Gen. Math. Notes, Vol. 25, No. 1, November 2014, pp.19-26
ISSN 2219-7184; Copyright ©ICSRS Publication, 2014
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Notes on Some Schatten's Equation

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(Received: 3-6-14 / Accepted: 22-7-14)

Abstract

We consider Schatten´s equation in connection with a Banach algebra and Banach modules. From this connection we infer a new proof of the fact that the second action of a bounded derivation is also a bounded derivation.

Keywords: *Schatten equation, Arens transpose of a bounded bilinear map, Arens products.*

1 Preliminaries and Notation

Throughout this article let X be a Banach bimodule over a complex Banach algebra $\mathcal{U}$. Let $\mathcal{B}_l(X)$ and $\mathcal{B}_r(X)$ be the Banach algebras of bounded linear operators on X, endowed with the usual composition of operators or reverse composition respectively. Consequently there are contractive homomorphisms $\lambda : \mathcal{U} \to \mathcal{B}_l(X)$ and $\rho : \mathcal{U} \to \mathcal{B}_r(X)$ so that $\lambda(a)$ and $\rho(b)$ commutes whenever $a, b \in \mathcal{U}$. If $a \in \mathcal{U}$ and $x \in X$ we shall write $\lambda(a)(x) = ax$ and $\rho(a)(x) = xa$ and also

$$\begin{aligned} \pi_l &: \mathcal{U} \times X \to X, \pi_l(a, x) = ax, \\ \pi_r &: X \times \mathcal{U} \to X, \pi_r(x, a) = xa. \end{aligned}$$

Hence the above bilinear maps depends intrinsically of λ and ρ and they lay the way in which $\mathcal{U}$ acts on X at each side. As usual, $\mathcal{Z}^1(\mathcal{U}, X)$ will be the Banach space of continuous derivations d from $\mathcal{U}$ into X, i.e. bounded linear operators $d : \mathcal{U} \to X$ that verifies the Leibnitz rule: $d(ab) = d(a)b + ad(b)$ for any $a, b \in \mathcal{U}$.

Let us consider the Arens adjoints of π_l and π_r, say

$$\pi_l^* : X^* \times \mathcal{U} \to X^*, \pi_l^{**} : X^{**} \times X^* \to \mathcal{U}^*, \pi_l^{***} : \mathcal{U}^{**} \times X^{**} \to X^{**},$$

$$\pi_r^* : X^* \times X \to \mathcal{U}^*, \pi_r^{**} : \mathcal{U}^{**} \times X^* \to X^*, \pi_r^{***} : X^{**} \times \mathcal{U}^{**} \to X^{**},$$

so that for $a \in \mathcal{U}$, $a^{**} \in \mathcal{U}^{**}$, $x \in X$, $x^* \in X^*$ and $x^{**} \in X^{**}$ is

$$\begin{aligned} \langle x, \pi_l^* (x^*, a) \rangle &= \langle \pi_l (a, x), x^* \rangle = \langle ax, x^* \rangle = \langle x, x^* a \rangle, \\ \langle a, \pi_l^{**} (x^{**}, x^*) \rangle &= \langle \pi_l^* (x^*, a), x^{**} \rangle = \langle x^* a, x^{**} \rangle = \langle a, x^{**} x^* \rangle, \\ \langle x^*, \pi_l^{***} (a^{**}, x^{**}) \rangle &= \langle \pi_l^{**} (x^{**}, x^*), a^{**} \rangle = \langle x^{**} x^*, a^{**} \rangle = \langle x^*, a^{**} x^{**} \rangle, \end{aligned}$$

$$\begin{aligned} \langle a, \pi_r^* (x^*, x) \rangle &= \langle \pi_r (x, a), x^* \rangle = \langle xa, x^* \rangle = \langle a, x^* x \rangle, \\ \langle x, \pi_r^{**} (a^{**}, x^*) \rangle &= \langle \pi_r^* (x^*, x), a^{**} \rangle = \langle x^* x, a^{**} \rangle = \langle x, a^{**} x^* \rangle, \\ \langle x^*, \pi_r^{***} (x^{**}, a^{**}) \rangle &= \langle \pi_r^{**} (a^{**}, x^*), x^{**} \rangle = \langle a^{**} x^*, x^{**} \rangle = \langle x^*, x^{**} a^{**} \rangle. \end{aligned}$$

A straightforward computation reveals that for all $a^{**}, b^{**} \in \mathcal{U}^{**}$ and $x^{**} \in X^{**}$ one gets

$$\begin{aligned} \pi_l^{***} (a^{**} \Box b^{**}, x^{**}) &= \pi_l^{***} (a^{**}, \pi_l^{***} (b^{**}, x^{**})), \\ \pi_r^{***} (\pi_r^{***} (x^{**}, a^{**}), b^{**}) &= \pi_r^{***} (x^{**}, a^{**} \Box b^{**}), \end{aligned}$$

where $\Box$ denotes the current first Arens product. Indeed,

$$\pi_l^{***} (a^{**}, \pi_r^{***} (x^{**}, b^{**})) = \pi_r^{***} (\pi_l^{***} (a^{**}, x^{**}), b^{**})$$

and X^{**} becomes a $(\mathcal{U}^{**}, \Box)$-Banach bimodule. We observe that for the reverse algebra $\mathcal{U}^r$ of $\mathcal{U}$ we get the identities

$$Hom(\mathcal{U}, \mathcal{B}_l(X)) = Hom(\mathcal{U}^r, \mathcal{B}_r(X)), Hom(\mathcal{U}, \mathcal{B}_r(X)) = Hom(\mathcal{U}^r, \mathcal{B}_l(X)).$$

So, by following the above lines with $\mathcal{U}^r$ instead of $\mathcal{U}$ we see that X^{**} admits a $((\mathcal{U}^r)^{**}, \Diamond)$-Banach bimodule structure.

2 Our Matter and Schatten´s Equation

Schatten´s equation is classic in the theory of tensor products of Banach spaces. It can be roughly written as $\mathcal{B}(E, X^{**}) = \mathcal{B}(E, X)^{**}$, where E and X are Banach spaces (cf. [5], pp. 40-41; [4], p. 13). For a derivation of Schatten´s formula when E is finite dimensional with connections with the principle of local reflexivity the reader can see [2]. Our goal is to consider this matter in a more algebraic context, allowing E to be a Banach algebra $\mathcal{U}$ and X a Banach $\mathcal{U}$-bimodule. In Th. 3.1 we shall prove the existence of a contractive homomorphism of Banach $\mathcal{U}$-bimodules between $\mathcal{B}(\mathcal{U}^\#, X)^{**}$ and $\mathcal{B}((\mathcal{U}^{**})^\#, X^{**})$, where $\mathcal{U}^\#$ and $(\mathcal{U}^{**})^\#$ are the unitization of $\mathcal{U}$ and $(\mathcal{U}^{**}, \Box)$ respectively. Afterwards in Th. 3.3 we give a new proof of the well known fact that the second action of a bounded derivation between a Banach algebra $\mathcal{U}$ with values in a Banach $\mathcal{U}$-bimodule is also a derivation (cf. [3], Prop. 1.7).

3 The Results

Theorem 3.1 *There exists a contractive bounded linear homomorphism of $\mathcal{U}$-Banach bimodules*

$$\Gamma : \mathcal{B}\left(\mathcal{U}^{\#}, X\right)^{**} \to \mathcal{B}\left(\left(\mathcal{U}^{**}\right)^{\#}, X^{**}\right).$$

Proof: The space $\mathcal{B}\left(\mathcal{U}^{\#}, X\right)$ becomes a Banach $\mathcal{U}$-bimodule by defining

$$aT : b^{\#} \to aT\left(b^{\#}\right) y Ta : b^{\#} \to T(ab^{\#})$$

if $a \in \mathcal{U}$, $b^{\#} \in \mathcal{U}^{\#}$ and $T \in \mathcal{B}\left(\mathcal{U}^{\#}, X\right)$. Hence it is apparent that $\mathcal{B}\left(\mathcal{U}^{\#}, X\right)^{**}$ and $\mathcal{B}\left(\left(\mathcal{U}^{**}\right)^{\#}, X^{**}\right)$ are Banach $\mathcal{U}$-bimodules.

Given $a^{**} + \alpha 1 \in \left(\mathcal{U}^{**}\right)^{\#}$, $x^* \in X^*$ and $T \in \mathcal{B}\left(\mathcal{U}^{\#}, X\right)$ let us write

$$\langle T, u\left(x^*, a^{**} + \alpha 1\right)\rangle \triangleq \langle (T \circ \mathfrak{h})^*\left(x^*\right), a^{**}\rangle + \alpha \langle T(1), x^*\rangle, \tag{1}$$

where $\mathfrak{h} : \mathcal{U} \hookrightarrow \mathcal{U}^{\#}$ is the natural injection. It is evident that $u\left(x^*, a^{**} + \alpha 1\right)$ is $\mathbb{C}$-linear and it is bounded because

$$\begin{aligned} \left|\langle T, u\left(x^*, a^{**} + \alpha 1\right)\rangle\right| &\leq \left\|a^{**}\right\| \left\|(T \circ \mathfrak{h})^*\left(x^*\right)\right\| + |\alpha| \left\|x^*\right\| \left\|T(1)\right\| \\ &\leq \left[\left\|a^{**}\right\| \left\|(T \circ \mathfrak{h})^*\right\| + |\alpha| \left\|T\right\|\right] \left\|x^*\right\| \\ &\leq \left[\left\|a^{**}\right\| + |\alpha|\right] \left\|T\right\| \left\|x^*\right\| \\ &= \left\|a^{**} + \alpha 1\right\| \left\|T\right\| \left\|x^*\right\|. \end{aligned} \tag{2}$$

Now, for $n \in \mathcal{B}\left(\mathcal{U}^{\#}, X\right)^{**}$ we set

$$\begin{aligned} \Gamma(n)\left(a^{**} + \alpha 1\right) &: X^* \to \mathbb{C}, \\ \langle x^*, \Gamma(n)\left(a^{**} + \alpha 1\right)\rangle &\triangleq \langle u\left(x^*, a^{**} + \alpha 1\right), n\rangle \, if x^* \in X^*. \end{aligned} \tag{3}$$

By (1) the $\mathcal{B}\left(\mathcal{U}^{\#}, X\right)^*$-valued map $\left(x^*, a^{**} + \alpha 1\right) \to u\left(x^*, a^{**} + \alpha 1\right)$ is $\mathbb{C}$-bilinear on $X^* \times \left(\mathcal{U}^{**}\right)^{\#}$. Further, if $\beta \in \mathbb{C}$, $x^*, y^* \in X^*$ then

$$\begin{aligned} \langle \beta x^* + y^*, \Gamma(n)\left(a^{**} + \alpha 1\right)\rangle &= \langle u\left(\beta x^* + y^*, a^{**} + \alpha 1\right), n\rangle \\ &= \beta \langle u\left(x^*, a^{**} + \alpha 1\right), n\rangle + \langle u\left(y^*, a^{**} + \alpha 1\right), n\rangle \\ &= \beta \langle x^*, \Gamma(n)\left(a^{**} + \alpha 1\right)\rangle + \langle y^*, \Gamma(n)\left(a^{**} + \alpha 1\right)\rangle, \end{aligned}$$

i.e. $\Gamma(n)\left(a^{**} + \alpha 1\right)$ is $\mathbb{C}$-linear on X^*. Besides by (2) we have

$$\begin{aligned} \left|\langle x^*, \Gamma(n)\left(a^{**} + \alpha 1\right)\rangle\right| &\leq \|n\| \left\|u\left(x^*, a^{**} + \alpha 1\right)\right\| \\ &\leq \|n\| \left\|a^{**} + \alpha 1\right\| \left\|x^*\right\|, \end{aligned} \tag{4}$$

i.e. $\Gamma(n)(a^{**}+\alpha 1) \in X^{**}$. If $\overline{a}^{\#} = a^{**}+\alpha 1$, $\overline{b}^{\#} = b^{**}+\beta 1$ in $(\mathcal{U}^{**})^{\#}$, $x^* \in X^*$ and $\gamma \in \mathbb{C}$ we have

$$\begin{aligned}
\left\langle x^*, \Gamma(n)\left(\gamma\overline{a}^{\#}+\overline{b}^{\#}\right)\right\rangle &= \langle u(x^*, \gamma a^{**}+b^{**}+(\gamma\alpha+\beta)1), n\rangle \\
&= \left\langle \gamma u(x^*, \overline{a}^{\#}) + u\left(x^*, \overline{b}^{\#}\right), n\right\rangle \\
&= \gamma\left\langle u(x^*, \overline{a}^{\#}), n\right\rangle + \left\langle u\left(x^*, \overline{b}^{\#}\right), n\right\rangle \\
&= \gamma\left\langle x^*, \Gamma(n)(\overline{a}^{\#})\right\rangle + \left\langle x^*, \Gamma(n)\left(\overline{b}^{\#}\right)\right\rangle,
\end{aligned}$$

i.e. $\Gamma(n)$ is $\mathbb{C}$-linear on $(\mathcal{U}^{**})^{\#}$. By (4) $\Gamma(n) \in \mathcal{B}\left((\mathcal{U}^{**})^{\#}, X^{**}\right)$ since

$$\|\Gamma(n)(a^{**}+\alpha 1)\| \leq \|n\|\,\|a^{**}+\alpha 1\|\, for any a^{**}+\alpha 1 \in (\mathcal{U}^{**})^{\#}. \tag{5}$$

Hence Γ is well defined, its linearity is immediate by (3) and it is contractive as follows by (5).

With the above notation let $a^{**} = w^*\text{-}\lim_{j\in J}\chi_{\mathcal{U}}(a_j)$ for some bounded net $\{a_j\}_{J\in J}$ of $\mathcal{U}$, where $\chi_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{U}^{**}$ is the natural immersion of $\mathcal{U}$ into $\mathcal{U}^{**}$. Then

$$\begin{aligned}
\langle T, u(x^*, a^{**}+\alpha 1)\, a\rangle &= \langle aT, u(a^{**}+\alpha 1)\rangle \\
&= \langle ((aT)\circ\mathfrak{h})^*(x^*), a^{**}\rangle + \alpha\langle (aT)(1), x^*\rangle \\
&= \lim_{j\in J}\langle a_j, ((aT)\circ\mathfrak{h})^*(x^*)\rangle + \alpha\langle aT(1), x^*\rangle \\
&= \langle (T\circ\mathfrak{h})^*(x^*a), a^{**}\rangle + \alpha\langle T(1), x^*a\rangle \\
&= \langle T, u(x^*a, a^{**}+\alpha 1)\rangle.
\end{aligned}$$

Since T is arbitrary we see that

$$\begin{aligned}
\left\langle x^*, \Gamma(an)(\overline{a}^{\#})\right\rangle &= \left\langle u(x^*, \overline{a}^{\#}), an\right\rangle \\
&= \left\langle u(x^*, \overline{a}^{\#})\, a, n\right\rangle \\
&= \left\langle u(x^*a, \overline{a}^{\#}), n\right\rangle \\
&= \left\langle x^*a, \Gamma(n)(\overline{a}^{\#})\right\rangle \\
&= \left\langle x^*, a\Gamma(n)(\overline{a}^{\#})\right\rangle \\
&= \left\langle x^*, (a\Gamma(n))(\overline{a}^{\#})\right\rangle.
\end{aligned} \tag{6}$$

On the other hand, if $x^* \in X^*$ then

$$\begin{aligned}
\langle x^*, (T\circ\mathfrak{h})^{**}(\chi_{\mathcal{U}}(a))\rangle &= \langle (T\circ\mathfrak{h})^*(x^*), \chi_{\mathcal{U}}(a)\rangle \\
&= \langle a, (T\circ\mathfrak{h})^*(x^*)\rangle \\
&= \langle T(\mathfrak{h}(a)), x^*\rangle \\
&= \langle (Ta)(1), x^*\rangle \\
&= \langle x^*, \chi_X((Ta)(1))\rangle,
\end{aligned}$$

i.e.
$$(T\circ\mathfrak{h})^{**}(\chi_{\mathcal{U}}(a)) = \chi_X((Ta)(1)).$$

Thus

$$\begin{aligned}
\langle T, u(x^*, aa^{**}+\alpha\chi_{\mathcal{U}}(a))\rangle &= \langle (T\circ\mathfrak{h})^*(x^*), aa^{**}+\alpha\chi_{\mathcal{U}}(a)\rangle \\
&= \langle (T\circ\mathfrak{h})^*(x^*)a, a^{**}\rangle + \alpha\langle x^*, (T\circ\mathfrak{h})^{**}(\chi_{\mathcal{U}}(a))\rangle \\
&= \lim_{j\in J}\langle T(\mathfrak{h}(aa_j), x^*\rangle + \alpha\langle x^*, \chi_X((Ta)(1))\rangle \\
&= \lim_{j\in J}\langle ((Ta)\circ\mathfrak{h})(a_j), x^*\rangle + \alpha\langle (Ta)(1), x^*\rangle \\
&= \langle ((Ta)\circ\mathfrak{h})^*(x^*), a^{**}\rangle + \alpha\langle (Ta)(1), x^*\rangle \\
&= \left\langle Ta, u\left(x^*, \overline{a}^{\#}\right)\right\rangle \\
&= \left\langle T, au\left(x^*, \overline{a}^{\#}\right)\right\rangle,
\end{aligned}$$

i.e.
$$au\left(x^*, \overline{a}^{\#}\right) = u(x^*, aa^{**}+\alpha\chi_{\mathcal{U}}(a)).$$

Now

$$\begin{aligned}
\left\langle x^*, \Gamma(na)\left(\overline{a}^{\#}\right)\right\rangle &= \left\langle u\left(x^*, \overline{a}^{\#}\right), na\right\rangle \\
&= \langle u(x^*, aa^{**}+\alpha\chi_{\mathcal{U}}(a)), n\rangle \\
&= \langle x^*, \Gamma(n)(aa^{**}+\alpha\chi_{\mathcal{U}}(a))\rangle \\
&= \left\langle x^*, \Gamma(n)\left(a\overline{a}^{\#}\right)\right\rangle \\
&= \left\langle x^*, (\Gamma(n)a)\left(\overline{a}^{\#}\right)\right\rangle.
\end{aligned} \tag{7}$$

Since (6) and (7) hold for all $x^*\in X^*$ and $\overline{a}^{\#}\in(\mathcal{U}^{**})^{\#}$ then Γ is a homomorphism of Banach $\mathcal{U}$-bimodules.

Remark 3.2 *Let $\mathfrak{K}: X\to\mathcal{B}\left(\mathcal{U}^{\#}, X\right)$, $\mathfrak{K}(x)\left(a^{\#}\right) = xa^{\#}$ if $x\in X$, $a^{\#}\in\mathcal{U}^{\#}$. It is word mentioning that a bounded linear operator $t:\mathcal{U}\to X$ is a derivation if and only if $\mathfrak{K}\circ t=\delta_{t^{\#}}$, where $t^{\#}\in\mathcal{B}\left(\mathcal{U}^{\#}, X\right)$ is the extensión of t to $\mathcal{U}^{\#}$ so that $t^{\#}(1)=0$ and*

$$\delta_{t^{\#}}:\mathcal{U}\to\mathcal{B}\left(\mathcal{U}^{\#}, X\right), \delta_{t^{\#}}(a) = t^{\#}a - at^{\#}\ for\ a\in\mathcal{U},$$

is the inner derivation implemented by $t^{\#}$. (cf. [3], Prop. 1.1).

Theorem 3.3 **(i)** *Let $\mathfrak{K}: X\to\mathcal{B}\left(\mathcal{U}^{\#}, X\right)$ so that $\mathfrak{K}(x)\left(a^{\#}\right) = xa^{\#}$ if $x\in X$ and $a^{\#}\in\mathcal{U}^{\#}$. Then $\mathfrak{K}$ is an isometric homomorphism of Banach $\mathcal{U}$-bimodules and*

$$\Gamma(\mathfrak{K}^{**}(x^{**}))\left(\overline{a}^{\#}\right) = x^{**}\overline{a}^{\#} if x^{**}\in X^{**}, \overline{a}^{\#}\in(\mathcal{U}^{**})^{\#}. \tag{8}$$

(ii) *If* $d \in \mathcal{Z}^1(\mathcal{U}, X)$ *then* $d^{**} \in \mathcal{Z}^1(\mathcal{U}^{**}, X^{**})$.

(iii) *If* Γ *is injective then*

$$\left\{t \in \mathcal{B}(\mathcal{U}, X) : t^{**} \in \mathcal{Z}^1(\mathcal{U}^{**}, X^{**})\right\} \subseteq \mathcal{Z}^1(\mathcal{U}, X).$$

Proof:

(i) The first claim is immediate. Let $\overline{a}^{\#} = a^{**} + \alpha 1$ in $(\mathcal{U}^{**})^{\#}$, $x^{**} \in X^{**}$, say $a^{**} = w^*\text{-}\lim_{j\in J} \chi_{\mathcal{U}}(a_j)$ and $x^{**} = w^*\text{-}\lim_{i\in I} \chi_X(x_i)$ for some bounded nets $\{a_j\}_{j\in J}$ in $\mathcal{U}$ and $\{x_i\}_{i\in I}$ in X. For $x^* \in X^*$ we see that

$$\begin{aligned}
\left\langle x^*, \Gamma\left(\mathfrak{K}^{**}(x^{**})\right)\left(\overline{a}^{\#}\right)\right\rangle &= \left\langle u\left(x^*, \overline{a}^{\#}\right), \mathfrak{K}^{**}(x^{**})\right\rangle \\
&= \left\langle \mathfrak{K}^*\left(u\left(x^*, \overline{a}^{\#}\right)\right), x^{**}\right\rangle \\
&= \lim_{i\in I} \left\langle x_i, \mathfrak{K}^*\left(u\left(x^*, \overline{a}^{\#}\right)\right)\right\rangle \\
&= \lim_{i\in I} \left\langle \mathfrak{K}(x_i), u\left(x^*, \overline{a}^{\#}\right)\right\rangle \\
&= \lim_{i\in I} \left[\langle (\mathfrak{K}(x_i) \circ \mathfrak{h})^*(x^*), a^{**}\rangle + \alpha \langle \mathfrak{K}(x_i)(1), x^*\rangle\right] \\
&= \lim_{i\in I} \left[\lim_{i\in J} \langle a_j, (\mathfrak{K}(x_i) \circ \mathfrak{h})^*(x^*)\rangle + \alpha \langle x_i, x^*\rangle\right] \\
&= \lim_{i\in I} \lim_{j\in J} \langle x_i a_j, x^*\rangle + \alpha \langle x^*, x^{**}\rangle \\
&= \left\langle x^*, x^{**}\overline{a}^{\#}\right\rangle.
\end{aligned}$$

(ii) If we write $\overline{\mathfrak{K}} = \Gamma \circ \mathfrak{K}$ given $d \in \mathcal{Z}^1(\mathcal{U}, X)$ by (i) and Remark 3.2 we have

$$\overline{\mathfrak{K}} \circ d^{**} = (\Gamma \circ \mathfrak{K}^{**}) \circ d^{**} = \Gamma \circ (\mathfrak{K}^{**} \circ d^{**}) = \Gamma \circ (\mathfrak{K} \circ d)^{**} = \Gamma \circ \delta^{**}_{d^{\#}}.$$

Thus the result will follow once we prove that

$$\Gamma \circ \delta^{**}_{d^{\#}} = \delta_{(d^{**})^{\#}}. \tag{9}$$

For, let $a^{**} \in \mathcal{U}^{**}$, $\overline{b}^{\#} \in (\mathcal{U}^{**})^{\#}$, $\overline{b}^{\#} = b^{**} + \beta 1$. Further, let

$$a^{**} = w^* - \lim_{i\in I} \chi_{\mathcal{U}}(a_i) \text{ and } b^{**} = w^* - \lim_{j\in J} \chi_{\mathcal{U}}(b_j)$$

for some bounded nets $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ of $\mathcal{U}$. If $x^* \in X^*$ we obtain

$$\begin{aligned}
\left\langle x^*, \Gamma(\delta^{**}_{d^{\#}}(a^{**})(\overline{b}^{\#})\right\rangle &= \left\langle u(x^*, \overline{b}^{\#}), \delta^{**}_{d^{\#}}(a^{**})\right\rangle \\
&= \lim_{i\in I} \left\langle a_i, \delta^*_{d^{\#}}\left[u(x^*, \overline{b}^{\#})\right]\right\rangle \\
&= \lim_{i\in I} \left\langle \delta_{d^{\#}}(a_i), u(x^*, \overline{b}^{\#})\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{i\in I}\left[\left\langle \left(\delta_{d^{\#}}\left(a_i\right)\circ \mathfrak{h}\right)^*\left(x^*\right), b^{**}\right\rangle + \beta\left\langle \delta_{d^{\#}}\left(a_i\right)(1), x^*\right\rangle\right] \\
&= \lim_{i\in I}\left[\lim_{j\in J}\left\langle b_j, \left(\delta_{d^{\#}}\left(a_i\right)\circ \mathfrak{h}\right)^*\left(x^*\right)\right\rangle + \beta\left\langle d\left(a_i\right), x^*\right\rangle\right] \\
&= \lim_{i\in I}\lim_{j\in J}\left\langle d\left(a_i b_j\right) - a_i d\left(b_j\right), x^*\right\rangle + \beta\left\langle d^*\left(x^*\right), a^{**}\right\rangle \\
&= \lim_{i\in I}\lim_{j\in J}\left\langle b_j, d^*\left(x^*\right)a_i - d^*\left(x^* a_i\right)\right\rangle + \beta\left\langle d^*\left(x^*\right), a^{**}\right\rangle \\
&= \lim_{i\in I}\left\langle d^*\left(x^*\right)a_i - d^*\left(x^* a_i\right), b^{**}\right\rangle + \beta\left\langle d^*\left(x^*\right), a^{**}\right\rangle \\
&= \left\langle b^{**}d^*\left(x^*\right) - d^{**}\left(b^{**}\right)x^*, a^{**}\right\rangle + \beta\left\langle d^*\left(x^*\right), a^{**}\right\rangle \\
&= \left\langle d^*\left(x^*\right), a^{**}\Box b^{**}\right\rangle + \left\langle \beta d^*\left(x^*\right) - d^{**}\left(b^{**}\right)x^*, a^{**}\right\rangle \\
&= \left\langle x^*, d^{**}\left(a^{**}\Box b^{**} + \beta a^{**}\right) - a^{**}d^{**}\left(b^{**}\right)\right\rangle \\
&= \left\langle x^*, \left[\left(d^{**}\right)^{\#}a^{**} - a^{**}\left(d^{**}\right)^{\#}\right]\left(\overline{b}^{\#}\right)\right\rangle \\
&= \left\langle x^*, \delta_{\left(d^{**}\right)^{\#}}\left(a^{**}\right)\left(\overline{b}^{\#}\right)\right\rangle
\end{aligned}
$$

and (9) holds.

(iii) Let $t \in \mathcal{B}(\mathcal{U}, X)$ so that $t^{**} \in \mathcal{Z}^1(\mathcal{U}^{**}, X^{**})$. For $a \in \mathcal{U}$ let us write $T_a = \delta_{t^{\#}}(a) - \mathfrak{K}(t(a))$. Given $\overline{b}^{\#} \in (\mathcal{U}^{**})^{\#}$ as above and $x^* \in X^*$ we see that

$$
\begin{aligned}
\left\langle x^*, \Gamma(\chi_{\mathcal{B}(\mathcal{U}^{\#},X)}(T_a))(\overline{b}^{\#})\right\rangle &= \left\langle u\left(x^*, \overline{b}^{\#}\right), \chi_{\mathcal{B}(\mathcal{U}^{\#},X)}(T_a)\right\rangle \\
&= \left\langle T_a, u\left(x^*, \overline{b}^{\#}\right)\right\rangle \\
&= \left\langle \left(T_a\circ \mathfrak{h}\right)^*\left(x^*\right), b^{**}\right\rangle \\
&= \lim_{j\in J}\left\langle b_j, \left(T_a\circ \mathfrak{h}\right)^*\left(x^*\right)\right\rangle \\
&= \lim_{j\in J}\left\langle t\left(ab_j\right) - at\left(b_j\right) - t(a)b_j, x^*\right\rangle \\
&= \left\langle t^*\left(x^*\right)a - t^*\left(x^* a\right) - x^* t(a), b^{**}\right\rangle \\
&= \left\langle x^*, t^{**}\left(ab^{**}\right) - at^{**}\left(b^{**}\right)\right\rangle - \left\langle x^* t(a), b^{**}\right\rangle \\
&= 0.
\end{aligned}
$$

Therefore $\Gamma(\chi_{\mathcal{B}(\mathcal{U}^{\#},X)}(T_a)) = 0$ and as Γ and $\chi_{\mathcal{B}(\mathcal{U}^{\#},X)}$ are injective our claim follows by Remark 3.2.

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