bar{M}$.

From (8), it can be seen that

$$\bar{\nabla}_X \xi = -\bar{\phi}X. \tag{9}$$

Example 2.1 *Let* $\bar{M} = R^{2n+1}$ *be the* $(2n+1)-$*dimensional real number space with standard coordinate system* $(x_1, y_1, x_2, y_2, ..., x_n, y_n, z)$*. Defining*

$$\begin{aligned}
\bar{\phi}\frac{\partial}{\partial x_\alpha} &= \frac{\partial}{\partial y_\alpha}, \qquad \bar{\phi}\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \qquad \bar{\phi}\frac{\partial}{\partial z} = 0, \\
\xi &= \frac{\partial}{\partial z}, \qquad \overline{\eta} = dz, \\
\overline{g} &= \eta \otimes \eta + \sum_{\alpha=1}^{n} dx_\alpha \otimes dx_\alpha - \sum_{\alpha=1}^{n} dy_\alpha \otimes dy_\alpha,
\end{aligned} \tag{10}$$

where $\alpha = 1, 2, ..., n$, *then the set* $(\bar{\phi}, \xi, \eta, \bar{g})$ *is an almost paracontact metric structure on* R^{2n+1}.

2.2 Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds [1].

Let $(\bar{M}, \bar{g})$ be a real $(n+m)-$dimensional semi-Riemannian manifold with index q, such that $m, n \geq 1, 1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $\bar{M}$, where g is the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a *lightlike submanifold* of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}, \tag{11}$$

is a degenerate $n-$dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad\,T_xM = T_xM \cap T_xM^{\perp}$ which is known as *radical (null) space*. If the mapping $Rad\,TM : x \in M \to RadT_xM$, defines a smooth distribution, called *radical distribution*, on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an $r-$*lightlike submanifold*.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\,TM$ in TM. This means that

$$TM = S(TM) \perp Rad\,TM, \tag{12}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad\,TM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and $Rad\,TM$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{13}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = \{Rad\,TM \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{14}$$

Theorem 2.2 *[1] Let* $(M, g, S(TM), S(TM^{\perp}))$ *be an* $r-$*lightlike submanifold of a semi-Riemannian manifold* $(\bar{M}, \bar{g})$. *Suppose* U *is a coordinate neighbourhood of* M *and* E_i, $i \in \{1, ..., r\}$ *is a basis of* $\Gamma(Rad\,TM))|_U$.*Then, there exist a complementary vector subbundle* $ltr(TM)$ *of* $Rad\,TM$ *in* $S(TM^{\perp})^{\perp}$ *and a basis* $\{N_i\}$, $i \in \{1, ..., r\}$ *of* $\Gamma(ltr(TM)\mid_U$ *such that*

$$\bar{g}(N_i, E_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{15}$$

for any $i, j \in \{1, ..., r\}$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^{\perp})$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to the decomposition (14), The Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{16}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \quad \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{17}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. $\bar{\nabla}$ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. According to (13), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (16) and (17) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{18}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{19}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{20}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, and $W \in \Gamma(S(TM^{\perp}))$, where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $\nabla^l_X N, D^l(X, W) \in \Gamma(ltr(TM))$, $\nabla^s_X W, D^s(X, N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

Let P be a projection of TM on $S(TM)$. Then, using the decomposition in (12) we can write

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{21}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E, \tag{22}$$

for any $X, Y \in \Gamma(TM)$ and $E \in \Gamma(Rad\,TM)$, where $\{\nabla^*_X PY, A^*_E X\}$ and $\{h^*(X, PY), \nabla^{*t}_X E\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad\,TM)$, respectively.

By using the equtions given above, we obtain

$$\bar{g}(h^l(X, PY), E) = \bar{g}(A^*_E X, PY), \tag{23}$$

$$\bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY), \tag{24}$$

$$\bar{g}(h^l(X, E), E) = 0, \quad A^*_E E = 0. \tag{25}$$

In general, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (18), we get

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y). \tag{26}$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$.

3 Invariant Submanifolds

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of $(\bar{M}, \bar{g})$. For any vector field X tangent to M, we put

$$\bar{\phi}X = PX + FX, \tag{27}$$

where PX and FX are tangential and transversal components of $\bar{\phi}X$, respectively. It is known that [15] if the structure vector field ξ is tangent to the submanifold M, then $\xi \in S(TM)$. It follows that M is called *invariant* in $\bar{M}$ if $\bar{\phi}X \in \Gamma(TM)$, that is, $\bar{\phi}X = PX$, for all $X \in \Gamma(TM)$.

For any $U \in \Gamma(tr(TM))$, we put

$$\bar{\phi}U = tU + fU, \tag{28}$$

where tU and fU are tangential and transversal components of $\bar{\phi}U$, respectively. Clearly, the submanifold M which is tangent to the structure vector field ξ is invariant in $\bar{M}$ if $\bar{\phi}U = fU$. Therefore, if M is an invariant submanifold of a para-Sasakian manifold $\bar{M}$, then we have

$$F = 0 \quad and \quad t = 0.$$

For any vector fields $U, U' \in \Gamma(tr(TM))$, we have $\bar{g}(\bar{\phi}U, U') = \bar{g}(fU, U')$, which shows that $\bar{g}(fU, U') = -\bar{g}(U, fU')$. Also, for any $X \in \Gamma(TM)$, we have

$$\bar{g}(FX, U) + \bar{g}(X, tU) = 0. \tag{29}$$

Now, define covariant derivatives of P, T, F and f, respectively as

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{30}$$

$$(\nabla_X t)U = \nabla_X tU - t\nabla^t_X U, \tag{31}$$

$$(\nabla_X F)Y = \nabla^t_X FY - F\nabla_X Y, \tag{32}$$

$$(\nabla_X f)U = \nabla^t_X fU - f\nabla^t_X U. \tag{33}$$

From (8) we have

$$\begin{aligned} -\bar{g}(X,Y)\xi + \eta(Y)X &= \bar{\nabla}_X \bar{\phi}Y - \bar{\phi}\bar{\nabla}_X Y \\ &= \nabla_X PY + h(X, PY) - A_{FX}Y + \nabla^t_X FY \\ &\quad -P\nabla_X Y - F\nabla_X Y - th(X,Y) - fh(X,Y). \end{aligned}$$

Using (30), (32) and comparing the tangential and transversal components, we get

$$(\nabla_X P)Y = -\bar{g}(X,Y)\xi + \eta(Y)X + A_{FX}Y + th(X,Y), \tag{34}$$

$$(\nabla_X F)Y = -h(X, PY) + fh(X,Y). \tag{35}$$

Lemma 3.1 *Let M be an invariant lightlike submanifold of a para-Sasakian manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$. Then we have*

$$h^l(X,\xi) = 0, \quad h^s(X,\xi) = 0, \quad A_N\xi = 0, \quad A_W\xi = 0, \tag{36}$$

$$\bar{\phi}h(X,Y) = h(\bar{\phi}X, Y) = h(X, \bar{\phi}Y), \tag{37}$$

for all $X, Y \in \Gamma(TM)$.

Proof. For an invariant lightlike submanifold, from (9) and (18) we have

$$\nabla_X\xi = -PX, \quad h^l(X,\xi) = 0, \quad h^s(X,\xi) = 0. \tag{38}$$

Now, let $N \in \Gamma(ltr(TM))$, then $-\bar{g}(N, \bar{\phi}X) = \bar{g}(N, \bar{\nabla}_X\xi)$. Since M is tangent to the structure vector field ξ and $\bar{\nabla}$ is a metric connection, we have

$$\bar{g}(\bar{\nabla}_X N, \xi) + \bar{g}(N, \bar{\nabla}_X\xi) = 0.$$

Then, we write

$$\bar{g}(N, \bar{\phi}X) = \bar{g}(\bar{\nabla}_X N, \xi) = \bar{g}(A_N X, \xi). \tag{39}$$

Also, using (18) we have

$$\bar{g}(N, \bar{\phi}X) = -\bar{g}(N, \bar{\nabla}_X\xi) - \bar{g}(h^l(X,\xi), N). \tag{40}$$

Therefore, from (39) and (40), we obtain

$$\bar{g}(A_N X, \xi) = \bar{g}(N, \bar{\nabla}_X\xi) + \bar{g}(h^l(X,\xi), N). \tag{41}$$

Using (38) in (41), we get

$$\bar{g}(A_N X, \xi) = -\bar{g}(N, PX). \tag{42}$$

Replacing X by ξ gives

$$A_N\xi = 0. \tag{43}$$

Similarly, let $W \in \Gamma(S(TM^\perp))$. Then, we obtain

$$\bar{g}(W, \bar{\phi}X) = -\bar{g}(A_W X, \xi), \tag{44}$$

and

$$\bar{g}(W, \bar{\phi}X) = \bar{g}(h^s(X,\xi), W). \tag{45}$$

Thus, from (44) and (45), we have

$$\bar{g}(A_W X, \xi) = -\bar{g}(h^s(X,\xi), W). \tag{46}$$

By using (38) in (46), we get

$$A_W X = 0.$$

In particular, we have

$$A_W\xi = 0.$$

For an invariant lightlike submanifold, since $F = 0$, then (32) implies (37).

Proposition 3.2 *Let $(M, g, S(TM), S(TM^{\perp}))$ be an invariant lightlike submanifold of a para-Sasakian manifold $\bar{M}$. If the second fundamental forms h^l and h^s of M are parallel, then M is totally geodesic.*

Proof. Assume that h^l is parallel. Then, we have

$$(\nabla_X^t h^l)(Y, \xi) = \nabla_X h^l(Y, \xi) - h^l(\nabla_X Y, \xi) - h^l(Y, \nabla_X \xi) = 0. \tag{47}$$

Using (36) and (9), we get $h^l(Y, PX) = 0$. Similarly, we have $h^s(Y, PX) = 0$, which completes the proof.

Definition 3.3 *[16] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that for all $X, Y \in \Gamma(TM)$,*

$$h(X, Y) = Hg(X, Y). \tag{48}$$

Using (18) and (48), it is easy to see that M is totally umbilical if and only if on each coordinate neighborhood U, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y), \quad D^l(X, W) = 0, \tag{49}$$

for all $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Theorem 3.4 *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of a para-Sasakian manifold $\bar{M}$ such that the structure vector field ξ is tangent to M. If M is totally umbilical, then M is totally geodesic and invariant.*

Proof. Using (9), (18), (27) and taking into account the transversal parts, we get

$$h^l(X, \xi) + h^s(X, \xi) = FX, \ \ \forall X \in \Gamma(TM). \tag{50}$$

$\bar{\phi}\xi = 0$ implies that $P\xi = 0$ and $F\xi = 0$. Thus, from (50) we have $h^l(\xi, \xi) = 0$ and $h^s(\xi, \xi) = 0$. Since ξ is nonnull, if M is totally umbilical, then (49) implies that $h^l = 0$ and $h^s = 0$, which show that M is totally geodesic. Also, $h^l(X, \xi) + h^s(X, \xi) = FX$ implies that M is invariant in $\bar{M}$, which completes the proof.

4 Paracontact CR-Lightlike Submanifolds

Definition 4.1 *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of a para-Sasakian manifold $(\bar{M}, \bar{g})$ such that the structure vector field ξ is tangent to M. Then M is said to be a paracontact CR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

i) Rad TM is a distribution on M such that $Rad\,TM \cap \bar{\phi} Rad\,TM = \{0\}$.

ii) There exist vector bundles D_0 and D' over M such that

$$S(TM) = \{\bar{\phi} Rad\,TM \oplus D'\} \perp D_0 \perp \{\xi\}, \tag{51}$$

$$\bar{\phi} D_0 = D_0, \quad \bar{\phi} D' = L_1 \perp ltr(TM), \tag{52}$$

where D_0 is nondegenerate and L_1 is a vector subbundle of $S(TM^{\perp})$.

In this case, we have the following decompositions:

$$TM = \{D \oplus D\} \perp \{\xi\}, \tag{53}$$

$$D = Rad\,TM \perp \bar{\phi} Rad\,TM \perp D_0. \tag{54}$$

A paracontact CR-lightlike submanifold is said to be proper if $D_0 \neq \{0\}$ and $L_1 \neq \{0\}$. If $D_0 = \{0\}$, then M is said to be totally real lightlike submanifold.

Example 4.2 *Let M be a lightlike hypersurface of $\bar{M}$. Then, for $E \in \Gamma(Rad\,TM)$ we have $\bar{g}(\bar{\phi}E, E) = 0$, which implies $\bar{\phi}E \in \Gamma(TM)$. Thus, we get a distribution $\bar{\phi}TM^{\perp}$ of rank 1 on M such that $\bar{\phi}TM^{\perp} \cap TM^{\perp} = \{0\}$. So, we have $\bar{\phi}TM^{\perp} \in S(TM)$. Now, let $N \in \Gamma(ltr(TM))$ such that $\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0$ and $\bar{g}(\bar{\phi}N, N) = 0$. Then, it is obvious that $\bar{\phi}N \in \Gamma(S(TM))$. Assume that $D' = \bar{\phi}(tr(TM))$. Thus, we obtain*

$$S(TM) = \{\bar{\phi}TM^{\perp} \oplus D'\} \perp D_0,$$

where D_0 is a nondegenerate distribution and $\bar{\phi}D' = tr(TM)$. Hence, M is a paracontact CR-lightlike hypersurface.

Also, using (1), (27) and (28) we have

$$P^2 = I - \eta \otimes \xi - tF, \tag{55}$$

$$FP + fF = 0, \tag{56}$$

$$f^2 = I - Ft, \tag{57}$$

$$Pt + tf = 0. \tag{58}$$

Lemma 4.3 *In a paracontact CR-lightlike submanifold M of a para-Sasakian manifold $\bar{M}$, a vector field X tangent to M belongs to $D \oplus \{\xi\}$ if and only if $FX = 0$.*

Lemma 4.4 *The distribution $D \oplus \{\xi\}$ in a paracontact CR-lightlike submanifold of a para-Sasakian manifold has an almost paracontact metric structure (P, ξ, η, g).*

Proof. From (55) we have

$$P^2 X = X - \eta(X)\xi - tFX.$$

Let X belongs to $D \oplus \{\xi\}$. Then, we obtain

$$P^2 X = X - \eta(X)\xi. \tag{59}$$

Since $\bar{\phi}\xi = 0$, then we have

$$P\xi = 0 \tag{60}$$

and

$$g(PX, PY) = \bar{g}(\bar{\phi}X, \bar{\phi}Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{61}$$

for any $X, Y \in D \oplus \{\xi\}$. Hence, (59)-(61) complete the proof.

Define the orthogonal complement subbundle to the vector subbundle L_1 in $S(TM^\perp)$ by $L_1^\perp$. Then, we write

$$tr(TM) = \bar{\phi}D' \oplus L_1^\perp. \tag{62}$$

For a paracontact CR-lightlike submanifold M, we put

$$\bar{\phi}X = PX + FX, \quad \forall X \in \Gamma(TM), \tag{63}$$

where $PX \in \Gamma(D)$ and $FX \in \Gamma(L_1 \perp ltr(TM))$. Similarly, we write

$$\bar{\phi}W = BW + CW, \quad \forall W \in \Gamma(S(TM^\perp)), \tag{64}$$

where $BW \in \Gamma(\bar{\phi}L_1)$ and $CW \in \Gamma(L_1^\perp)$.

Lemma 4.5 *For a paracontact CR-lightlike submanifold of a para-Sasakian manifold, the subbundle $L_1^\perp$ has an almost paracomplex structure.*

Proof. For any $X \in \Gamma(L_1^\perp)$, from (57) we have

$$f^2 X = X - FtX,$$

which completes the proof.

Lemma 4.6 *If $tU = 0$, then we have $U \in \Gamma(L_1^{\perp})$, for any $U \in \Gamma(tr(TM))$.*

Proof. For any $U \in \Gamma(tr(TM))$, put

$$\bar{\phi}U = tU + fU,$$

and let $tU = 0$. Then, for $X \in \Gamma(D^{'})$ we have

$$\bar{g}(\bar{\phi}X, U) = -\bar{g}(X, \bar{\phi}U) = -\bar{g}(X, fU) = 0.$$

This completes the proof.

Lemma 4.7 *The almost paracontact structure (P, ξ, η, g) in a paracontact CR-lightlike submanifold of a para-Sasakian manifold is para-Sasakian if and only if either $th(X, Y) = 0$ or $h(X, Y) \in \Gamma(L_1^{\perp})$, for any $X, Y \in \Gamma(D)$.*

Proof. By virtue of Lemma 5 and equation (34), the proof follows.

Proposition 4.8 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, D and $D \oplus D^{'}$ are not integrable.*

Proof. Suppose that D is integrable. Then, we have $g([X, Y], \xi) = 0$, for any $X, Y \in \Gamma(D)$. Also, we derive

$$g([X, Y], \xi) = \bar{g}(\bar{\nabla}_X Y, \xi) - \bar{g}(\bar{\nabla}_Y X, \xi).$$

Since $\bar{\nabla}$ is a metric connection, by using (9) we get

$$g([X, Y], \xi) = \bar{g}(Y, \bar{\phi}X) - \bar{g}(X, \bar{\phi}Y),$$

which gives

$$g([X, Y], \xi) = 2\bar{g}(Y, \bar{\phi}X).$$

It is well known that, there exist no isotropic and totally lightlike paracontact CR-lightlike submanifolds. So, M is proper and D_0 is nondegenerate. Then, we can choose nonnull vector fields $X, Y \in \Gamma(D)$ such that $\bar{g}(Y, \bar{\phi}X) \neq 0$, which is a contradiction. Hence, D is not integrable. By a similar way it is easy to see that $D \oplus D^{'}$ is not integrable. This completes the proof.

Proposition 4.9 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, $D \perp \{\xi\}$ is integrable if and only if*

$$h(X, \bar{\phi}Y) = h(\bar{\phi}X, Y), \quad \forall X, Y \in \Gamma(D \perp \{\xi\}).$$

Proof. From (18), (27), (64) and (8) we obtain

$$F(\nabla_X Y) = -Ch^s(X,Y) + h(X,\bar{\phi}Y),$$

for all $X, Y \in \Gamma(D\perp\{\xi\})$. Replacing X by Y in the last equation, we get

$$F(\nabla_Y X) = -Ch^s(X,Y) + h(\bar{\phi}X, Y).$$

Consequently, we have

$$F[X,Y] = h(X,\bar{\phi}Y) - h(\bar{\phi}X, Y),$$

which completes the proof.

Proposition 4.10 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, $D\perp\{\xi\}$ defines a totally geodesic foliation if and only if*

$$h^l(X,\bar{\phi}Y) = 0 \quad \textit{and} \quad h^s(X,Y) \quad \textit{has no components in } L_1. \tag{65}$$

Proof. By the definition of paracontact CR-lightlike submanifold, $D\perp\{\xi\}$ defines a totally geodesic foliation if and only if

$$g(\nabla_X Y, \bar{\phi}E) = g(\nabla_X Y, W) = 0,$$

for $X, Y \in \Gamma(D\perp\{\xi\})$ and $W \in \Gamma(\bar{\phi}L_1)$. Then, from (18) we have

$$g(\nabla_X Y, \bar{\phi}E) = -\bar{g}(\bar{\phi}\bar{\nabla}_X Y, E).$$

Using (8) and (18), we get

$$g(\nabla_X Y, \bar{\phi}E) = -\bar{g}(h^l(X,\bar{\phi}Y), E). \tag{66}$$

Similarly, we derive

$$g(\nabla_X \bar{\phi}Y, W) = -\bar{g}(h^s(X,Y), \bar{\phi}W). \tag{67}$$

Thus, from (66) and (67), we obtain (65), which completes the proof.

Proposition 4.11 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, D' defines a totally geodesic foliation if and only if $A_N Z$ and $A_{\bar{\phi}W} Z$ have no components in $\bar{\phi}L_1 \perp \bar{\phi}(Rad\,TM)$ and $D_0 \perp \bar{\phi}Rad\,TM$, respectively, for any $Z, W \in \Gamma(D')$.*

Proof. $D^{'}$ defines a totally geodesic foliation if and only if

$$\bar{g}(\nabla_Z W, N) = g(\nabla_Z W, \bar{\phi}N) = g(\nabla_Z W, X) = g(\nabla_Z W, \xi) = 0, \qquad (68)$$

for $Z, W \in \Gamma(D^{'})$, $N \in \Gamma(ltr(TM))$ and $X \in \Gamma(D_0)$. From (8) and (19), we get

$$g(\nabla_Z W, \xi) = 0. \qquad (69)$$

On the other hand, $\bar{\nabla}$ is a metric connection and (7) implies that

$$\bar{g}(\nabla_Z W, N) = g(W, A_Z N). \qquad (70)$$

By using (8) and (53), we obtain

$$g(\nabla_Z W, \bar{\phi}N) = g(A_{\bar{\phi}W} Z, N). \qquad (71)$$

By a similar way, we have

$$g(\nabla_Z W, \bar{\phi}X) = g(A_{\bar{\phi}W} Z, X). \qquad (72)$$

From (69)-(72), we complete the proof.

Proposition 4.12 *In a paracontact CR-lightlike submanifold of a para-Sasakian manifold, the distribution $D^{'}$ is integrable.*

Proof. For any $X, Z \in \Gamma(D^{'})$, from (8), we get

$$\bar{\phi}([X, Z]) = A_{\bar{\phi}X} Z - A_{\bar{\phi}Z} X + \nabla^t_X \bar{\phi}Z - \nabla^t_Z \bar{\phi}X. \qquad (73)$$

Let $Y \in D^{'}$. Then, we get

$$\bar{g}(A_{\bar{\phi}Z} \hat{X}, Y) = \bar{g}(\bar{\nabla}_{\hat{X}} Z, \bar{\phi}Y),$$

for any $\hat{X} \in \Gamma(TM)$, $Z \in \Gamma(D^{'})$. From the last equation we obtain

$$\bar{g}(A_{\bar{\phi}Z} \hat{X}, Y) = \bar{g}(A_{\bar{\phi}Y} \hat{X}, Z). \qquad (74)$$

Also, from (21), we write

$$\bar{g}(h^*(PX, PY), N) = \bar{g}(A_N PX, PY). \qquad (75)$$

Since h^* is bilinear and symmetric, then we have

$$\bar{g}(A_N PX, PY) = \bar{g}(PX, A_N PY). \qquad (76)$$

Choose $\hat{X} \in \Gamma(D_0)$. Then, from (74) and (76), we obtain

$$\bar{g}(\hat{X}, A_{\bar{\phi}Z} Y) = \bar{g}(\hat{X}, A_{\bar{\phi}Y} Z).$$

Thus, non-degeneracy of D_0 implies that

$$A_{\bar{\phi}Z} Y = A_{\bar{\phi}Y} Z,$$

for any $Y, Z \in \Gamma(D^{'})$. This completes the proof.

Theorem 4.13 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, D_0 is integrable if and only if*

i) $\bar{g}(h^*(X,\bar{\phi}Y),N)=\bar{g}(h^*(Y,\bar{\phi}X),N)$,

ii) $g(\nabla^*_X Y,\bar{\phi}E)=g(\nabla^*_Y X,\bar{\phi}E)$,

iii) $\bar{g}(h^s(X,\bar{\phi}Y),W)=\bar{g}(h^s(Y,\bar{\phi}X),W)$,

iv) $\bar{g}(\nabla^*_X Y,\xi)=\bar{g}(\nabla^*_Y X,\xi)$,

v) $\bar{g}(h^*(X,Y),N)=\bar{g}(h^*(Y,X),N)$,

for any $X,Y\in\Gamma(D_0)$, $N\in\Gamma(ltr(TM))$ and $E\in\Gamma(Rad\,TM)$.

Proof. Using (8), (18) and (21) we have

$$\begin{aligned}\bar{g}([X,Y],\bar{\phi}N) &= -\bar{g}(\bar{\nabla}_X\bar{\phi}Y,N)+\bar{g}(\bar{\nabla}_Y\bar{\phi}X,N)\\ &= -\bar{g}(\nabla_X\bar{\phi}Y,N)+\bar{g}(\nabla_Y\bar{\phi}X,N)\\ &= -\bar{g}(h^*(X,\bar{\phi}Y),N)+\bar{g}(h^*(Y,\bar{\phi}X),N). \qquad (77)\end{aligned}$$

From (18) and (21) we get

$$\begin{aligned}\bar{g}([X,Y],\bar{\phi}E) &= \bar{g}(\bar{\nabla}_XY,\bar{\phi}E)-\bar{g}(\bar{\nabla}_YX,\bar{\phi}E)\\ &= g(\nabla_XY,\bar{\phi}E)-g(\nabla_YX,\bar{\phi}E)\\ &= g(\nabla^*_XY,\bar{\phi}E)-g(\nabla^*_YX,\bar{\phi}E). \qquad (78)\end{aligned}$$

On the other hand (8), (18) and (21) give

$$\begin{aligned}\bar{g}([X,Y],\bar{\phi}W) &= -\bar{g}(\bar{\nabla}_X\bar{\phi}Y,W)+\bar{g}(\bar{\nabla}_Y\bar{\phi}X,W)\\ &= -\bar{g}(\nabla_X\bar{\phi}Y,W)+\bar{g}(\nabla_Y\bar{\phi}X,W)\\ &= -\bar{g}(h^s(X,\bar{\phi}Y),W)+\bar{g}(h^s(Y,\bar{\phi}X),W). \qquad (79)\end{aligned}$$

Finally, from (21) we obtain

$$\begin{aligned}\bar{g}([X,Y],\xi) &= \bar{g}(\bar{\nabla}_XY,\xi)-\bar{g}(\bar{\nabla}_YX,\xi)\\ &= \bar{g}(\nabla^*_XY,\xi)-\bar{g}(\nabla^*_YX,\xi), \qquad (80)\end{aligned}$$

$$\begin{aligned}\bar{g}([X,Y],N) &= \bar{g}(\bar{\nabla}_XY,N)-\bar{g}(\bar{\nabla}_YX,N)\\ &= \bar{g}(h^*(X,Y),N)-\bar{g}(h^*(Y,X),N). \qquad (81)\end{aligned}$$

From the definition of CR-lightlike submanifold, D_0 is integrable if and only if

$$\bar{g}([X,Y],\bar{\phi}N)=\bar{g}([X,Y],\bar{\phi}E)=0,$$

$$\bar{g}([X,Y],\bar{\phi}W)=\bar{g}([X,Y],\xi)=\bar{g}([X,Y],N)=0,$$

for any $X,Y\in\Gamma(D_0)$, $N\in\Gamma(ltr(TM))$, $W\in\Gamma(L_1)$ and $E\in\Gamma(Rad\,TM)$. Thus, from (77)-(81) the proof is complete.

Theorem 4.14 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, $Rad\,TM$ is integrable if and only if*

i) $\bar{g}(\acute{E}, h^l(\hat{E}, \bar{\phi}E)) = \bar{g}(\hat{E}, h^l(\acute{E}, \bar{\phi}E))$,

ii) $\bar{g}(h^s(\acute{E}, \bar{\phi}\hat{E}), W) = \bar{g}(h^s(\hat{E}, \bar{\phi}\acute{E}), W)$,

iii) $g(h^*(\acute{E}, \bar{\phi}\hat{E}), N) = g(h^*(\hat{E}, \bar{\phi}\acute{E}), N)$,

iv) $g(A^*_{\hat{E}}\acute{E}, \xi) = g(A^*_{\acute{E}}\hat{E}, \xi)$,

v) $\bar{g}(\hat{E}, h^l(\acute{E}, X)) = \bar{g}(\acute{E}, h^l(\hat{E}, X))$,

for any $E, \acute{E}, \hat{E} \in \Gamma(Rad\,TM)$, $X \in \Gamma(D_0)$, $N \in \Gamma(ltr(TM))$.

Proof. Using (18) we have

$$\begin{aligned}
\bar{g}([\acute{E}, \hat{E}], \bar{\phi}E) &= \bar{g}(\bar{\nabla}_{\acute{E}}\hat{E}, \bar{\phi}E) - \bar{g}(\bar{\nabla}_{\hat{E}}\acute{E}, \bar{\phi}E) \\
&= \acute{E}\bar{g}(\hat{E}, \bar{\phi}E) - \bar{g}(\hat{E}, \bar{\nabla}_{\acute{E}}\bar{\phi}E) - \hat{E}\bar{g}(\acute{E}, \bar{\phi}E) + \bar{g}(\acute{E}, \bar{\nabla}_{\hat{E}}\bar{\phi}E) \\
&= -\bar{g}(\hat{E}, \bar{\nabla}_{\acute{E}}\bar{\phi}E) + \bar{g}(\acute{E}, \bar{\nabla}_{\hat{E}}\bar{\phi}E) \\
&= -\bar{g}(\hat{E}, h^l(\acute{E}, \bar{\phi}E)) + \bar{g}(\acute{E}, h^l(\hat{E}, \bar{\phi}E)). \qquad (82)
\end{aligned}$$

From (8), (18) and (21) we get

$$\begin{aligned}
\bar{g}([\acute{E}, \hat{E}], \bar{\phi}W) &= -\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E}, \bar{\phi}W) + \bar{g}(\bar{\nabla}_{\hat{E}}\acute{E}, \bar{\phi}W) \\
&= -\bar{g}(h^s(\acute{E}, \bar{\phi}\hat{E}), W) + \bar{g}(h^s(\hat{E}, \bar{\phi}\acute{E}), W) \qquad (83)
\end{aligned}$$

and

$$\begin{aligned}
\bar{g}([\acute{E}, \hat{E}], \bar{\phi}N) &= -\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E}, \bar{\phi}N) + \bar{g}(\bar{\nabla}_{\hat{E}}\acute{E}, \bar{\phi}N) \\
&= -\bar{g}(h^*(\acute{E}, \bar{\phi}\hat{E}), N) + \bar{g}(h^*(\hat{E}, \bar{\phi}\acute{E}), N). \qquad (84)
\end{aligned}$$

Similarly, from (18) and (22) we obtain

$$\begin{aligned}
\bar{g}([\acute{E}, \hat{E}], \xi) &= \bar{g}(\bar{\nabla}_{\acute{E}}\hat{E}, \xi) - \bar{g}(\bar{\nabla}_{\hat{E}}\acute{E}, \xi) \\
&= \bar{g}(\nabla_{\acute{E}}\hat{E}, \xi) - \bar{g}(\nabla_{\hat{E}}\acute{E}, \xi) \\
&= -g(A^*_{\hat{E}}\acute{E}, \xi) + g(A^*_{\acute{E}}\hat{E}, \xi) \qquad (85)
\end{aligned}$$

and

$$\begin{aligned}
\bar{g}([\acute{E}, \hat{E}], X) &= \bar{g}(\bar{\nabla}_{\acute{E}}\hat{E}, X) - \bar{g}(\bar{\nabla}_{\hat{E}}\acute{E}, X) \\
&= \bar{g}(\nabla_{\acute{E}}\hat{E}, X) - \bar{g}(\nabla_{\hat{E}}\acute{E}, X) \\
&= -g(A^*_{\hat{E}}\acute{E}, X) + g(A^*_{\acute{E}}\hat{E}, X) \\
&= -\bar{g}(\hat{E}, h^l(\acute{E}, X)) + \bar{g}(\acute{E}, h^l(\hat{E}, X)). \qquad (86)
\end{aligned}$$

By the definition of paracontact CR-lightlike submanifold, $Rad\,TM$ is integrable if and only if

$$\bar{g}([\acute{E},\hat{E}],\bar{\phi}E) = \bar{g}([\acute{E},\hat{E}],\bar{\phi}W) = \bar{g}([\acute{E},\hat{E}],\bar{\phi}N) = 0,$$
$$\bar{g}([\acute{E},\hat{E}],\xi) = \bar{g}([\acute{E},\hat{E}],X) = 0,$$

for any $E,\acute{E},\hat{E} \in \Gamma(Rad\,TM)$, $X \in \Gamma(D_0)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(L_1)$. Thus, from (82)-(86) the proof is complete.

Theorem 4.15 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, $\bar{\phi}Rad\,TM$ is integrable if and only if*

i) $\bar{g}(E,h^l(\hat{E},\bar{\phi}\acute{E})) = \bar{g}(E,h^l(\acute{E},\bar{\phi}\hat{E}))$,

ii) $\bar{g}(h^s(\bar{\phi}\acute{E},\hat{E}),W) = \bar{g}(h^s(\bar{\phi}\hat{E},\acute{E}),W)$,

iii) $\bar{g}(A_N\bar{\phi}\acute{E},\bar{\phi}\hat{E}) = \bar{g}(A_N\bar{\phi}\hat{E},\bar{\phi}\acute{E})$,

iv) $g(A^*_{\hat{E}}\bar{\phi}\acute{E},\bar{\phi}X) = g(A^*_{\acute{E}}\bar{\phi}\hat{E},\bar{\phi}X)$,

for any $E,\acute{E},\hat{E} \in \Gamma(Rad\,TM)$, $X \in \Gamma(D_0)$, $N \in \Gamma(ltr(TM))$.

Proof. Using (8), (16) and (22), we have

$$\begin{aligned}\bar{g}([\bar{\phi}\acute{E},\bar{\phi}E],\bar{\phi}E) &= -\bar{g}(\bar{\nabla}_{\bar{\phi}\acute{E}}\hat{E},E) + \bar{g}(\bar{\nabla}_{\bar{\phi}\hat{E}}\acute{E},E)\\ &= -\bar{g}(E,h^l(\bar{\phi}\acute{E},\hat{E})) + \bar{g}(E,h^l(\acute{E},\bar{\phi}\hat{E})),\end{aligned} \tag{87}$$

$$\begin{aligned}\bar{g}([\bar{\phi}\acute{E},\bar{\phi}E],\bar{\phi}W) &= -\bar{g}(\bar{\nabla}_{\bar{\phi}\acute{E}}\hat{E},W) + \bar{g}(\bar{\nabla}_{\bar{\phi}\hat{E}}\acute{E},W)\\ &= -\bar{g}(W,h^s(\bar{\phi}\acute{E},\hat{E})) + \bar{g}(W,h^s(\acute{E},\bar{\phi}\hat{E})),\end{aligned} \tag{88}$$

$$\begin{aligned}\bar{g}([\bar{\phi}\acute{E},\bar{\phi}E],N) &= \bar{g}(\bar{\nabla}_{\bar{\phi}\acute{E}}\bar{\phi}\hat{E},N) - \bar{g}(\bar{\nabla}_{\bar{\phi}\hat{E}}\bar{\phi}\acute{E},N)\\ &= -\bar{g}(\bar{\phi}\hat{E},\bar{\nabla}_{\bar{\phi}\acute{E}}N) + \bar{g}(\bar{\phi}\acute{E},\bar{\nabla}_{\bar{\phi}\hat{E}}N)\\ &= \bar{g}(A_N\bar{\phi}\acute{E},\bar{\phi}\hat{E}) - \bar{g}(A_N\bar{\phi}\hat{E},\bar{\phi}\acute{E}),\end{aligned} \tag{89}$$

$$\begin{aligned}\bar{g}([\bar{\phi}\acute{E},\bar{\phi}\hat{E}],\xi) &= -\bar{g}(\bar{\phi}\hat{E},\bar{\nabla}_{\bar{\phi}\acute{E}}\xi) + \bar{g}(\bar{\phi}\acute{E},\bar{\nabla}_{\bar{\phi}\hat{E}}\xi)\\ &= -\bar{g}(\bar{\phi}\hat{E},\bar{\phi}^2\acute{E}) + \bar{g}(\bar{\phi}\acute{E},\bar{\phi}^2\hat{E})\\ &= \bar{g}(\bar{\phi}\hat{E},\acute{E}) - \bar{g}(\bar{\phi}\acute{E},\hat{E}),\end{aligned} \tag{90}$$

and finally

$$\begin{aligned}\bar{g}([\bar{\phi}\acute{E},\bar{\phi}\hat{E}],X) &= \bar{g}(\bar{\nabla}_{\bar{\phi}\acute{E}}\bar{\phi}\hat{E},X) - \bar{g}(\bar{\nabla}_{\bar{\phi}\hat{E}}\bar{\phi}\acute{E},X)\\ &= -\bar{g}(\bar{\nabla}_{\bar{\phi}\acute{E}}\hat{E},\bar{\phi}X) + \bar{g}(\bar{\nabla}_{\bar{\phi}\hat{E}}\acute{E},\bar{\phi}X)\\ &= g(A^*_{\hat{E}}\bar{\phi}\acute{E},\bar{\phi}X) - g(A^*_{\acute{E}}\bar{\phi}\hat{E},\bar{\phi}X).\end{aligned} \tag{91}$$

Using the definition of paracontact CR-lightlike submanifold, note that $\bar{\phi}Rad\,TM$ is integrable if and only if

$$\bar{g}([\bar{\phi}\acute{E},\bar{\phi}\hat{E}],\bar{\phi}E)=\bar{g}([\bar{\phi}\acute{E},\bar{\phi}\hat{E}],\bar{\phi}W)=\bar{g}([\bar{\phi}\acute{E},\bar{\phi}\hat{E}],N)=0,$$
$$\bar{g}([\bar{\phi}\acute{E},\bar{\phi}\hat{E}],\xi)=\bar{g}([\bar{\phi}\acute{E},\bar{\phi}\hat{E}],X)=0,$$

for any $E,\acute{E},\hat{E}\in\Gamma(Rad\,TM)$, $X\in\Gamma(D_0)$, $N\in\Gamma(ltr(TM))$ and $W\in\Gamma(L_1)$. Thus, from (87)-(91) the proof is complete.

Theorem 4.16 *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, each leaf of radical distribution is totally geodesic if and only if*

i) $A^*_{\acute{E}}\hat{E}\notin,\Gamma(D_0\perp M^1)$,

ii) $\bar{g}(h^s(\acute{E},\bar{\phi}\hat{E}),W)=0$,

iii) $\bar{g}(h^*(\acute{E},\bar{\phi}\hat{E}),N)=0$,

where $M^1=\bar{\phi}ltr(TM)$ for any $E,\acute{E},\hat{E}\in\Gamma(Rad\,TM)$, $X\in\Gamma(D_0)$, $N\in\Gamma(ltr(TM))$ and $W\in\Gamma(L_1)$.

Proof. By using (18) and (22), we have

$$\begin{aligned} g(\nabla_{\acute{E}}\hat{E},\xi) &= \bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\xi)=\acute{E}\bar{g}(\hat{E},\xi)-\bar{g}(\hat{E},\bar{\nabla}_{\acute{E}}\xi)\\ &= \bar{g}(\hat{E},\bar{\phi}\acute{E})=0 \end{aligned} \tag{92}$$

and

$$\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\bar{\phi}E)=-\bar{g}(A^*_{\hat{E}}\acute{E},\bar{\phi}E),\quad \bar{g}(\nabla_{\acute{E}}\hat{E},X)=-\bar{g}(A^*_{\hat{E}}\acute{E},X). \tag{93}$$

On the other hand, (8), (18) and (21) give

$$\begin{aligned} \bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\bar{\phi}W) &= -\bar{g}(\bar{\phi}\bar{\nabla}_{\acute{E}}\hat{E},W)\\ &= -\bar{g}(\bar{\nabla}_{\acute{E}}\bar{\phi}\hat{E},W)\\ &= \bar{g}(h^s(\acute{E},\bar{\phi}\hat{E}),W). \end{aligned} \tag{94}$$

Finally, we get

$$\begin{aligned} \bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\bar{\phi}N) &= -\bar{g}(\bar{\phi}\bar{\nabla}_{\acute{E}}\hat{E},N)\\ &= -\bar{g}(\bar{\nabla}_{\acute{E}}\bar{\phi}\hat{E},N)\\ &= \bar{g}(h^*(\acute{E},\bar{\phi}\hat{E}),N). \end{aligned} \tag{95}$$

From the definition of paracontact CR-lightlike submanifold, each leaf of $Rad\,TM$ defines totally geodesic foliation in M if and only if

$$\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\bar{\phi}E)=\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\bar{\phi}W)=\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\bar{\phi}N)=\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},\xi)=\bar{g}(\bar{\nabla}_{\acute{E}}\hat{E},X)=0,$$

for any $E,\acute{E},\hat{E}\in\Gamma(Rad\,TM)$, $X\in\Gamma(D_0)$, $N\in\Gamma(ltr(TM))$ and $W\in\Gamma(L_1)$. Hence, from (92)-(95) we complete the proof.

5 Radical Transversal Lightlike Submanifolds

Definition 5.1 *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of a para-Sasakian manifold $(\bar{M}, \bar{g})$ such that $\xi \in \Gamma(TM)$. We say that M is a radical transversal lightlike submanifold of $\bar{M}$ if*

$$\bar{\phi} Rad\,TM = ltr(TM), \tag{96}$$

and

$$\bar{\phi}(S(TM)) = S(TM). \tag{97}$$

Example 5.2 *Let $\bar{M}$ be a $9-$dimensional almost paracontact metric manifold with the structure $(\bar{\phi}, \xi, \eta, \bar{g})$ given in Example 2.1. Suppose that M is a submanifold of $\bar{M}$ defined by*

$$x_1 = y_3,\ x_2 = -y_4,\ x_3 = y_1,\ x_4 = -y_2.$$

Then, the tangent bundle TM of M is spanned by

$$\left\{ \begin{array}{lll} Z_1 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_1}, & Z_2 = -\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_2}, & Z_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_3}, \\ Z_4 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_4}, & Z_5 = \frac{\partial}{\partial z} & \end{array} \right\}. \tag{98}$$

Thus, $Rad\,TM = Span\{Z_1, Z_2\}$ and the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = -\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1},\quad N_2 = -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_3}.$$

It follows that $\bar{\phi} Z_1 = -N_2$ and $\bar{\phi} Z_3 = -N_1$, which imply that $\bar{\phi} Rad\,TM = ltr(TM)$. Also, $\bar{\phi} Z_2 = -Z_4$. Then, we have $\bar{\phi}(S(TM)) = S(TM)$. Hence, M is a radical transversal 2-lightlike submanifold.

Proposition 5.3 *There do not exist 1-lightlike radical transversal lightlike submanifold of a para-Sasakian manifold.*

Proof. Let M be an 1-lightlike radical transversal lightlike submanifolds of a para-Sasakian manifold $\bar{M}$. Then, we have

$$Rad\,TM = Span\{E\},$$

which implies that $ltr(TM) = Span\{N\}$. Using (5) we have

$$\begin{aligned} \bar{g}(\bar{\phi}E, E) &= -\bar{g}(\bar{\phi}^2 E, \bar{\phi}E) + \eta(\bar{\phi}E)\eta(E) \\ &= -\bar{g}(E - \eta(E)\xi, \bar{\phi}E). \end{aligned}$$

Since ξ belongs to $S(TM)$, then we get

$$\bar{g}(\bar{\phi}E, E) = -\bar{g}(E, \bar{\phi}E),$$

which implies that $\bar{g}(\bar{\phi}E, E) = 0$. On the other hand, from (96) we have $\bar{\phi}E = N \in ltr(TM)$. Therefore, $\bar{g}(\bar{\phi}E, E) = \bar{g}(N, E) = 1$, which is a contradiction. Hence, M can not be an 1-lightlike radical transversal submanifold.

Proposition 5.4 *There exist no isotropic or totally lightlike radical transversal lightlike submanifolds of a para-Sasakian manifold.*

Theorem 5.5 *Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, the distribution $S(TM^{\perp})$ is invariant with respect to $\bar{\phi}$.*

Proof. Let $W \in \Gamma(S(TM^{\perp}))$, $E \in \Gamma(Rad\,TM)$ and $N \in \Gamma(ltr(TM))$. From (5) we have

$$\bar{g}(\bar{\phi}W, E) = -\bar{g}(W, \bar{\phi}E) = 0, \tag{99}$$

$$\bar{g}(\bar{\phi}W, N) = -\bar{g}(W, \bar{\phi}N) = 0, \tag{100}$$

which imply that

$$\bar{\phi}(S(TM^{\perp})) \cap Rad\,TM = \{0\}$$

and

$$\bar{\phi}(S(TM^{\perp})) \cap ltr(TM) = \{0\}.$$

Choosing $X \in \Gamma(S(TM))$ and using (1), (5) and (9), we get

$$\bar{g}(\bar{\phi}W, X) = -\bar{g}(W, \bar{\phi}X) = 0, \tag{101}$$

which shows that $\bar{\phi}(S(TM^{\perp})) \cap S(TM) = \{0\}$. Thus, our assertion follows from (99), (100) and (101).

Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold. Let Q and T be the projection morphisms on $Rad\,TM$ and $S(TM)$, respectively. Then, for any $X \in \Gamma(TM)$, we have

$$X = TX + QX, \tag{102}$$

where $TX \in \Gamma(S(TM))$ and $QX \in \Gamma(Rad\,TM)$. Applying $\bar{\phi}$ to (102), we obtain

$$\bar{\phi}X = \bar{\phi}TX + \bar{\phi}QX. \tag{103}$$

If we put $\bar{\phi}TX = SX$ and $\bar{\phi}QX = LX$ in (103), we write

$$\bar{\phi}X = SX + LX, \tag{104}$$

where $SX \in \Gamma(S(TM))$ and $LX \in \Gamma(ltr(TM))$. Then, using (8), (18), (19) and (104), we get

$$\begin{aligned} -\bar{g}(X,Y)\xi + \eta(Y)X &= \nabla_X SY + h^l(X, SY) + h^s(X, SY) \\ &\quad - A_{LY}X + \nabla^l_X LY + D^s(X, LY) \\ &\quad - S\nabla_X Y - L\nabla_X Y - \bar{\phi}h^l(X,Y) \\ &\quad - \bar{\phi}h^s(X,Y). \end{aligned}$$

Considering the tangential, lightlike transversal and screen transversal components of the above equation, we get

$$(\nabla_X S)Y = -\bar{g}(X,Y)\xi + \eta(Y)X + \bar{\phi}h^l(X,Y) + A_{LY}X, \tag{105}$$

$$h^l(X,SY) + \nabla^l_X LY - L\nabla_X Y = 0, \tag{106}$$

$$h^s(X,SY) + D^s(X,LY) - \bar{\phi}h^s(X,Y) = 0, \tag{107}$$

respectively.

It is well known that in general the induced connection of a lightlike submanifold is not a metric connection. Note that the induced connection is a metric connection if and only if $\nabla_X Y \in \Gamma(Rad\,TM)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad\,TM)$ [1].

Theorem 5.6 *Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, the induced connection ∇ on M is a metric connection if and only if $A_{\bar{\phi}Y}X$ has no component in $S(TM)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad\,TM)$.*

Proof. Assume that ∇ is a metric connection. Then, using (18), for any $Z \in \Gamma(S(TM))$, we get

$$\bar{g}(\bar{\nabla}_X Y, Z) = 0.$$

Taking into account the above equation together with (5), we obtain

$$\bar{g}(\bar{\phi}\bar{\nabla}_X Y, \bar{\phi}Z) - \eta(\bar{\nabla}_X Y)\eta(Z) = 0,$$

which implies that

$$\bar{g}(-(\bar{\nabla}_X\bar{\phi})Y + \bar{\nabla}_X\bar{\phi}Y, \bar{\phi}Z) = 0.$$

By using (8) and (19), we get

$$\bar{g}(A_{\bar{\phi}Y}X, \bar{\phi}Z) = 0.$$

Conversely, suppose that $A_{\bar{\phi}Y}X$ has no component in $S(TM)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad\,TM)$. Then from (19), we have

$$\bar{g}(\bar{\nabla}_X\bar{\phi}Y, Z) = 0.$$

By using (8) and (18), we get

$$\bar{g}(\nabla_X Y, \bar{\phi}Z) = 0.$$

This implies that $\nabla_X Y \in \Gamma(Rad\,TM)$, which proves our assertion.

Regarding the integrability of the distributions which are involved in the definition of a radical transversal lightlike submanifold, we have the following.

Theorem 5.7 *Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, $S(TM)$ is integrable if and only if*

$$h^l(X, SY) = h^l(Y, SX),$$

for all $X, Y \in \Gamma(S(TM))$.

Proof. By interchanging the roles of X and Y in (106), we get

$$h^l(Y, SX) + \nabla^l_Y LX - L\nabla_Y X = 0. \tag{108}$$

Combining (107) together with (108), we get

$$h^l(X, SY) - h^l(Y, SX) = L[X, Y],$$

from which our assertion follows.

Theorem 5.8 *Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then $Rad\,TM$ is integrable if and only if*

$$A_{LX}Y = A_{LY}X,$$

for all $X, Y \in \Gamma(Rad\,TM)$.

Proof. Using (105), we get

$$(\nabla_X S)Y = \bar{\phi}h^l(X, Y) + A_{LY}X,$$

which implies that

$$-S\nabla_X Y = \bar{\phi}h^l(X, Y) + A_{LY}X. \tag{109}$$

By interchanging the roles of X and Y in (109), we get

$$-S\nabla_Y X = \bar{\phi}h^l(Y, X) + A_{LX}Y. \tag{110}$$

From (109), (110) and the fact that h^l is symmetric, we obtain

$$A_{LX}Y - A_{LY}X = S[X, Y],$$

which completes the proof.

Theorem 5.9 *Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, radical distribution defines a totally geodesic foliation on M, if and only if $A_{\bar{\phi}Y}X$ has no component in $S(TM)$, for $X, Y \in \Gamma(Rad\,TM)$.*

Proof. By the definition of radical transversal lightlike submanifolds, $Rad\,TM$ defines a totally geodesic foliation if and only if $g(\nabla_X Y, Z) = 0$, for $X, Y \in \Gamma(Rad\,TM)$ and $Z \in \Gamma(S(TM))$. Using (18) and the fact that $\bar{\nabla}$ is a metric connection, we get

$$g(\nabla_X Y, Z) = X\bar{g}(Y, Z) - g(Y, \bar{\nabla}_X Z),$$

which implies that

$$g(\nabla_X Y, Z) = -g(Y, \bar{\nabla}_X Z).$$

By using (5), (8) and (18), we obtain

$$g(\nabla_X Y, Z) = -g(\bar{\phi}Y, X)\eta(Z) + g(\bar{\phi}Y, \nabla_X \bar{\phi}Z).$$

The last equation together with (18) gives

$$g(\nabla_X Y, Z) = g(A_{\bar{\phi}Y} X, \bar{\phi}Z),$$

from which our assertion follows.

Theorem 5.10 *Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, screen distribution $S(TM)$ defines a totally geodesic foliation on M, if and only if $A^*_{\bar{\phi}N} X$ has no component in $S(TM)$, for $X \in \Gamma(S(TM))$, $N \in \Gamma(ltr(TM))$.*

Proof. Let M be a radical transversal lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then, $S(TM)$ defines a totally geodesic foliation if and only if $g(\nabla_X Y, N) = 0$ for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Hence, by using (18), we get

$$g(\nabla_X Y, N) = g(\bar{\nabla}_X Y, N).$$

From (5), we obtain

$$g(\bar{\nabla}_X Y, N) = g(\bar{\nabla}_X \bar{\phi}Y, \bar{\phi}N).$$

Using (18) and (22) gives

$$g(\bar{\nabla}_X Y, N) = -g(A^*_{\bar{\phi}N} X, \bar{\phi}Y),$$

which completes the proof.

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