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A New Approach on Type-3 Slant Helix in E^4

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Abstract

In this paper, we introduce type-3 slant helix according to parallel transport frame in 4-dimensional Euclidean Space E^4 and we use a constant angle ϕ between a unit and fixed direction vector field X and the last relatively parallel vector field M_3 of the curve, that is,

$$\langle M_3, X \rangle = \cos\phi = const.$$

where $\langle , \rangle$ is Euclidean inner product. Since the relatively parallel vector field M_3 of the curve makes a constant angle with the unit and fixed direction vector field X we call this curve as a type-3 slant helix in 4-dimensional Euclidean Space E^4. Also we define new harmonic curvature functions and we give a vector field D which we call Darboux vector field for type-3 slant helix. And then we obtain some characterizations for type-3 slant helix in terms of the harmonic curvature functions and the Darboux vector field D.

Keywords: *Slant Helices, Harmonic Curvature Functions, Curves in Euclidean Space.*

1 Introduction

A helix is a curve that lies on the surface with a constant distance between adjacent coils. There are many examples of helices. They are even more common, both in natural and artificial situations. A coiled spring is a helix that could be wrapped around a cylinder and while the vortex caused by water going down a plughole is a helix that could be wrapped around a cone. Perhaps the best known helix of all is the double helix of DNA. It is called a double helix because it has two strands wound around each other, each of which is a helix. Some bacteria, such as spirochetes are helical in shape and move due to a helical wave moving along their length and some bacteria have helical flagella to move them forward. The filamentous green algae spirogyra has a helical chloroplast running the length of each cylindrical cell... Thus, this curve is very important for understand to nature. So, lots of author interested in the helices and they published many papers in Euclidean 3 and 4 space. Such as, in [7, 8], a classical consequence for necessary and sufficient condition noted by M. A. Lancret and B. de Saint Venant first proved that a curve is a general helix *iff* the ratio curvature to torsion be constant. Then, In [6], harmonic curvature functions are defined by Özdamar and Hacisalihoğlu by using these harmonic curvature functions they generalized inclined (general helix) curves of E^3 to E^n and gave a characterization for the inclined curves in E^n " if a curve α is an inclined curve, then: $\sum_{i=1}^{n-2} H_i^2 = constant.$" After them, in [3], Izumiya and Takeuchi consider that the principle normal vector field of the curve instead of the tangent vector field and they defined a new kind of helix which is called slant helix. Also, they gave some characterizations for the slant helix in 3-dimensional Euclidean space E^3. In 2008, Önder et al. defined a new kind of slant helix in Euclidean 4-space E^4 and it is called $B_2 - slant\ helix$ [5]. And then, Gök et al. generalized the B_2 slant helix of E^4 to E^n in 2009 [4]. Then, many studies have been reported in Euclidean space using the Frenet frame. However, the Frenet frame is constructed for k times continuously differentiable non-degenerate curves. Curvatures may be zero at some points on the curve. That is, the $i-th$ ($1 < i < k$) curvature of the curve may be zero. In this situation, we need an alternative frame in E^n. Therefore in [1], Bishop defined a new frame for a curve and he called it Bishop frame which is well defined even when the curve has vanishing second derivative in $3-$dimensional Euclidean space. Then, in [2] Gökçelik et al. defined a new frame which is well defined even when the curve has vanishing $i-th$ $(1 < i < 4)$ derivative in E^4 .

In this paper, we give type-3 slant helix according to parallel transport frame in $4-$dimensional Euclidean space E^4, where we use the constant angle ϕ between a unit and fixed direction X and the last parallel vector field M_3 of the curve, that is, $\langle M_3, X \rangle = \cos\phi = \text{constant}$. Where $\langle , \rangle$ is standard inner product. Since the last parallel vector field M_3 of a curve makes a constant angle with the fixed direction vector field X, we call this curve type-3 slant helix in $4-$dimensional Euclidean space E^4. Moreover, if parallel vector field M_2 (resp. M_1) of a curve makes a constant angle with the fixed direction vector field *X*, we call this curve type-2 (resp. type-1) slant helix in $4-$ dimensional Euclidean space E^4. We know that the parallel transport frame on a helix spin along the helix. Thus, unit and fixed direction vector field X of the type-3 slant helix may makes a constant angle with the parallel transport frame M_1 and M_2. So, the type-3 slant helix can be type-1 and type-2 slant helix. Also we define new harmonic curvature functions and we give a vector field *D* which we call Darboux vector field for type-3 slant helix. And then we obtain some characterizations for type-3 slant helix in terms of the harmonic curvature functions and the Darboux vector field *D*.

2 Preliminaries

Let $\alpha : I \subset \mathbb{R} \to E^4$ be an arbitrary curve in E^4. Recall that the curve is said to be a unit speed curve (or parameterized by arc length functions) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle , \rangle$ denotes the standard inner product of E^4 given by

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in E^4$. In particular, the norm of a vector $X \in E^4$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$. Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along the unit speed curve α. Then, Frenet frame formulas are given by

$$T' = \bar{k}_1 N,\ N' = -\bar{k}_1 T + \bar{k}_2 B_1,\ B_1' = -\bar{k}_2 N + \bar{k}_3 B_2,\ B_2' = -\bar{k}_3 B_1$$

where $\bar{k}_i$ $(i = 1,2,3)$ denotes the $i-th$ curvature function of the curve α. The Frenet frame is constructed for the curve of 4-time continuously differentiable non-degenerate curves. Curvature may vanish at some points on the curve. That is, $i-th$ $(1 < i < 4)$ derivative of the curve may be zero. In this situation, we need an alternative frame. Thus, in [2], Gökçelik et al. defined a new frame for a curve it is called as parallel transport frame which is well defined even when the curve has vanishing $i-th$ $(1 < i < 4)$ derivative in $4-$dimensional Euclidean space. Let $\alpha(s)$ be a arbitrary curve parameterized by arc length s and $V(s)$ be any normal vector field which is perpendicular to the tangent vector field $T(s)$ of the curve $\alpha(s)$ said to be relatively parallel vector field if its derivative is tangential along the curve $\alpha(s)$. If $T(s)$ is an unique vector field for a given curve, we can choose any convenient arbitrary basis $\{M_1(s), M_2(s), M_3(s)\}$ of the frame, they are perpendicular to $T(s)$ at each point. The relation between the Frenet frame and parallel transport frame may be expressed as:

$$N = \cos\theta(s)\cos\psi(s)M_1 + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))M_2$$
$$+ (\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))M_3$$

$$B_1 = \cos\theta(s)\sin\psi(s)M_1 + (\cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))M_2$$
$$+ (-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))M_3$$

$$B_2 = -\sin\theta(s)M_1 + \sin\phi(s)\cos\theta(s)M_2 + \cos\phi(s)\cos\theta(s)M_3$$

Then, the alternative parallel frame equations are given as

$$T' = k_1M_1 + k_2M_2 + k_3M_3,\ M_1' = -k_1T,\ M_2' = -k_2T,\ M_3' = -k_3T \tag{1}$$

where k_1, k_2, k_3 are principal curvature functions according to parallel transport frame of the curve α. They defined as follows:

$$k_1 = \bar{k}_1\cos\theta\cos\psi,\ k_2 = \bar{k}_1(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi),$$
$$k_3 = \bar{k}_1(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi),$$

$$\bar{k}_1(s) = \sqrt{k_1^2 + k_2^2 + k_3^2},\ \bar{k}_2(s) = -\psi'(s) + \phi'(s)\sin\theta(s),\ \bar{k}_3(s) = \frac{\theta'(s)}{\sin\psi(s)},$$
$$\phi'(s) = \frac{-\theta'(s)\cot\psi(s)}{\cos\theta(s)}.$$

where

$$\psi'(s) = -(\bar{k}_2 + \bar{k}_3\frac{\sqrt{\bar{k}_3^2 - \theta'^2}}{\sqrt{\bar{k}_1^2 + \bar{k}_2^2}}),\ \phi'(s) = -\frac{\sqrt{\bar{k}_3^2(s) - \theta'(s)^2}}{\cos\theta(s)},\ \theta'(s) = \frac{\bar{k}_3(s)}{\sqrt{\bar{k}_1^2(s) + \bar{k}_2^2(s)}}.$$

Definition 2.1: *Let $\alpha : I \subset \mathbb{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1, 2, ..., n)$ in E^n. Then, we say that α is a $W-$curve if it has constant curvatures (i.e., $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3, ...,$ $\bar{k}_n$ are constant) [4].*

Theorem 2.1: *Let $\alpha : I \subset \mathbb{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1, 2, 3)$ in E^4. Then, α lies on the on $3-$sphere S^3 iff $ak_1 + bk_2 + ck_3 + 1 = 0$ where a, b and c are constant [2].*

3 A New Approach on Type-3 Slant Helix

In this section, we give same characterizations for be a type-3 slant helix according to parallel transport frame by using the harmonic curvature functions in E^4.

Definition 3.1: *Let* $\alpha : I \subset \mathbb{R} \to E^4$ *be a unit speed curve with nonzero curvatures* $k_i \ \ (i=1,2,3)$ *in* E^4. *Let* $\{T, M_1, M_2, M_3\}$ *be the parallel transport frame along the unit speed curve* α. *We call* α *is a type-3 slant helix if* M_3 *makes a constant angle* ϕ *with the fixed direction X, that is* $\langle M_3, X\rangle = \cos\phi \,(\neq \frac{\pi}{2}), \phi = \text{constant}$ *along the curve* α.

Definition 3.2: *Let* α *be a unit speed curve with nonzero curvatures* k_i $(i=1,2,3)$ *in* E^4. *Then harmonic curvature functions of the curve* α *are defined as follows:* $H_0 = 0,\ H_1 = -\left(\frac{k_3}{k_2}\right)' / \left(\frac{k_1}{k_2}\right)',\ H_2 = -\left(\frac{k_3}{k_1}\right)' / \left(\frac{k_2}{k_1}\right)'$

Theorem 3.1: *Let* $\alpha : I \subset \mathbb{R} \to E^4$ *be a unit speed curve with nonzero curvatures* $k_i \ \ (i=1,2,3)$ *in* E^4 *and* X *be a unit vector field of* E^4. $\{T, M_1, M_2, M_3\}$ *be the parallel transport frame along the* α *and* $\{H_0, H_1, H_2\}$ *denote the harmonic curvature of the curve* α. *If* $\alpha : I \subset \mathbb{R} \to E^4$ *is a type-3 slant helix with axis* X, *then following equations are satisfied:*

$$\langle T, X\rangle = H_0\langle M_3, X\rangle, \langle M_1, X\rangle = H_1\langle M_3, X\rangle, \langle M_2, X\rangle = H_2\langle M_3, X\rangle \tag{2}$$

Proof: Since α is a type-3 slant helix with fixed axis X, then $\langle M_3, X\rangle = \cos\phi$ ($\phi =$ constant). If we differentiating this equation with respect to s, then $\langle M_3', X\rangle = 0$, from parallel transport frame we obtain $-k_3\langle T, X\rangle = 0$, where $k_3 \neq 0$, then

$$\langle T, X\rangle = 0 \tag{3}$$

So, we have $$\langle T, X\rangle = H_0\langle M_3, X\rangle.$$

If we differentiating Eq. (3) with respect to s and if we use parallel transport frame, then we obtain

$$k_1\langle M_1, X\rangle + k_2\langle M_2, X\rangle + k_3\langle M_3, X\rangle = 0 \tag{4}$$

If we differentiating Eq. (4) with respect to s and if we use parallel transport frame, then we obtain

$$k_1'\langle M_1, X\rangle + k_2'\langle M_2, X\rangle + k_3'\langle M_3, X\rangle = 0 \tag{5}$$

From Eq. (4) and Eq. (5) we get following equations

$$\langle M_1, X\rangle = [\left(\frac{k_3}{k_2}\right)' / \left(\frac{k_1}{k_2}\right)']\langle M_3, X\rangle,\ \langle M_2, X\rangle = [\left(\frac{k_3}{k_1}\right)' / \left(\frac{k_2}{k_1}\right)']\langle M_3, X\rangle \tag{6}$$

and using the Definition 3.2 we have

$$\langle M_1, X\rangle = H_1\langle M_3, X\rangle \text{ and } \langle M_2, X\rangle = H_2\langle M_3, X\rangle.$$

Corollary 3.2: *Let $\alpha : I \subset \mathrm{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit and fixed vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and $\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. Then, $\alpha : I \subset \mathrm{R} \to E^4$ is a type-3 slant helix with axis X iff the harmonic curvatures $H_1 = -\left(\frac{k_3}{k_2}\right)' / \left(\frac{k_1}{k_2}\right)'$, $H_2 = -\left(\frac{k_3}{k_1}\right)' / \left(\frac{k_2}{k_1}\right)'$ of the curve α are constant ($\neq 0$).*

Proof: If we differentiate last following equation with respect to s and if we use parallel transport frame and Eq.(3) we have $-k_1\langle T, X\rangle = H_1'\langle M_3, X\rangle = 0$. Since $\langle M_3, X\rangle \neq 0$, H_1' must be zero. So, H_1 is a constant function. Conversely, let H_1 be constant, then if we differentiate the equation $\langle M_1, X\rangle = H_1\langle M_3, X\rangle$ with respect to s we get $-k_1\langle T, X\rangle = H_1'\langle M_3, X\rangle + H_1(\langle M_3, X\rangle)' = 0$. If we consider H_1 is a non-zero constant then via Eq.(3) we obtain $\langle M_3, X\rangle = \|M_3\|\|X\|\cos\phi =$ constant. Similarly, we can easily see that H_2 is a constant function.

Corollary 3.3: *Let $\alpha : I \subset \mathrm{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit and fixed vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and $\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. Then, the axis of the type-3 slant helix can be written as $X = \{H_1M_1 + H_2M_2 + M_3\}\cos\phi$*

Proof: If the axis of type-3 slant helix α in E^4 is *X*, then we can write

$X = c_1T + c_2M_1 + c_3M_2 + c_4M_3$, then using Theorem 3.1 we have

$$c_1 = \langle T, X\rangle = H_0\langle M_3, X\rangle = 0,\ c_2 = \langle M_1, X\rangle = H_1\langle M_3, X\rangle$$
$$c_3 = \langle M_2, X\rangle = H_2\langle M_3, X\rangle,\ c_4 = \langle M_3, X\rangle = \cos\phi$$

Therefore easily we obtain $X = \{H_1M_1 + H_2M_2 + M_3\}\cos\phi.$

Theorem 3.4: *Let $\alpha : I \subset \mathrm{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit and fixed vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and $\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. Then, the axis X of the type-3 slant helix α makes a constant angle with the vector fields $\{M_1, M_2\}$. That is, $\langle M_1, X\rangle = const.$ and $\langle M_2, X\rangle = const.$*

Proof: From Theorem 3.1 we know that $\langle M_1, X\rangle = H_1\cos\phi$ and $\langle M_2, X\rangle = H_2\cos\phi$. Also, we know that from Corollary 3.2 the harmonic curvatures $\{H_1, H_2\}$ of the type-3 slant helix α are constant. So, if we differentiate the last two equations we get $\langle M_1, X\rangle' = (H_1\cos\phi)' = 0$ and $\langle M_2, X\rangle' = (H_2\cos\phi)' = 0$. Therefore $\langle M_1, X\rangle$ and $\langle M_2, X\rangle$ are constant.

Corollary 3.5: *Let $\alpha : I \subset \mathrm{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and $\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. Then, α is a type-3 slant helix then the curve α is type-1 and type-2 slant helix.*

Proof: It is obvious from the last theorem.

Definition 3.3: *Let $\alpha : I \subset \mathrm{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit and fixed vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and $\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. Then, the Darboux vector of the type-3 slant helix α is given by $D = H_1M_1 + H_2M_2 + M_3$.*

Theorem 3.6: *Let $\alpha : I \subset \mathrm{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and*

$\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. Then, α is a type-3 slant helix iff D is constant vector field.

Proof: Let $\alpha : I \subset \mathbb{R} \to E^4$ be a type-3 slant helix with axis X. From Corollary 3.3 we know $X = \{H_1 M_1 + H_2 M_2 + M_3\}\cos\phi$. Conversely, let D be a constant vector field. Then we have $\langle D, M_3 \rangle = 1$. So, $\|D\|\cos\phi = 1$. Thus, we get $\cos\phi = \frac{1}{\|D\|}$, where ϕ is a constant angle between D and M_3. In this case we can define a unique axis of the type-3 slant helix as $X = \cos\phi D$, where $\langle M_3, X \rangle = \frac{1}{\|D\|} = \cos\phi$. Thus, X is a fixed vector and α is a type-3 slant helix.

Theorem 3.7: *Let $\alpha : I \subset \mathbb{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit and fixed vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and $\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. If α is a type-3 slant helix, then* $\left[(\frac{k_3}{k_2})' / (\frac{k_1}{k_2})' \right]^2 + \left[(\frac{k_3}{k_1})' / (\frac{k_2}{k_1})' \right]^2 = \tan^2\phi = const.$

where ϕ is a constant angle between X and M_3.

Proof: Let α be a type-3 slant helix, since the axis X of the curve α is a unit vector field $\left(H_1^2 + H_2^2\right)\cos^2\phi + \cos^2\phi = 1$. Then,

$$H_1^2 + H_2^2 = \left[(\frac{k_3}{k_2})' / (\frac{k_1}{k_2})' \right]^2 + \left[(\frac{k_3}{k_1})' / (\frac{k_2}{k_1})' \right]^2 = \frac{1 - \cos^2\phi}{\cos^2\phi} = \tan^2\phi = const.$$

Theorem 3.8: *Let $\alpha : I \subset \mathbb{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,3)$ in E^4 and X be a unit and fixed vector field of E^4. $\{T, M_1, M_2, M_3\}$ be the parallel transport frame along the curve α and $\{H_0, H_1, H_2\}$ denote the harmonic curvature of the curve α. Then, α is a type-3 slant helix iff $k_1 H_1 + k_2 H_2 + k_3 = 0$.*

Proof: Differentiating the Darboux vector D along the type-3 slant helix α, we obtain $D' = H_1' M_1 + H_1 M_1' + H_2' M_2 + H_2 M_2' + M_3' = 0$. Then, using the parallel transport frame and Corollary 3.2, easily we get $k_1 H_1 + k_2 H_2 + k_3 = 0$. Conversely, if $k_1 H_1 + k_2 H_2 + k_3 = 0$, then $D' = 0$, i.e, $D =$ constant. Thus, from Theorem 3.6 α is a type-3 slant helix.

Theorem 3.10: *Let* $\alpha : I \subset \mathbb{R} \to E^4$ *be a unit speed curve with nonzero curvatures* k_i $(i = 1,2,3)$ *in* E^4. $\{T, M_1, M_2, M_3\}$ *be the parallel transport frame along the curve* α *and* $\{H_0, H_1, H_2\}$ *denote the harmonic curvature of the curve* α. *The parallel transport frame matrix* $M_3(s)$ *is given as*

$$M_3(s) = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^3_3.$$

Then, α *is a type-3 slant helix iff* $D = [H_0 \quad H_1 \quad H_2 \quad 1] \in \mathbb{R}^3_1$ *satisfies the following equation:*

$$M_3(s)[H_0 \quad H_1 \quad H_2 \quad 1]^T = \frac{d}{ds}[H_0 \quad H_1 \quad H_2 \quad 1].$$

Proof: Using the Corollary 3.2 and Theorem 3.8 direct substitution shows that

$$\begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ H_2 \\ 1 \end{bmatrix} = k_1 H_1 + k_2 H_2 + k_3 = \frac{d}{ds}[H_0 \quad H_1 \quad H_2 \quad 1] = 0$$

Corollary 3.11: *Let* $\alpha : I \subset \mathbb{R} \to E^4$ *be a unit speed curve with non-zero curvatures* k_i $(i = 1,2,3)$ *in* E^4. $\{T, M_1, M_2, M_3\}$ *be the parallel transport frame along the curve* α *and* $\{H_0, H_1, H_2\}$ *denote the harmonic curvature of the curve* α. *Then, the curve* α *is a type-3 slant helix iff the Darboux vector* D *lies in the kernel of the parallel transport frame matrix* $M_3(s)$ *in* E^4.

Theorem 3.12: *Let* $\alpha : I \subset \mathbb{R} \to E^4$ *be a unit speed curve with non-zero curvatures* k_i $(i = 1,2,3)$ *in* E^4. *If* α *is a type-3 slant helix then* α *cannot be a W-curve in* E^4.

Proof: Let the curve type-3 slant helix α be a $W-$curve. Then, from Eq. (6) we know that

$$\cos\phi = \frac{\langle M_1, X\rangle (\frac{k_1}{k_2})'}{(\frac{k_3}{k_2})'} \neq 0. \tag{7}$$

So, we get $\frac{k_1}{k_2}$ and $\frac{k_3}{k_2}$ cannot be constant. On the other hand if α is a $W-$curve then $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$ are constant then we get $\frac{k_1}{k_2}=const$. Since $\phi \neq \frac{\pi}{2}$ using the Eq. (7) α cannot be a $W-$curve.

Theorem 3.13: *Let $\alpha : I \subset \mathbb{R} \to E^4$ be a unit speed curve with nonzero curvatures k_i $(i=1,2,3)$ in E^4. If α is a type-3 slant helix then α cannot lie on $3-$ sphere S^3.*

Proof: Let α lies on a sphere with center P and radius r then from Theorem 2.1 $\{k_1, k_2, k_3\}$ satisfies the equation $ak_1+bk_2+ck_3+1=0$ where a, b and c are constant. But we know from Theorem 3.8 if α is a type-3 slant helix with nonzero curvatures k_i $(i=1,2,3)$ in E^4 *iff* $k_1H_1+k_2H_2+k_3=0$ where H_1, H_2 are constant, that is, $\{k_1, k_2, k_3\}$ don't satisfies the equation $ak_1+bk_2+ck_3+1=0$. So, α cannot lie on $3-$ sphere S^3.

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