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A Solution of Generalized Fractional Volterra Type Integral Equation Involving K_4 - Function

Kishan Sharma[1], Renu Jain[2] and V.S. Dhakar[3]

[1]Department of Mathematics, NRI Institute of Technology and Management,
Gwalior-474001, India
Address: B-3, Krishna Puri, Taraganj, Lashkar,
Gwalior (M.P.)-474001, India
E-mail: drkishansharma2006@rediffmail.com,
drkishan010770@yahoo.com
[2]School of Mathematics and Allied Sciences, Jiwaji University,
Gwalior (M.P.)-474011, India
E-mail: renujain3@rediffmail.com
[3]Research Scholar, Suresh Gyan Vihar University,
Jagatpura, Jaipur(Raj.), India
E-mail: viroo4u@gmail.com

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Abstract

The main objective of this paper is to derive a solution of a generalized fractional Volterra integral equation involving K_4*-function with the help of the Sumudu transform. Several special cases are also mentioned.*

Keywords: *Sumudu transform, Fractional differential operator, Fractional integral operator,* K_4*-function, R- and G-functions of Lorenzo-Hartley(L-H).*

1 Introduction and Definitions

Fractional Calculus represents a generalization of the ordinary differentiation and integration to arbitrary order. During the last three decades the subject has been widely used in the various fields of science and engineering. Many applications of Fractional Calculus can be found in Turbulence and Fluid Dynamics, Stochastic Dynamical System, Plasma Physics and Controlled Thermonuclear Fusion, Non-linear Control Theory, Image Processing, Non-linear Biological Systems and Astrophysics. The Mittag-Leffler function has gained importance and popularity during the last one decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by Mittag-Leffler[8,9] in terms of the power series

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+1)},\quad (\alpha>0) \tag{1.1}$$

A generalization of this series in the following form

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+\beta)},\qquad (\alpha,\beta>0) \tag{1.2}$$

has been studied by several authors notably by Mittag-Leffler[8,9],Wiman[3], Agrawal[20], Humbert and Agrawal[18] and Dzrbashjan[15,16,17]. It is shown in [19] that the function defined by (1.1) and (1.2) are both entire functions of order $\rho=1$ and type $\sigma=1$. A detailed account of the basic properties of these two functions are given in the third volume of Bateman manuscript project[1] and an account of their various properties can be found in [16,21].

The F-function of Robotnov and Hartley [22] is defined by the power series

$$F_{q}[a,x]=\sum_{n=0}^{\infty}\frac{a^{n}x^{(n+1)q-1}}{\Gamma((n+1)q)},q>0 \tag{1.3}$$

This function effect the direct solution of the fundamental linear fractional order differential equation.

Recently, the interest in the R- and G-functions of Lorenzo-Hartley[4,5] and their popularity have sharply increased in view of their important role and applications in Fractional Calculus and related integral and differential equations of fractional order.

The R- and the G-functions (but not the Meijer's G-function) introduced by Lorenzo-Hartley[4] are defined by the power series

$$R_{q,\nu}[a,c,x]=\sum_{n=0}^{\infty}\frac{a^n(x-c)^{(n+1)q-1-\nu}}{\Gamma((n+1)q-\nu)}, \tag{1.4}$$

where $x>c\geq 0, q\geq 0, R(q-\nu)>0,$

and

$$G_{\alpha,\beta,\gamma}[a,c,x]=\sum_{n=0}^{\infty}\frac{(\gamma)_n\, a^n(x-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)}, \tag{1.5}$$

where $R(\alpha\gamma-\beta)>0$ and $(\gamma)_n$ is the Pochhammer's symbol given by

$$(\gamma)_n=\begin{cases}1, n=0\\ \gamma(\gamma+1)....(\gamma+n-1), n\in N\end{cases}$$

Particular cases:

If we put $c=0$ in above equations (1.4) and (1.5), we get

$$R_{q,\nu}[a,x]=\sum_{n=0}^{\infty}\frac{a^n x^{(n+1)q-1-\nu}}{\Gamma((n+1)q-\nu)} \tag{1.6}$$

and

$$G_{\alpha,\beta,\gamma}[a,x]=\sum_{n=0}^{\infty}\frac{(\gamma)_n\, a^n x^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)} \tag{1.7}$$

The Riemann-Liouville operator of fractional integral of order υ is given by

$$D_x^{-\upsilon}\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\int_0^x (x-t)^{\upsilon-1}f(t)dt \tag{1.8}$$

provided that the integral exists.

The Riemann-Liouville operator of fractional derivative of order υ is defined [2,10,11,12] in the following form

$$D_x^{\upsilon}\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\frac{d^n}{dx^n}\int_0^x\frac{f(t)}{(x-t)^{\upsilon+n-1}}dt, (n-1<\upsilon<n) \tag{1.9}$$

provided that the integral exists.

Watugala[7] introduced a new integral transform, called the Sumudu transform defined for the functions of exponential order, over the set of the functions,

$$A = \{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0,\infty) \},$$

by

$$G(s) = S\{f(t)\} = \int_0^\infty f(st)e^{-t}dt, s \in (\tau_1, \tau_2). \tag{1.10}$$

For further details of this transform, please see([6,14]).

The K_4-function[13] is defined as

$$K_4^{(\alpha,\beta,\gamma),(a,c):(p;q)}(a_1,\ldots,a_p; b_1,\ldots,b_q; x) =$$

$$K_4^{(\alpha,\beta,\gamma),(a,c):(p;q)}(x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\gamma)_n \, a^n (x-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)} \tag{1.11}$$

where $R(\alpha\gamma-\beta)>0$ and $(a_i)_n, i=1,2,\ldots,p$ and $(b_j)_n, j=1,2,\ldots,q$ are the Pochhammer symbols.

Particularly for $c=0$, equation(1.11) reduces into the following form

$$K_4^{(\alpha,\beta,\gamma),(a,0):(p;q)}(a_1,\ldots,a_p; b_1,\ldots,b_q; x) =$$

$$K_4^{(\alpha,\beta,\gamma),(a,0):(p;q)}(x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\gamma)_n \, a^n x^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)} \tag{1.12}$$

Further details of this function are given by [13].
In order to prove our main results, we shall required the following lemma stated below:

Lemma 1.1. *The Sumudu transform of the* K_4*-function defined by (1.12) is given by*

$$S\{K_4(t)\} = s^{\alpha\gamma-\beta-1} \, {}_{p+1}F_q(a_1,\ldots,a_p,\gamma; b_1,\ldots,b_q; as^\alpha) \tag{1.13}$$

provided that $R(\alpha\gamma-\beta)>0$.

Proof.

Using (1.10) and (1.12) and evaluating the inner integral, we arrive at the result

$$S\{K_4(t)\} = s^{\alpha\gamma-\beta-1} \, {}_{p+1}F_q(a_1, \ldots, a_p, \gamma; b_1, \ldots, b_q; as^{\alpha}), R(\alpha\gamma-\beta) > 0. \quad (1.14)$$

This proves (1.1).

2 Solution of the Generalized Fractional Volterra Integral Equation

Theorem 2.1. *The Volterra type integral equation*

$$D_x^{-\lambda}\{h(\tau)\} = \kappa\int_0^{\tau} h(\xi) \, K_4^{(\alpha,\beta,\gamma),(a,0):(p;q)}(a_1, \ldots, a_p; b_1, \ldots, b_q; \xi) d\xi + \eta f(\tau) \quad (2.1)$$

has its solution given explicitly by

$$h(\tau) = \eta\sum_{r=0}^{\infty} \kappa^r \int_0^{\tau} F(\tau-\xi) \, K_4^{(\alpha,\beta r,\gamma r),(a,0):(p;q)}(a_1, \ldots, a_p; b_1, \ldots, b_q; \xi) d\xi \quad (2.2)$$

where $0 \le \tau \le 1; \kappa, \alpha, \beta, \gamma, \eta \in C$ *and* $R(\alpha\gamma-\beta) > 0$.

Proof. Now taking the Sumudu transform on both the sides of (2.1) and then using the inverse Sumudu transform and Lemma1.1, we obtain

$$h(\tau) = \eta\sum_{r=0}^{\infty} \kappa^r \int_0^{\tau} F(\tau-\xi) \, K_4^{(\alpha,\beta r,\gamma r),(a,0):(p;q)}(a_1, \ldots, a_p; b_1, \ldots, b_q; \xi) d\xi \quad (2.3)$$

where $0 \le \tau \le 1; \kappa, \alpha, \beta, \gamma, \eta \in C$ and $R(\alpha\gamma-\beta) > 0$.

If we put $r = s = 0$ in (2.1), we get[23]
Corollary 2.1. *The Volterra type integral equation*

$$D_x^{-\lambda}\{h(\tau)\} = \kappa\int_0^{\tau} h(\xi) \, G_{\alpha,\beta,\gamma}[a,\xi] d\xi + \eta f(\tau) \quad (2.4)$$

has its solution given explicitly by

$$h(\tau) = \eta\sum_{r=0}^{\infty} \kappa^r \int_0^{\tau} F(\tau-\xi) \, G_{\alpha,\beta r,\gamma r}[a,\xi] d\xi \quad (2.5)$$

where $G_{\alpha,\beta,\gamma}[a,\xi]$ *is given by (1.7) and*
$0 \le \tau \le 1; \kappa, \alpha, \beta, \gamma, \eta \in C$ *and* $R(\alpha), R(\beta), R(\alpha-\beta) > 0.$
If we take $\gamma = 1$ *in Cor.(2.1), we get[23]*

Corollary 2.2. *The Volterra type integral equation*

$$D_x^{-\lambda}\{h(\tau)\} = \kappa\int_0^{\tau} h(\xi) \quad R_{\alpha,\beta}[a,\xi]d\xi + \eta f(\tau) \tag{2.6}$$

has its solution given explicitly by

$$h(\tau) = \eta\sum_{r=0}^{\infty} \kappa^r \int_0^{\tau} F(\tau-\xi) \quad G_{\alpha,\beta r,r}[a,\xi]d\xi \tag{2.7}$$

where $R_{\alpha,\beta}[a,\xi]$ *is given by (1.6) and* $R(\alpha), R(\beta), R(\alpha-\beta) > 0.$
If we set $\beta = 0, \gamma = 1$ *and replace* a *by* $-a$ *in Cor.(2.1), we arrive[23] at*

Corollary 2.3. *The Volterra type integral equation*

$$D_x^{-\lambda}\{h(\tau)\} = \kappa\int_0^{\tau} h(\xi) \quad F_{\alpha}[-a,\xi]d\xi + \eta f(\tau) \tag{2.8}$$

has its solution given explicitly by

$$h(\tau) = \eta\sum_{r=0}^{\infty} \kappa^r \int_0^{\tau} F(\tau-\xi) \quad G_{\alpha,\lambda(r+1),r}[-a,\xi]d\xi \tag{2.9}$$

where $F_{\alpha}[-a,\xi]$ *is the F-function defined by Robotnov and Hartley[22] and* $R(\alpha) > 0.$

Conclusion

In this paper, we have presented a solution of a generalized fractional Volterra integral equation involving K_4-function with the help of the Sumudu transform. It is expected that some of the results derived in this survey may find applications in the solution of certain fractional order differential and integral equations arising problems of physical sciences and engineering areas.

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