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Idempotent Elements of the Semigroups $B_X(D)$ Defined by Semilattices of the Class $\Sigma_2(X, 8)$, When $Z_7 \cap Z_6 = \varnothing$

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Abstract

By the symbol $\Sigma_2(X, 8)$ we denote the class of all X- semilattices of unions whose every element is isomorphic to an X- semilattice of form $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\}$, where

$$Z_6 \subset Z_3 \subset Z_1 \subset \breve{D},\quad Z_6 \subset Z_4 \subset Z_1 \subset D,\quad Z_6 \subset Z_4 \subset Z_2 \subset \breve{D},\quad Z_7 \subset Z_4 \subset Z_1 \subset \breve{D},$$
$$Z_7 \subset Z_4 \subset Z_2 \subset \breve{D},\quad Z_7 \subset Z_5 \subset Z_2 \subset \breve{D};$$
$$Z_i \setminus Z_j \neq \varnothing,\ (i, j) \in \{(7,6),(6,7),(5,4),(4,5),(5,3),(3,5),(4,3),(3,4),(2,1),(1,2)\}.$$

The paper gives description of idempotent elements of the semigroup $B_X(D)$ which are defined by semilattices of the class $\Sigma_2(X, 8)$, for which intersection the minimal elements is empty. When X is a finite set, the formulas are derived, by means of which the number of idempotent elements of the semigroup is calculated.

Keywords: *Semilattice, Semigroup, Binary Relation, Idempotent Element.*

1 Introduction

Let X be an arbitrary nonempty set, D be a $X-$semilattice of unions, i.e. a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation α_f on the set X that satisfies the condition $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such α_f $(f: X\to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a $X-$ semilattice of unions D (see ([1], Item 2.1).

By $\varnothing$ we denote an empty binary relation or empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $T\in D$, $\varnothing\neq D'\subseteq D$ and $t\in \breve{D}=\bigcup_{Y\in D} Y$. Then by symbols we denote the following sets:

$$y\alpha=\{x\in X \mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y} y\alpha,\ V(D,\alpha)=\{Y\alpha \mid Y\in D\},$$
$$X^*=\{T\mid \varnothing\neq T\subseteq X\},\ D'_t=\{Z'\in D' \mid t\in Z'\},\ Y_T^\alpha=\{x\in X \mid x\alpha=T\},$$
$$D'_T=\{Z'\in D' \mid T\subseteq Z'\},\ \ddot{D}'_T=\{Z'\in D' \mid Z'\subseteq T\}.$$

By symbol $\wedge(D,D_t)$ we mean an exact lower bound of the set D' in the semilattice D.

Definition 1.1: *Let $\varepsilon\in B_X(D)$. If $\varepsilon\circ\varepsilon=\varepsilon$, then ε is called an idempotent element of the semigroup $B_X(D)$ and ε is called right unit if $\alpha\circ\varepsilon=\alpha$ for any $\alpha\in B_X(D)$ (see [1], [2], [3]).*

Definition 1.2: *We say that a complete X- semilattice of unions D is an $XI-$ semilattice of unions if it satisfies the following two conditions:*

a) $\wedge(D,D_t)\in D$ for any $t\in\breve{D}$;

b) $Z=\bigcup_{t\in Z}\wedge(D,D_t)$ for any nonempty element Z of D. (see ([1], definition 1.14.2), ([2] definition 1.14.2), [3], or [4]).

Definition 1.3: *Let $\alpha\in B_X(D)$, $T\in V(X^*,\alpha)$ and $Y_T^\alpha=\{y\in X \mid y\alpha=T\}$. A representation of a binary relation α of the form $\alpha=\cup_{T\in V(X^*,\alpha)}(Y_T^\alpha\times T)$ is called quasinormal.*

Note that, if $\alpha = \cup_{T\in V(X^*,\alpha)} \left(Y_T^\alpha \times T\right)$ is a quasinormal representation of a binary relation α, then the following conditions are true:

1) $X = \cup_{T\in V(X^*,\alpha)} Y_T^\alpha$;

2) $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$, for $T, T' \in V\left(X^*,\alpha\right)$ and $T \neq T'$;

Let $\sum_n (X, m)$ denote the class of all complete $X-$semilattice of unions where every element is isomorphic to a fixed semilattice D (see [1]).

Definition 1.4: *We say that a nonempty element T is a nonlimiting element of the set D' if $T \setminus l(D',T) \neq \varnothing$ and a nonempty element T is a limiting element of the set D' if $T \setminus l(D',T) = \varnothing$ (see ([1], Definition 1.13.1 and 1.13.2), ([2], Definition 1.13.1 and 1.13.2]).*

Theorem 1.1: *Let D be a complete $X-$semilattice of unions. The semigroup $B_X(D)$ possesses right unit iff D is an $XI-$semilattice of unions (see ([1], Theorem 6.1.3), ([2] Theorem6.1.3), or [5]).*

Theorem 1.2: *Let X be a finite set and $D(\alpha)$ be the set of all those elements T of the semilattice $Q = V(D,\alpha)\setminus\{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. A binary relation α having a quasinormal representation $\alpha = \bigcup\limits_{T\in V(D,\alpha)} \left(Y_T^\alpha \times T\right)$ is an idempotent element of this semigroup iff*

a) $V(D,\alpha)$ is complete $XI-$semilattise of unions;

b) $\bigcup\limits_{T'\in \ddot{D}(\alpha)_T} Y_{T'}^\alpha \supseteq T$ for any $T \in D(\alpha)$;

c) $Y_T^\alpha \cap T \neq \varnothing$ for any nonlimiting element of the set $\ddot{D}(\alpha)_T$ (see ([1], Theorem 6.3.9), ([2], Theorem 6.3.9) or [5]).

Theorem1.3. *Let D, $\Sigma(D)$, $E_X^{(r)}(D')$ and I denote respectively the complete $X-$semilattice of unions, the set of all $XI-$subsemilatices of the semilattice D, the set of all right units of the semigroup $B_X(D')$ and the set of all idempotents of the semigroup $B_X(D)$. Then for the sets $E_X^{(r)}(D')$ and I the following statements are true:*

b) if $\varnothing \notin D$, then

1) $E_X^{(r)}(D') \cap E_X^{(r)}(D'') = \varnothing$ for any elements D' and D'' of the set $\Sigma(D)$ that satisfy the condition $D' \neq D''$;

2) $I = \bigcup\limits_{D'\in\Sigma(D)} E_X^{(r)}(D')$;

3) The equality $|I| = \sum_{D' \in \Sigma(D)} \left| E_X^{(r)}(D') \right|$ is fulfilled for the finite set X (see ([1], statement b) Theorem (6.2.3) , ([2] statement b) Theorem 6.2.3), or [5]).

2 Results

By the symbol $\Sigma_2(X,8)$ we denote the class of all X – semilattices of unions whose every element is isomorphic to an X – semilattice of form $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, where

$$\begin{array}{lll} Z_6 \subset Z_3 \subset Z_1 \subset \breve{D}, & Z_6 \subset Z_4 \subset Z_1 \subset D, & Z_6 \subset Z_4 \subset Z_2 \subset \breve{D}, \\ Z_7 \subset Z_4 \subset Z_1 \subset \breve{D}, & Z_7 \subset Z_4 \subset Z_2 \subset \breve{D}, & Z_7 \subset Z_5 \subset Z_2 \subset \breve{D}, \\ Z_1 \setminus Z_2 \neq \varnothing, \quad Z_2 \setminus Z_1 \neq \varnothing, & Z_3 \setminus Z_4 \neq \varnothing, \quad Z_4 \setminus Z_3 \neq \varnothing, & Z_3 \setminus Z_5 \neq \varnothing, \\ Z_5 \setminus Z_3 \neq \varnothing, \quad Z_4 \setminus Z_5 \neq \varnothing, & Z_5 \setminus Z_4 \neq \varnothing, \quad Z_6 \setminus Z_7 \neq \varnothing, & Z_7 \setminus Z_6 \neq \varnothing. \end{array} \tag{1}$$

The semilattice satisfying the conditions (1) is shown in Figure 1. Let $C(D) = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ is a family sets, where $P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7$ are pairwise disjoint subsets of the set X and

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 \end{pmatrix}$$

is a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form:

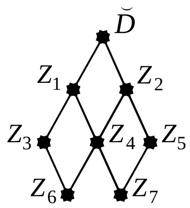

Fig. 1

$$\begin{aligned} \breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ Z_3 &= P_0 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ Z_4 &= P_0 \cup P_3 \cup P_5 \cup P_6 \cup P_7 \\ Z_5 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_6 \cup P_7 \\ Z_6 &= P_0 \cup P_5 \cup P_7 \\ Z_7 &= P_0 \cup P_3 \cup P_6 \end{aligned} \tag{2}$$

Here the elements P_1, P_2, P_3, P_5 are basis sources, the element P_0, P_4, P_6, P_7 is sources of completenes of the semilattice D. Therefore $|X| \geq 4$ and $\delta = 4$ (see ([1], Item 11.4), ([2], Item 11.4) or [3]).

Now assume that $D \in \Sigma_2(X,8)$. We introduce the following notation:

1) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 in figure 2);

2) $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$ (see diagram 2 in figure 2);

3) $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 in figure 2);

4) $Q_4 = \{T, T', T'', \breve{D}\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T'' \subset \breve{D}$ (see diagram 4 in figure 2);

5) $Q_5 = \{T, T', T'', T' \cup T''\}$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$ and $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ (see diagram 5 in figure 2);

6) $Q_6 = \{T, Z_4, Z, Z', \breve{D}\}$, where $T \in \{Z_7, Z_6\}$, $Z, Z' \in \{Z_2, Z_1\}$, $Z \neq Z'$, $Z \setminus Z' \neq \varnothing$, $Z' \setminus Z \neq \varnothing$ (see diagram 6 in figure 2);

7) $Q_7 = \{T, T', T'', T' \cup T'', \breve{D}\}$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$ and $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ (see diagram 7 in figure 2);

8) $Q_8 = \{T, T', Z_4, Z_4 \cup T', Z, \breve{D}\}$, where $T \in \{Z_7, Z_6\}$, $T' \in \{Z_5, Z_3\}$, $Z_4 \cup T'$, $Z \in \{Z_2, Z_1\}$, $Z_4 \cup T' \neq Z$, $T \subset T'$ and $T' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T' \neq \varnothing$, $(Z_4 \cup T') \setminus Z \neq \varnothing$, $Z \setminus (Z_4 \cup T') \neq \varnothing$ (see diagram 8 in figure 2);

9) $Q_9 = \{T, T', T \cup T'\}$, where $T, T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$ and $T \cap T' = \varnothing$ (see diagram 9 in figure 2);

10) $Q_{10} = \{T, T', T \cup T', T''\}$, where $T, T', T'' \in D$, $T \cup T' \subset T''$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$ and $T \cap T' = \varnothing$ (see diagram 10 in figure2);

11) $Q_{11} = \{Z_7, Z_6, Z_4, Z, \breve{D}\}$, where $Z \in \{Z_2, Z_1\}$ and $Z_7 \cap Z_6 = \varnothing$ (see diagram 11 in figure 2);

12) $Q_{12} = \{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\}$, where $Z_7 \cap Z_6 = \varnothing$ (see diagram 12 in figure 2);

13) $Q_{13} = \{T, T', T \cup T', T'', Z\}$, where $T, T', T'', Z \in D$, $(T \cup T') \subset Z$, $T' \subset T'' \subset Z$, $(T \cup T') \setminus T'' \neq \varnothing$, $T'' \setminus (T \cup T') \neq \varnothing$ and $T \cap T'' = \varnothing$ (see diagram 13 in figure 2);

14) $Q_{14} = \{T, T', Z_4, Z, Z', \breve{D}\}$, where $T, T', Z, Z' \in D$, $(T \cup T') \subset Z'$, $T' \subset Z \subset Z' \subset \breve{D}$, $Z_4 \setminus Z \neq \varnothing$, $Z \setminus Z_4 \neq \varnothing$ and $T \cap Z = \varnothing$ (see diagram 14 in figure 2);

15) $Q_{15} = \{T', T, Z_4, T'', Z, T'' \cup Z_4, \breve{D}\}$, where $T, T' \in \{Z_7, Z_6\}$, $T \neq T'$, $T \subset T''$, $T'' \in \{Z_5, Z_3\}$, $Z_4 \subset Z$, $Z \cup T'' \cup Z_4 = \breve{D}$, $(T'' \cup Z_4) \setminus Z \neq \varnothing$, $Z \setminus (T'' \cup Z_4) \neq \varnothing$, $T'' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T'' \neq \varnothing$ and $T' \cap T'' = \varnothing$ (see diagram 15 in figure 2);

16) $Q_{16} = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, where $Z_5 \cap Z_3 = \varnothing$ (see diagram 16 in figure 2).

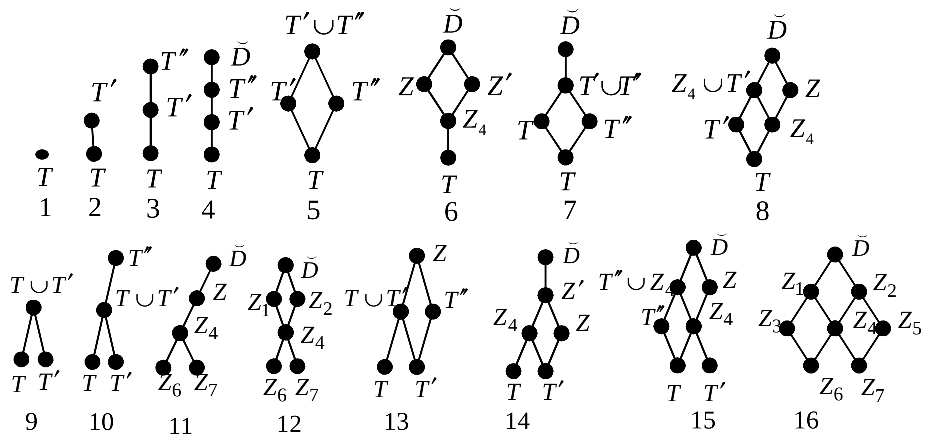

Fig.2

Denote by the symbol $\Sigma(Q_i)$ $(i=1,2,...,16)$ the set of all XI-subsemilattices of the semilattice D isomorphic to Q_i. Assume that $D'\in\Sigma(Q_i)$ and denote by the symbol $I(D')$ the set of all idempotent elements α of the semigroup $B_X(D')$, for which the semilattices $V(D,\alpha)$ and Q_i are mutually α isomorphic and $V(D,\alpha)=Q_i$.

Definition 2.1: *Let the symbol* $\Sigma'_{XI}(X,D)$ *denote the set of all* XI*-subsemilattices of the semilattice* D.

Let, further, $D,D'\in\Sigma'(X,D)$ and $\vartheta_{XI}\subseteq\Sigma'_{XI}(X,D)\times\Sigma'_{XI}(X,D)$. It is assumed that $D\vartheta_{XI}D'$ if and only if there exists some complete isomorphism φ between the semilattices D and D'. One can easily verify that the binary relation ϑ_{XI} is an equivalence relation on the set $\Sigma'_{XI}(X,D)$.

Let D' be an XI-subsemilattices of the semilattice . By $I(D')$ we denoted the set of all idempotent elements of the semigroup $B_X(D')$ and $\left|I^*(Q_i)\right|=\sum_{D'\in Q_i\vartheta_{XI}}\left|I(D')\right|$, where $i=1,2,...,16$.

Lemma 2.1: If $D\in\Sigma_2(X,8)$, then the following equalities are true:

1) $\left|I(Q_1)\right|=1$;

2) $\left|I(Q_2)\right|=\left(2^{|T'\setminus T|}-1\right)\cdot 2^{|X\setminus T'|}$;

3) $\left|I(Q_3)\right|=\left(2^{|T'\setminus T|}-1\right)\cdot\left(3^{|T''\setminus T'|}-2^{|T''\setminus T'|}\right)\cdot 3^{|X\setminus T''|}$;

4) $\left|I(Q_4)\right|=\left(2^{|T'\setminus T|}-1\right)\cdot\left(3^{|T''\setminus T'|}-2^{|T''\setminus T'|}\right)\cdot\left(4^{|\breve{D}\setminus T''|}-3^{|\breve{D}\setminus T''|}\right)\cdot 4^{|X\setminus\breve{D}|}$;

5) $\left|I(Q_5)\right|=\left(2^{|T'\setminus T''|}-1\right)\cdot\left(2^{|T''\setminus T'|}-1\right)\cdot 4^{|X\setminus(T'\cup T'')|}$;

6) $\left|I(Q_6)\right|=\left(2^{|Z_4\setminus T|}-1\right)\cdot 2^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(3^{|Z\setminus Z'|}-2^{|Z\setminus Z'|}\right)\cdot\left(3^{|Z'\setminus Z|}-2^{|Z'\setminus Z|}\right)\cdot 5^{|X\setminus\breve{D}|}$;

7) $\left|I(Q_7)\right|=\left(2^{|T'\setminus T''|}-1\right)\cdot\left(2^{|T''\setminus T'|}-1\right)\cdot\left(5^{|\breve{D}\setminus(T'\cup T'')|}-4^{|\breve{D}\setminus(T'\cup T'')|}\right)\cdot 5^{|X\setminus\breve{D}|}$;

8) $\left|I(Q_8)\right|=\left(2^{|T'\setminus Z|}-1\right)\cdot\left(2^{|Z_4\setminus T'|}-1\right)\cdot\left(3^{|Z\setminus(Z_4\cup T')|}-2^{|Z\setminus(Z_4\cup T')|}\right)\cdot 6^{|X\setminus\breve{D}|}$;

9) $\left|I(Q_9)\right|=3^{|X\setminus(T\cup T')|}$;

10) $\left|I(Q_{10})\right|=\left(4^{|T''\setminus(T\cup T')|}-3^{|T''\setminus(T\cup T')|}\right)\cdot 4^{|X\setminus T''|}$;

11) $\left|I(Q_{11})\right|=\left(4^{|Z\setminus Z_4|}-3^{|Z\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z|}-4^{|\breve{D}\setminus Z|}\right)\cdot 5^{|X\setminus\breve{D}|}$;

12) $\left|I(Q_{12})\right|=\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}$;

13) $\left|I(Q_{13})\right|=\left(2^{|T''\setminus(T'\cup T)|}-1\right)\cdot 5^{|X\setminus Z|}$;

14) $\left|I(Q_{14})\right|=\left(2^{|Z\setminus Z_4|}-1\right)\cdot\left(6^{|\breve{D}\setminus Z'|}-5^{|\breve{D}\setminus Z'|}\right)\cdot 6^{|X\setminus\breve{D}|}$;

15) $\left|I\left(Q_{15}\right)\right|=\left(2^{|T''\setminus Z|}-1\right)\cdot\left(4^{|Z\setminus(T''\cup Z_4)|}-3^{|Z\setminus(T''\cup Z_4)|}\right)\cdot 7^{|X\setminus\breve{D}|}$;

16) $\left|I\left(Q_{16}\right)\right|=\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 8^{|X\setminus\breve{D}|}$ (see [6] Lemma 3.3).

Theorem 2.1: *Let* $D\in\Sigma_2(X,8)$, $Z_7\cap Z_6=\varnothing,\ Z_7\cap Z_3\neq\varnothing,\ Z_6\cap Z_5\neq\varnothing$ *and* $\alpha\in B_X(D)$. *The binary relation* α *is an idempotent relation of the semmigroup* $B_X(D)$ *iff binary relation* α *satisfies only one condition of the following conditions:*

1) $\alpha=X\times T$, where $T\in D$;

2) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)$, where $T,T'\in D, T\subset T'$, $Y_T^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\cap T'\neq\varnothing$;

3) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)$, where $T,T',T''\in D$, $T\subset T'\subset T''$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

4) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T,T',T''\in D$ and $T\subset T'\subset T''\subset\breve{D}$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

5) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times T'\cup T''\right)$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

6) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_Z^\alpha\times Z\right)\cup\left(Y_{Z'}^\alpha\times Z'\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_7,Z_6\}$, $Z,Z'\in\{Z_2,Z_1\}$, $Z\neq Z'$, $Z\setminus Z'\neq\varnothing$, $Z'\setminus Z\neq\varnothing$, $Y_T^\alpha,Y_4^\alpha,Y_Z^\alpha,Y_{Z'}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z$, $Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha\supseteq Z'$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_Z^\alpha\cap Z_Z\neq\varnothing$, $Y_{Z'}^\alpha\cap Z'\neq\varnothing$

7) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T,T',T''\in D$ and $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

8) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_{T'\cup Z_4}^\alpha\times\left(T'\cup Z_4\right)\right)\cup\left(Y_Z^\alpha\times Z\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_7,Z_6\}$, $T'\in\{Z_5,Z_3\}$, $Z_4\cup T'$, $Z\in\{Z_2,Z_1\}$, $Z_4\cup T'\neq Z$, $T\subset T'$, $T'\setminus Z_4\neq\varnothing$, $Z_4\setminus T'\neq\varnothing$, $\left(Z_4\cup T'\right)\setminus Z\neq\varnothing$, $Z\setminus\left(Z_4\cup T'\right)\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_4^\alpha,Y_Z^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_Z^\alpha\cap Z\neq\varnothing$;

9) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T\cup T'}^\alpha\times(T\cup T')\right)$, where $T,T'\in D$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\supseteq T'$;

10) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T\cup T'}^\alpha\times(T\cup T')\right)\cup\left(Y_{T''}^\alpha\times T''\right)$, where $T,T',T''\in D$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\supseteq T'$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

11) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_Z^\alpha\times Z\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z\in\{Z_2,Z_1\}$, $Y_7^\alpha,Y_6^\alpha,Y_Z^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions:
$Y_7^\alpha\supseteq Z_7$, $Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z$, $Y_Z^\alpha\cap Z\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

12) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_7^\alpha,Y_6^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_2^\alpha\cap Z_2\neq\varnothing$, $Y_1^\alpha\cap Z_1\neq\varnothing$.

Proof: In this case, when $Z_7\cap Z_6=\varnothing$, from the Lemma 2.3 in [6] it follows that diagrams 1-12 given in fig.1 exhibit all diagrams of $XI-$subsemilattices of the semilattices D, a quasinormal representation of idempotent elements of the semigroup $B_X(D)$, which are defined by these $XI-$semilattices, may have one of the forms listed above. The statements 1)-4) immediately follows from the Corollary 13.1.1 in [1], Corollary 13.1.1 in [2], the statements 5)-7) immediately follows from the Corollary 13.3.1 in [1], Corollary 13.3.1 in [2] and the statement 8) immediately follows from the Theorems 13.7.2 in [1], Theorems 13.7.2 in [2], The statements 9)-11) immediately follows from the Corollary 13.2.1 in [1], 13.2.1 in [2], the statement 12) immediately follows from the Corollary 13.5.1 in [1], 13.5 in [2].

The Theorem is proved.

Lemma 2.2: *Let* $D\in\Sigma_2(X,8)$ *and* $Z_7\cap Z_6=\varnothing$, $Z_7\cap Z_3\neq\varnothing$, $Z_6\cap Z_5\neq\varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_9)\right|$ *can be calculated by the formula*

$$\left|I^*(Q_9)\right|=3^{\left|X\setminus Z_4\right|}.$$

Proof: By definition of the given semilattice D we have $Q_9\vartheta_{XI}=\{\{Z_7,Z_6,Z_4\}\}$. If the following equality is hold $D_1'=\{Z_7,Z_6,Z_4\}$, then $\left|I^*(Q_9)\right|=\left|I(D_1')\right|$. From this equality and statement (9) of Lemma 2.1 we obtain validity of Lemma 2.2.

The Lemma is proved.

Lemma 2.3: *Let* $D\in\Sigma_2(X,8)$ *and* $Z_7\cap Z_6=\varnothing$, $Z_7\cap Z_3\neq\varnothing$, $Z_6\cap Z_5\neq\varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_{10})\right|$ *can be calculated by the formula*

$$\left|I^*(Q_{10})\right| = \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|} + \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|} + \left(4^{|\breve{D}\setminus Z_4|} - 3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus \breve{D}|}.$$

Proof: By definition of the given semilattice D we have

$$Q_{10}\theta_{XI} = \left\{\{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_6, Z_4, \breve{D}\}\right\}.$$

If the following equalities are hold

$D_1' = \{Z_7, Z_6, Z_4, Z_2\}, D_2' = \{Z_7, Z_6, Z_4, Z_1\}, D_3' = \{Z_7, Z_6, Z_4, \breve{D}\}$, then

$$\left|I^*(Q_{10})\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right|$$

From this equality and statement (10) of Lemma 2.1 we obtain validity of Lemma 2.3.

The Lemma is proved.

Lemma 2.4: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 \neq \varnothing$, $Z_6 \cap Z_5 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_{11})\right|$ *can be calculated by the formula*

$$\left|I^*(Q_{11})\right| = \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|} - 4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus \breve{D}|} + \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|} - 4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}.$$

Proof: By definition of the given semilattice D we have

$$Q_{11}\theta_{XI} = \left\{\{Z_7, Z_6, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_1, \breve{D}\}\right\}.$$

If the following equalities are hold $D_1' = \{Z_7, Z_6, Z_4, Z_2, \breve{D}\}, D_2' = \{Z_7, Z_6, Z_4, Z_1, \breve{D}\}$, then

$$\left|I^*(Q_{11})\right| = \left|I(D_1')\right| + \left|I(D_2')\right|$$

From this equality and statement (11) of Lemma 2.1 we obtain validity of Lemma 2.4.

The Lemma is proved.

Lemma 2.5: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 \neq \varnothing$, $Z_6 \cap Z_5 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_{12})\right|$ *can be calculated by the formula*

$$\left|I^*(Q_{12})\right| = 3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|} - 3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}.$$

Proof: By definition of the given semilattice D we have

$Q_{12}\theta_{XI} = \left\{\left\{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\right\}\right\}$.

If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\right\}$, then $\left|I^*(Q_{12})\right| = \left|I(D_1')\right|$

From this equality and statement (12) of Lemma 2.1 we obtain validity of Lemma 2.5.

The Lemma is proved.

It was seen in [6] that $k_1 = \sum_{i=1}^{8}\left|I^*(Q_i)\right|$. Now, let us assume that

$$\begin{aligned} k_2 &= \left|I^*(Q_9)\right| + \left|I^*(Q_{10})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{12})\right| = \\ &= 3^{|X\setminus Z_4|} + \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|} + \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|} + \left(4^{|\breve{D}\setminus Z_4|} - 3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus \breve{D}|} + \\ &+ \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|} - 4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus \breve{D}|} + \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|} - 4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|} + \\ &+ 3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|} - 3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|} \end{aligned}$$

Theorem 2.2: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 \neq \varnothing$, $Z_6 \cap Z_5 \neq \varnothing$. *If* X *is a finite set and* I_D *is the set of all idempotent elements of the semigroup* $B_X(D)$, *then* $|I_D| = k_1 + k_2$.

Proof: This Theorem immediately follows from the Theorem 2.1.

The Theorem is proved.

Theorem 3.1: *Let* $D \in \Sigma_2(X,8)$, $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 = \varnothing$, $Z_6 \cap Z_5 \neq \varnothing$ *and* $\alpha \in B_X(D)$. *The binary relation* α *is an idempotent relation of the semmigroup* $B_X(D)$ *iff binary relation* α *satisfies only one condition of the Theorem 2.1 and only one following conditions:*

13) $\alpha = \left(Y_7^\alpha \times Z_7\right) \cup \left(Y_6^\alpha \times Z_6\right) \cup \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3 \times Z_3\right) \cup \left(Y_1^\alpha \times Z_1\right)$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq Z_7$, $Y_6^\alpha \supseteq Z_6$, $Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_3^\alpha \cap Z_3 \neq \varnothing$;

14) $\alpha = \left(Y_7^\alpha \times Z_7\right) \cup \left(Y_6^\alpha \times Z_6\right) \cup \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions:

$Y_7^\alpha \supseteq Z_7$, $Y_6^\alpha \supseteq Z_6$, $Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

15) $\alpha = \left(Y_7^\alpha \times Z_7\right) \cup \left(Y_6^\alpha \times Z_6\right) \cup \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$ and satisfies the conditions:

$Y_7^\alpha \supseteq Z_7,\ Y_6^\alpha \supseteq Z_6,\ Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3,\ Y_6^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \supseteq Z_2,\ Y_3^\alpha \cap Z_3 \neq \varnothing,\ Y_2^\alpha \cap Z_2 \neq \varnothing;$

Proof: In this case, when $Z_7 \cap Z_6 = \varnothing,\ Z_7 \cap Z_3 = \varnothing,\ Z_6 \cap Z_5 \neq \varnothing$, from the Lemma 2.4 in [6] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of $XI-$ subsemilattices of the semilattices D, a quasinormal representation of idempotent elements of the semigroup $B_X(D)$, which are defined by these $XI-$ semilattices, may have one of the forms listed above. The statements 13), 14) immediately follows from the Corollary 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.2 in [1].

The Theorem is proved.

Lemma 3.1: *Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 = \varnothing,\ Z_7 \cap Z_3 = \varnothing,\ Z_6 \cap Z_5 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_9)\right|$ can be calculated by the formula*

$$\left|I^*(Q_9)\right| = 3^{|X \setminus Z_1|} + 3^{|X \setminus Z_4|}.$$

Proof: By definition of the given semilattice D we have

$$Q_9\theta_{XI} = \{\{Z_7, Z_6, Z_4\}, \{Z_7, Z_3, Z_1\}\}.$$

If the following equalities are hold $D_1' = \{Z_7, Z_6, Z_4\}, D_2' = \{Z_7, Z_3, Z_1\}$, then $\left|I^*(Q_9)\right| = \left|I(D_1')\right| + \left|I(D_2')\right|$. From this equality and statement (9) of Lemma 2.1 we obtain validity of Lemma 3.1.

The Lemma is proved.

Lemma 3.2: *Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 = \varnothing,\ Z_7 \cap Z_3 = \varnothing,\ Z_6 \cap Z_5 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{10})\right|$ can be calculated by the formula*

$$\left|I^*(Q_{10})\right| = \left(4^{|\breve{D} \setminus Z_1|} - 3^{|\breve{D} \setminus Z_1|}\right) \cdot 4^{|X \setminus \breve{D}|} + \left(4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|}\right) \cdot 4^{|X \setminus Z_2|} + \\ + \left(4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}\right) \cdot 4^{|X \setminus Z_1|} + \left(4^{|\breve{D} \setminus Z_4|} - 3^{|\breve{D} \setminus Z_4|}\right) \cdot 4^{|X \setminus \breve{D}|}$$

Proof: By definition of the given semilattice D we have

$$Q_{10}\theta_{XI} = \{\{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_6, Z_4, \breve{D}\}, \{Z_7, Z_3, Z_1, \breve{D}\}\}.$$

If the following equalities are hold

$$D_1' = \{Z_7, Z_6, Z_4, Z_2\}, D_2' = \{Z_7, Z_6, Z_4, Z_1\}, D_3' = \{Z_7, Z_6, Z_4, \breve{D}\}, D_4' = \{Z_7, Z_3, Z_1, \breve{D}\},$$

Then $\left|I^{*}(Q_{10})\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right|$. From this equality and statement (10) of Lemma 2.1 we obtain validity of Lemma 3.2.

The Lemma is proved.

Lemma 3.3: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_7 \cap Z_3 = \varnothing,\ Z_6 \cap Z_5 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^{*}(Q_{13})\right|$ *can be calculated by the formula*

$$\left|I^{*}(Q_{13})\right| = \left(2^{\left|Z_3 \setminus Z_4\right|} - 1\right) \cdot 5^{\left|X \setminus Z_1\right|}.$$

Proof: By definition of the given semilattice D we have $Q_{13}\theta_{XI} = \left\{\left\{Z_7, Z_6, Z_4, Z_3, Z_1\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_4, Z_3, Z_1\right\}$, then $\left|I^{*}(Q_{13})\right| = \left|I(D_1')\right|$. From this equality and statement (13) of Lemma 2.1 we obtain validity of Lemma 3.3.

The Lemma is proved.

Lemma 3.4: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_7 \cap Z_3 = \varnothing,\ Z_6 \cap Z_5 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^{*}(Q_{14})\right|$ *can be calculated by the formula*

$$\left|I^{*}(Q_{14})\right| = \left(2^{\left|Z_3 \setminus Z_4\right|} - 1\right) \cdot \left(6^{\left|\breve{D} \setminus Z_1\right|} - 5^{\left|\breve{D} \setminus Z_1\right|}\right) \cdot 6^{\left|X \setminus \breve{D}\right|}.$$

Proof: By definition of the given semilattice D we have $Q_{14}\theta_{XI} = \left\{\left\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\right\}$, then $\left|I^{*}(Q_{14})\right| = \left|I(D_1')\right|$. From this equality and statement (14) of Lemma 2.1 we obtain validity of Lemma 3.4.

The Lemma is proved.

Lemma 3.5: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_7 \cap Z_3 = \varnothing,\ Z_6 \cap Z_5 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^{*}(Q_{15})\right|$ *can be calculated by the formula*

$$\left|I^{*}(Q_{15})\right| = \left(2^{\left|Z_3 \setminus Z_2\right|} - 1\right) \cdot \left(4^{\left|Z_2 \setminus Z_1\right|} - 3^{\left|Z_2 \setminus Z_1\right|}\right) \cdot 7^{\left|X \setminus \breve{D}\right|}.$$

Proof: By definition of the given semilattice D we have

$$Q_{15}\theta_{XI} = \left\{\left\{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}\right\}.$$

If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$, then $\left|I^{*}(Q_{15})\right| = \left|I(D_1')\right|$. From this equality and statement (15) of Lemma 2.1 we obtain validity of Lemma 3.5.

The Lemma is proved.

Let us assume that

$$\begin{aligned}
k_3 &= \left|I^*(Q_9)\right| + \left|I^*(Q_{10})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{12})\right| + \left|I^*(Q_{13})\right| + \left|I^*(Q_{14})\right| + \left|I^*(Q_{15})\right| = \\
&= 3^{|X\setminus Z_1|} + 3^{|X\setminus Z_4|} + \left(4^{|\breve{D}\setminus Z_1|} - 3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus \breve{D}|} + \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|} + \\
&+ \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|} + \left(4^{|\breve{D}\setminus Z_4|} - 3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus \breve{D}|} + \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|} - 4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus \breve{D}|} + \\
&+ \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|} - 4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|} + 3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|} - 3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|} + \\
&+ \left(2^{|Z_3\setminus Z_4|} - 1\right)\cdot 5^{|X\setminus Z_1|} + \left(2^{|Z_3\setminus Z_4|} - 1\right)\cdot\left(6^{|\breve{D}\setminus Z_1|} - 5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|} + \left(2^{|Z_3\setminus Z_2|} - 1\right)\cdot\left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right)\cdot 7^{|X\setminus \breve{D}|}
\end{aligned}$$

Theorem 3.2: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_7 \cap Z_3 = \varnothing,\ Z_6 \cap Z_5 \neq \varnothing$. *If* X *is a finite set and* I_D *is the set of all idempotent elements of the semigroup* $B_X(D)$, *then* $|I_D| = k_1 + k_3$.

Proof: This Theorem immediately follows from the Theorem 3.1.

The Theorem is proved.

Theorem 4.1: Let $D \in \Sigma_2(X,8)$, $Z_7 \cap Z_6 = \varnothing,\ Z_6 \cap Z_5 = \varnothing,\ Z_7 \cap Z_3 \neq \varnothing$ and $\alpha \in B_X(D)$. The binary relation α is an idempotent relation of the semmigroup $B_X(D)$ iff binary relation α satisfies only one condition of the Theorem 2.1 and only one following conditions:

13) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2)$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq Z_7,\ Y_6^\alpha \supseteq Z_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5,\ Y_5^\alpha \cap Z_5 \neq \varnothing$;

14) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D})$,
where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions:

$$Y_7^\alpha \supseteq Z_7,\ Y_6^\alpha \supseteq Z_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5,\ Y_5 \cap Z_5 \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing;$$

15) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where

$Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and satisfies the conditions:

$$Y_7^\alpha \supseteq Z_7,\ Y_6^\alpha \supseteq Z_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_1^\alpha \supseteq Z_1,\ Y_5^\alpha \cap Z_5 \neq \varnothing,\ Y_1^\alpha \cap Z_1 \neq \varnothing.$$

Proof: In this case, when $Z_7 \cap Z_6 = \varnothing,\ Z_6 \cap Z_5 = \varnothing,\ Z_7 \cap Z_3 \neq \varnothing$, from the Lemma 2.5 in [6] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of $XI-$ subsemilattices of the semilattices *D*, a quasinormal representation of idempotent elements of the semigroup $B_X(D)$, which are defined by these XI – semilattices, may have one of the forms listed above. The statements 13), 14) immediately follows from the Corollary 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.2 in [1].

The Theorem is proved.

Lemma 4.1: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_6 \cap Z_5 = \varnothing,\ Z_7 \cap Z_3 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_9)\right|$ *can be calculated by the formula*

$$\left|I^*(Q_9)\right| = 3^{|X \setminus Z_2|} + 3^{|X \setminus Z_4|}.$$

Proof: By definition of the given semilattice D we have $Q_9\theta_{XI} = \{\{Z_7, Z_6, Z_4\}, \{Z_6, Z_5, Z_2\}\}$. If the following equalities are hold $D_1' = \{Z_7, Z_6, Z_4\}, D_2' = \{Z_6, Z_5, Z_2\}$, then $\left|I^*(Q_9)\right| = \left|I(D_1')\right| + \left|I(D_2')\right|$. From this equality and statement (9) of Lemma 2.1 we obtain validity of Lemma 4.1.

The Lemma is proved.

Lemma 4.2: Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 = \varnothing,\ Z_6 \cap Z_5 = \varnothing,\ Z_7 \cap Z_3 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{10})\right|$ can be calculated by the formula

$$\begin{aligned}\left|I^*(Q_{10})\right| &= \left(4^{|\breve{D} \setminus Z_2|} - 3^{|\breve{D} \setminus Z_2|}\right) \cdot 4^{|X \setminus \breve{D}|} + \left(4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|}\right) \cdot 4^{|X \setminus Z_2|} + \\ &+ \left(4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}\right) \cdot 4^{|X \setminus Z_1|} + \left(4^{|\breve{D} \setminus Z_4|} - 3^{|\breve{D} \setminus Z_4|}\right) \cdot 4^{|X \setminus \breve{D}|}.\end{aligned}$$

Proof: By definition of the given semilattice D we have

$$Q_{10}\theta_{XI} = \left\{\{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_6, Z_4, \breve{D}\}, \{Z_6, Z_5, Z_2, \breve{D}\}\right\}.$$

If the following equalities are hold

$$D_1' = \{Z_7, Z_6, Z_4, Z_2\}, D_2' = \{Z_7, Z_6, Z_4, Z_1\}, D_3' = \{Z_7, Z_6, Z_4, \breve{D}\}, D_4' = \{Z_6, Z_5, Z_2, \breve{D}\},$$

then $\left|I^*(Q_{10})\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right|$. From this equality and statement (10) of Lemma 2.1 we obtain validity of Lemma 4.2.

The Lemma is proved.

Lemma 4.3: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_6 \cap Z_5 = \varnothing,\ Z_7 \cap Z_3 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_{13})\right|$ *can be calculated by the formula*

$$\left|I^*(Q_{13})\right| = \left(2^{|Z_5\setminus Z_4|}-1\right)\cdot 5^{|X\setminus Z_2|}.$$

Proof: By definition of the given semilattice D we have $Q_{13}\theta_{XI} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2\}\}$. If the following equality is hold $D_1' = \{Z_7, Z_6, Z_5, Z_4, Z_2\}$, then $\left|I^*(Q_{13})\right| = \left|I(D_1')\right|$. From this equality and statement (13) of Lemma 2.1 we obtain validity of Lemma 4.3.

The Lemma is proved.

Lemma 4.4: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_6 \cap Z_5 = \varnothing,\ Z_7 \cap Z_3 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_{14})\right|$ *can be calculated by the formula*

$$\left|I^*(Q_{14})\right| = \left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus \breve{D}|}.$$

Proof: By definition of the given semilattice D we have $Q_{14}\theta_{XI} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, \breve{D}\}\}$. If the following equality is hold $D_1' = \{Z_7, Z_6, Z_5, Z_4, Z_2, \breve{D}\}$, then $\left|I^*(Q_{14})\right| = \left|I(D_1')\right|$. From this equality and statement (14) of Lemma 2.1 we obtain validity of Lemma 4.4.

The Lemma is proved.

Lemma 4.5: *Let* $D \in \Sigma_2(X,8)$ *and* $Z_7 \cap Z_6 = \varnothing,\ Z_6 \cap Z_5 = \varnothing,\ Z_7 \cap Z_3 \neq \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_{15})\right|$ *can be calculated by the formula*

$$\left|I^*(Q_{15})\right| = \left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot 7^{|X\setminus \breve{D}|}.$$

Proof: By definition of the given semilattice D we have $Q_{15}\theta_{XI} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}\}$. If the following equality is hold $D_1' = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, then $\left|I^*(Q_{15})\right| = \left|I(D_1')\right|$. From this equality and statement (15) of Lemma 2.1 we obtain validity of Lemma 4.5.

The Lemma is proved.

Let us assume that

$$\begin{aligned}
k_4 &= \left|I^*(Q_9)\right| + \left|I^*(Q_{10})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{12})\right| + \left|I^*(Q_{13})\right| + \left|I^*(Q_{14})\right| + \left|I^*(Q_{15})\right| = \\
&= 3^{|X\setminus Z_2|} + 3^{|X\setminus Z_4|} + \left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus \breve{D}|} + \left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|} + \\
&+ \left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|} + \left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus \breve{D}|} + \left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus \breve{D}|} + \\
&+ \left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|} + 3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|} + \\
&+ \left(2^{|Z_5\setminus Z_4|}-1\right)\cdot 5^{|X\setminus Z_2|} + \left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus \breve{D}|} + \left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot 7^{|X\setminus \breve{D}|}
\end{aligned}$$

Theorem 4.2: *Let* $D\in\Sigma_2(X,8)$ *and* $Z_7\cap Z_6=\varnothing,\ Z_6\cap Z_5=\varnothing,\ Z_7\cap Z_3\neq\varnothing$. *If* X *is a finite set and* I_D *is the set of all idempotent elements of the semigroup* $B_X(D)$, *then* $|I_D|=k_1+k_4$.

Proof: This Theorem immediately follows from the Theorem 4.1.

The Theorem is proved.

Theorem 5.1: Let $D\in\Sigma_2(X,8)$, $Z_7\cap Z_6=\varnothing,\ Z_6\cap Z_5=\varnothing,\ Z_7\cap Z_3=\varnothing,\ Z_5\cap Z_3\neq\varnothing$ and $\alpha\in B_X(D)$. The binary relation α is an idempotent relation of the semmigroup $B_X(D)$ iff binary relation α satisfies only one conditions of the Theorem 3.1 and only one conditions of the Theorem 4.1.

Proof: In this case, when $Z_7\cap Z_6=\varnothing,\ Z_6\cap Z_5=\varnothing,\ Z_7\cap Z_3=\varnothing,\ Z_5\cap Z_3\neq\varnothing$, from the Lemma 2.6 in [6] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of $XI-$subsemilattices of the semilattices D, a quasinormal representation of idempotent elements of the semigroup $B_X(D)$, which are defined by these $XI-$ semilattices, may have one of the forms listed above. This Theorem immediately follows from the Theorems 3.1 and 4.1.

The Theorem is proved.

Let us assume that

$$
\begin{aligned}
k_5 &= \left|I^*(Q_9)\right|+\left|I^*(Q_{10})\right|+\left|I^*(Q_{11})\right|+\left|I^*(Q_{12})\right|+\left|I^*(Q_{13})\right|+\left|I^*(Q_{14})\right|+\left|I^*(Q_{15})\right|=\\
&=3^{|X\setminus Z_1|}+3^{|X\setminus Z_2|}+3^{|X\setminus Z_4|}+\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|}+\\
&+\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|}+\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}+\\
&+\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}+3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}+\\
&+\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot 5^{|X\setminus Z_2|}+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 5^{|X\setminus Z_1|}+\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus\breve{D}|}+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}+\\
&+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 7^{|X\setminus\breve{D}|}+\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot 7^{|X\setminus\breve{D}|}
\end{aligned}
$$

Theorem 5.2: *Let* $D\in\Sigma_2(X,8)$ *and* $Z_7\cap Z_6=\varnothing,\ Z_7\cap Z_3=\varnothing,\ Z_6\cap Z_5=\varnothing,\ Z_5\cap Z_3\neq\varnothing$. *If* X *is a finite set and* I_D *is the set of all idempotent elements of the semigroup* $B_X(D)$, *then* $|I_D|=k_1+k_5$.

Proof: This Theorem immediately follows from the Theorem 5.1.

The Theorem is proved.

Theorem 6.1: *Let* $D\in\Sigma_2(X,8)$, $Z_5\cap Z_3=\varnothing$ *and* $\alpha\in B_X(D)$. *The Binary relation* α *is an idempotent relation of the semmigroup* $B_X(D)$ *iff binary relation* α *satisfies only one conditions of the Theorem 5.1 and only one following condition:*

13) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T\cup T'}^\alpha\times(T\cup T')\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_Z\times Z\right)$, where

$T,T',T'',Z\in D$, $(T\cup T')\subset Z$, $T'\subset T''\subset Z$, $(T\cup T')\setminus T''\neq\varnothing$, $T''\setminus(T\cup T')\neq\varnothing$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\supseteq T'$, $Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

16) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$,
where $Y_7^\alpha,Y_6^\alpha,Y_5^\alpha,Y_3^\alpha\notin\{\varnothing\}$ and satisfies the conditions:

$$Y_7^\alpha\supseteq Z_7,\quad Y_6^\alpha\supseteq Z_6,\quad Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5,\quad Y_6^\alpha\cup Y_3^\alpha\supseteq Z_3,\quad Y_5^\alpha\cap Z_5\neq\varnothing,\ Y_3^\alpha\cap Z_3\neq\varnothing.$$

Proof: In this case, when $Z_5\cap Z_3=\varnothing$, from the Lemma 2.7 in [6] it follows that diagrams 1-16 given in fig.1 exhibit all diagrams of $XI-$ subsemilattices of the semilattices D, a quasinormal representation of idempotent elements of the semigroup $B_X(D)$, which are defined by these $XI-$ semilattices, may have one of the forms listed above. The statement 13) immediately follows from the Corollary 13.4.1 in [1], 13.4.1 in [2]. Now we will proof the statement 16). It is easy to see, that the set $D(\alpha)=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ is a generating set of the semilattice D. Then the following equalities are hold:

$$\ddot{D}(\alpha)_{Z_7}=\{Z_7\},\ \ddot{D}(\alpha)_{Z_6}=\{Z_6\},\ \ddot{D}(\alpha)_{Z_5}=\{Z_7,Z_5\},\ \ddot{D}(\alpha)_{Z_4}=\{Z_7,Z_6,Z_4\},$$
$$\ddot{D}(\alpha)_{Z_3}=\{Z_6,Z_3\},\ddot{D}(\alpha)_{Z_2}=\{Z_7,Z_6,Z_5,Z_4,Z_2\},\ \ddot{D}(\alpha)_{Z_1}=\{Z_7,Z_6,Z_4,Z_3,Z_1\}.$$

By statement b) of the Theorem 1.2 follows that the following conditions are true:

$$Y_7^\alpha\supseteq Z_7,\ Y_6^\alpha\supseteq Z_6,\ Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5,\ Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\supseteq Z_4,\ Y_6^\alpha\cup Y_3^\alpha\supseteq Z_3,$$
$$Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2,\ Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_3^\alpha\cup Y_1^\alpha\supseteq Z_1;$$

$$Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\supseteq Z_7\cup Z_6\cup Y_4^\alpha=Z_4\cup Y_4^\alpha\supseteq Z_4,$$
$$Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha=\left(Y_7^\alpha\cup Y_5^\alpha\right)\cup\left(Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\right)\cup Y_2^\alpha\supseteq$$
$$\supseteq Z_5\cup Z_4\cup Y_2^\alpha=Z_2\cup Y_2^\alpha\supseteq Z_2,$$
$$Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_3^\alpha\cup Y_1^\alpha=\left(Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\right)\cup\left(Y_6^\alpha\cup Y_3^\alpha\right)\cup Y_1^\alpha\supseteq$$
$$\supseteq Z_4\cup Z_3\cup Y_1^\alpha=Z_1\cup Y_1^\alpha\supseteq Z_1,$$

i.e., the inclusions

$Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_3^\alpha\cup Y_1^\alpha\supseteq Z_1$ are always hold. Further, it is to see, that the following conditions are true:

$$l\left(\ddot{D}_{Z_7}, Z_7\right) = \cup\left(\ddot{D}_{Z_7} \setminus \{Z_7\}\right) = \varnothing,\ Z_7 \setminus l\left(\ddot{D}_{Z_7}, Z_7\right) = Z_7 \setminus \varnothing \neq \varnothing;$$
$$l\left(\ddot{D}_{Z_6}, Z_6\right) = \cup\left(\ddot{D}_{Z_6} \setminus \{Z_6\}\right) = \varnothing,\ Z_6 \setminus l\left(\ddot{D}_{Z_6}, Z_6\right) = Z_6 \setminus \varnothing \neq \varnothing;$$
$$l\left(\ddot{D}_{Z_5}, Z_5\right) = \cup\left(\ddot{D}_{Z_5} \setminus \{Z_5\}\right) = Z_7,\ Z_5 \setminus l\left(\ddot{D}_{Z_5}, Z_5\right) = Z_5 \setminus Z_7 \neq \varnothing;$$
$$l\left(\ddot{D}_{Z_3}, Z_3\right) = \cup\left(\ddot{D}_{Z_3} \setminus \{Z_3\}\right) = Z_6,\ Z_3 \setminus l\left(\ddot{D}_{Z_3}, Z_3\right) = Z_3 \setminus Z_6 \neq \varnothing;$$
$$l\left(\ddot{D}_{Z_4}, Z_4\right) = \cup\left(\ddot{D}_{Z_4} \setminus \{Z_4\}\right) = Z_4,\ Z_4 \setminus l\left(\ddot{D}_{Z_4}, Z_4\right) = Z_4 \setminus Z_4 = \varnothing;$$
$$l\left(\ddot{D}_{Z_2}, Z_2\right) = \cup\left(\ddot{D}_{Z_2} \setminus \{Z_2\}\right) = Z_2,\ Z_2 \setminus l\left(\ddot{D}_{Z_2}, Z_2\right) = Z_2 \setminus Z_2 = \varnothing;$$
$$l\left(\ddot{D}_{Z_1}, Z_1\right) = \cup\left(\ddot{D}_{Z_1} \setminus \{Z_1\}\right) = Z_1,\ Z_1 \setminus l\left(\ddot{D}_{Z_1}, Z_1\right) = Z_1 \setminus Z_1 = \varnothing.$$

We have the elements Z_7, Z_6, Z_5, Z_3 are nonlimiting elements of the sets $\ddot{D}(\alpha)_{Z_7}$, $\ddot{D}(\alpha)_{Z_6}$, $\ddot{D}(\alpha)_{Z_5}$ and $\ddot{D}(\alpha)_{Z_3}$ respectively. By statement $c)$ of the Theorem 1.2 it follows, that the conditions $Y_7^\alpha \cap Z_7 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_5^\alpha \cap Z_5 \neq \varnothing$ and $Y_3^\alpha \cap Z_3 \neq \varnothing$ are hold. Since $Z_7 \subset Z_5,\ Z_6 \subset Z_3$, we have $Y_5^\alpha \cap Z_5 \neq \varnothing$ and $Y_3^\alpha \cap Z_3 \neq \varnothing$. Therefore the following conditions are hold:

$$Y_7^\alpha \supseteq Z_7,\ Y_6^\alpha \supseteq Z_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5,\ Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3,$$
$$Y_5^\alpha \cap Z_5 \neq \varnothing,\ Y_3^\alpha \cap Z_3 \neq \varnothing.$$

The Theorem is proved.

Lemma 6.1: *Let* $D \in \Sigma_2(X, 8)$ *and* $Z_5 \cap Z_3 = \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_9)\right|$ *can be calculated by the formula*

$$\left|I^*(Q_9)\right| = 3^{|X \setminus Z_4|} + 3^{|X \setminus Z_2|} + 3^{|X \setminus Z_1|} + 3^{|X \setminus \breve{D}|}.$$

Proof: By definition of the given semilattice D we have

$$Q_9\theta_{XI} = \left\{\{Z_7, Z_6, Z_4\}, \{Z_6, Z_5, Z_2\}, \{Z_7, Z_3, Z_1\}, \{Z_5, Z_3, \breve{D}\}\right\}.$$

If the following equalities are hold

$D_1' = \{Z_7, Z_6, Z_4\}, D_2' = \{Z_6, Z_5, Z_2\}, D_3' = \{Z_7, Z_3, Z_1\}, D_4' = \{Z_5, Z_3, \breve{D}\}$, then

$\left|I^*(Q_9)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right|$. From this equality and statement (9) of Lemma 2.1 we obtain validity of Lemma 6.1.

The Lemma is proved.

Lemma 6.2: *Let* $D \in \Sigma_2(X, 8)$ *and* $Z_5 \cap Z_3 = \varnothing$. *If* X *is a finite set, then the number* $\left|I^*(Q_{16})\right|$ *can be calculated by the formula*

$$\left|I^*(Q_{16})\right| = \left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 8^{|X\setminus\breve{D}|}.$$

Proof: By definition of the given semilattice D we have $Q_{16}\theta_{XI} = \left\{\left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$, then $\left|I^*(Q_{16})\right| = \left|I(D_1')\right|$. From this equality and statement (16) of Lemma 2.1 we obtain validity of Lemma 6.2.

The Lemma is proved.

Let us assume that

$$\begin{aligned} k_6 &= \left|I^*(Q_9)\right| + \left|I^*(Q_{10})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{12})\right| + \left|I^*(Q_{13})\right| + \left|I^*(Q_{14})\right| + \left|I^*(Q_{15})\right| + \left|I^*(Q_{16})\right| = \\ &= 3^{|X\setminus Z_1|} + 3^{|X\setminus Z_2|} + 3^{|X\setminus Z_4|} + 3^{|X\setminus\breve{D}|} + \left(4^{|\breve{D}\setminus Z_1|} - 3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(4^{|\breve{D}\setminus Z_2|} - 3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|} + \\ &+ \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|} + \left(4^{|\breve{D}\setminus Z_4|} - 3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(4^{|Z_2\setminus Z_4|} - 3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|} - 4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|} + \\ &+ \left(4^{|Z_1\setminus Z_4|} - 3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|} - 4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|} + 3^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|} - 3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|} + \\ &+ \left(2^{|Z_5\setminus Z_4|} - 1\right)\cdot 5^{|X\setminus Z_2|} + \left(2^{|Z_3\setminus Z_4|} - 1\right)\cdot 5^{|X\setminus Z_1|} + \left(2^{|Z_3\setminus Z_2|} - 1\right)\cdot 5^{|X\setminus\breve{D}|} + \left(2^{|Z_5\setminus Z_1|} - 1\right)\cdot 5^{|X\setminus\breve{D}|} + \\ &+ \left(2^{|Z_5\setminus Z_4|} - 1\right)\cdot\left(6^{|\breve{D}\setminus Z_2|} - 5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus\breve{D}|} + \left(2^{|Z_3\setminus Z_4|} - 1\right)\cdot\left(6^{|\breve{D}\setminus Z_1|} - 5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|} + \\ &+ \left(2^{|Z_3\setminus Z_2|} - 1\right)\cdot\left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right)\cdot 7^{|X\setminus\breve{D}|} + \left(2^{|Z_5\setminus Z_1|} - 1\right)\cdot\left(4^{|Z_1\setminus Z_2|} - 3^{|Z_1\setminus Z_2|}\right)\cdot 7^{|X\setminus\breve{D}|} + \left(2^{|Z_5\setminus Z_1|} - 1\right)\cdot\left(2^{|Z_3\setminus Z_2|} - 1\right)\cdot 8^{|X\setminus\breve{D}|} \end{aligned}$$

Theorem 6.2: *Let $D \in \Sigma_2(X,8)$ and $Z_5 \cap Z_3 = \varnothing$. If X is a finite set and I_D is the set of all idempotent elements of the semigroup $B_X(D)$, then $|I_D| = k_1 + k_6$.*

Proof: This Theorem immediately follows from the Theorem 6.1.

The Theorem is proved.

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