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Riemann Extension of Minkowski Line Element in the Rindler Chart

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Abstract

In this paper, we discuss Riemannian extension of Minkowski metric in Rindler coordinates and its geodesics.

Keywords: *Riemannian curvature, Riemann extension, Flat metric, geodesic equations, constant positive curvature.*

1 Introduction

Patterson and Walker[7] have defined Riemann extensions and showed how a Riemannian structure can be given to the $2n$ dimensional tangent bundle of an $n-$ dimensional manifold with given non-Riemannian structure. This shows Riemann extension provides a solution of the general problem of embedding a manifold M carrying a given structure in a manifold M' carrying another structure, the embedding being carried out in such a way that the structure on M' induces in a natural way the given structure on M. The Riemann extension of Riemannian or non-Riemannian spaces can be constructed with the help of the Christoffel coefficients Γ^i_{jk} of corresponding Riemann space or with connection coefficients Π^i_{jk} in the case of the space of affine connection[5]. The theory of Riemann extensions has been extensively studied by Afifi[1] and Dryuma[2],[3], [4] [5].

2 Preliminaries

Let (M, g) be a Riemannian manifold and let g_{ij} be components of the metric tensor g.

The system of geodesic equations are

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0, \tag{1}$$

where Γ^k_{ij} are the Christoffel symbols constructed from g given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}(g_{mi,j} + g_{mj,i} - g_{ij,m}). \tag{2}$$

The geodesic deviation equations or the Jacobi equations are given by

$$\frac{D^2\eta^i}{ds^2} + R^i_{kjm}\frac{dx^k}{ds}\frac{dx^m}{ds}\eta^j = 0, \tag{3}$$

where R^i_{kjm} is Riemann curvature tensor of the manifold and

$$\frac{D\eta^i}{ds} = \frac{d\eta^i}{ds} + \Gamma^i_{jk}\eta^j\frac{dx^k}{ds}. \tag{4}$$

Using (1) and simplifying the equation (3) reduces to

$$\frac{d^2\eta^i}{ds^2} + 2\Gamma^i_{lm}\frac{dx^m}{ds}\frac{d\eta^l}{ds} + \frac{\partial\Gamma^i_{kl}}{\partial x^j}\frac{dx^k}{ds}\frac{dx^l}{ds}\eta^j = 0. \tag{5}$$

The Riemann extension[7] of the metric g is given by

$$ds^2 = -2\Gamma^k_{ij}(x^l)\psi_k dx^i dx^j + 2d\psi_k dx^k, \tag{6}$$

where ψ_k are the coordinates of additional space. The geodesic equations of (6) consists of two parts,

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{7}$$

and

$$\frac{\delta^2\psi_k}{ds^2} + R^l_{kji}\frac{dx^j}{ds}\frac{dx^i}{ds}\psi_l = 0, \tag{8}$$

where

$$\frac{\delta^2\psi_k}{ds^2} = \frac{d\psi_k}{ds} - \Gamma^l_{jk}\psi_l\frac{dx^j}{ds}. \tag{9}$$

The system of equations (7) is the system of geodesic equations for geodesics of basic space with local coordinates x^i and it does not contain the coordinates ψ_k.
The system of equations (8) is a $n \times n$ linear matrix system of second order in the form

$$\frac{d^2\psi}{ds^2} + A(s)\frac{d\psi}{ds} + B(s)\psi = 0, \tag{10}$$

where $A(s)$ and $B(s)$ are matrices.

3 Minkowski Metric in Rindler Coordinates

In relativistic physics, the Rindler coordinate chart is an important and useful coordinate chart representing part of flat spacetime, also called the Minkowski vacuum. The Rindler coordinate system or frame describes a uniformly accelerating frame of reference in Minkowski space. In special relativity, a uniformly accelerating particle undergoes hyperbolic motion. For each such particle a Rindler frame can be chosen in which it is at rest.

The Rindler chart is named after Wolfgang Rindler who popularised its use, although it was already well known in 1935[6].

In Rindler chart the Minkowski line element is given by

$$ds^2 = dx^2 + dy^2 + dz^2 - a^2x^2dt^2, \tag{11}$$

where x, y and z are space coordinates, t is time coordinate and 'a' is a constant called proper acceleration. The non vanishing Christoffel symbols of the metric (11) are

$$\Gamma^1_{44} = a^2x, \quad \Gamma^4_{14} = \tfrac{1}{x}. \tag{12}$$

The spatial metric of any four dimensional metric

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0dx^\alpha + g_{00}dx^0dx^0 \tag{13}$$

has the form

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \tag{14}$$

where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}} \tag{15}$$

is a three dimensional tensor determining the properties of the space.
In the case of the Minkowski metric in Rindler coordinates,

$$-dl^2 = dx^2 + dy^2 + dz^2, \tag{16}$$

which is a flat metric.

3.1 Geodesic Deviation Equations

The connection coefficients are clearly very simple. Using the Christoffel symbols (12) in equation (5) and simplifying we get a system of Jacobi equations.

$$\frac{d^2\eta^1}{ds^2} + 2a^2x\frac{dt}{ds}\frac{d\eta^4}{ds} + a^2\left(\frac{dt}{ds}\right)^2\eta^1 = 0, \tag{17}$$

$$\frac{d^2\eta^2}{ds^2} = 0, \tag{18}$$

$$\frac{d^2\eta^3}{ds^2} = 0, \tag{19}$$

$$\frac{d^2\eta^4}{ds^2} + \frac{2}{x}\frac{dt}{ds}\frac{d\eta^1}{ds} - \frac{1}{x^2}\frac{dx}{ds}\frac{dt}{ds}\eta^1 = 0, \tag{20}$$

where

$$\frac{d^2x}{ds^2} + a^2x\left(\frac{dt}{ds}\right)^2 = 0 \tag{21}$$

and

$$\frac{d^2t}{ds^2} + \frac{2}{x}\frac{dx}{ds}\frac{dt}{ds} = 0. \tag{22}$$

Equations (21) and (22) are the geodesic equations obtained from (1). Simplifying and substituting for 't' the system reduces to

$$\frac{d^2\eta^1}{ds^2} + \frac{2a^2k^2}{x}\frac{d\eta^4}{ds} + \frac{a^2k^4}{x^4}\eta^1 = 0 \tag{23}$$

$$\frac{d^2\eta^2}{ds^2} = 0 \tag{24}$$

$$\frac{d^2\eta^3}{ds^2} = 0 \tag{25}$$

$$\frac{d^2\eta^4}{ds^2} + \frac{2k^2}{x^3}\frac{d\eta^1}{ds} - \frac{k^2}{x^4}\frac{dx}{ds}\eta^1 = 0 \tag{26}$$

$$\frac{d^2x}{ds^2} + a^2\frac{k^4}{x^3} = 0. \tag{27}$$

Solving we get

$$\begin{aligned}\eta^1 &= x(c_1(\int \frac{(ak^2+\sqrt{a^2k^4+c^2x^2})^{\sqrt{5}}x^{-\sqrt{5}}}{\sqrt{a^2k^4+c^2x^2}}dx) \\ &\quad +c_2(\int \frac{(ak^2+\sqrt{a^2k^4+c^2x^2})^{-\sqrt{5}}x^{\sqrt{5}}}{\sqrt{a^2k^4+c^2x^2}}dx)) + c_3,\end{aligned} \tag{28}$$

$$\eta^2 = a_1 s + a_2, \tag{29}$$

$$\eta^3 = a_3 s + a_4, \tag{30}$$

$$\eta^4 = \int \frac{1}{\sqrt{a^2k^4+c^2x^2}}(x((\int \frac{1}{x^3\sqrt{a^2k^4+c^2x^2}}(2(ak^2+\sqrt{a^2k^4+c^2x^2})^{-\sqrt{5}} c_2 x^{1+\sqrt{5}} + 2(ak^2+\sqrt{a^2k^4+c^2x^2})^{\sqrt{5}} c_2 x^{1-\sqrt{5}} + \frac{1}{x}\eta^1\sqrt{a^2k^4+c^2x^2}dx)k^2+c_4)))dx + c_5, \tag{31}$$

and (32)

$$x^2 = \frac{1}{c^2}[c^4(s+d)^2 - a^2k^4], \tag{33}$$

where $a_1, a_2, a_3, a_4, c_1, c_2, c_3, c_4, c_5, d$ are arbitrary constants.

3.2 Riemann Extension

The metric in the extended space is given by

$$ds^2 = -2xa^2Pdt^2 - \frac{4}{x}Vdxdt + 2dxdP + 2dydQ + 2dzdU + 2dtdV. \tag{34}$$

The Christoffel symbols in the extended space are

$$\begin{array}{llll} \Gamma^5_{14} = \frac{2V}{x^2} & \Gamma^5_{44} = a^2P & \Gamma^5_{48} = -\frac{1}{x} & \Gamma^8_{11} = \frac{2V}{x^2} \\ \Gamma^8_{14} = a^2P & \Gamma^8_{18} = -\frac{1}{x} & \Gamma^8_{44} = 2Va^2 & \Gamma^8_{45} = -a^2x. \end{array} \tag{35}$$

The geodesic equations are given by

$$\frac{d^2x}{ds^2} + a^2x\left(\frac{dt}{ds}\right)^2 = 0 \tag{36}$$

$$\frac{d^2y}{ds^2} = 0 \tag{37}$$

$$\frac{d^2z}{ds^2} = 0 \tag{38}$$

$$\frac{d^2t}{ds^2}+\frac{2}{x}\frac{dx}{ds}\frac{dt}{ds}=0 \tag{39}$$

$$\frac{d^2P}{ds^2}+\frac{4V}{x^2}\frac{dx}{ds}\frac{dt}{ds}+a^2P\left(\frac{dt}{ds}\right)^2-\frac{2}{x}\frac{dt}{ds}\frac{dV}{ds}=0 \tag{40}$$

$$\frac{d^2Q}{ds^2}=0 \tag{41}$$

$$\frac{d^2U}{ds^2}=0 \tag{42}$$

$$\frac{d^2V}{ds^2}+\frac{2V}{x^2}(\frac{dx}{ds})^2+2a^2P\frac{dx}{ds}\frac{dt}{ds}-\frac{2}{x}\frac{dx}{ds}\frac{dV}{ds}+2a^2V\left(\frac{dt}{ds}\right)^2 \tag{43}$$

$$-2a^2x\frac{dt}{ds}\frac{dP}{ds}=0. \tag{44}$$

Solving the above equations, we obtain

$y=a_1s+b_1,\ z=a_2s+b_2,\ \ Q=a_3s+b_3,\ \ U=a_4s+b_4,$

$t=\frac{c^2}{2a}log\left(\frac{c^2(s+d)-ak^2}{c^2(s+d)+ak^2}\right)$ and $x^2=\frac{1}{c^2}[c^4(s+d)^2-a^2k^4].$

Transforming the remaining equations from s to x, we get

$$\begin{aligned}x^3(c^2x^2+a^2k^4)\frac{d^2P}{dx^2}-a^2k^4x^2\frac{dP}{dx}+a^2k^4xP\\=2k^2\sqrt{a^2k^4+c^2x^2}\left(x\frac{dV}{dx}-2V\right)\end{aligned} \tag{45}$$

and

$$\begin{aligned}x^2(c^2x^2+a^2k^4)\frac{d^2V}{dx^2}-(3a^2k^4x+2c^2x^3)\frac{dV}{dx}+(4a^2k^4+2c^2x^2)V\\=2a^2k^2x\sqrt{c^2x^2+a^2k^4}\left(x\frac{dP}{dx}-P\right).\end{aligned} \tag{46}$$

Solving (45) and (46) we get,

$$P(x)=\frac{b_1(a^2k^4+c^2x^2)+b_2\sqrt{a^2k^4+c^2x^2}+b_3}{x} \tag{47}$$

and

$$V(x)=x^2\left(b_4+\int\frac{2k^4a^2(k^4b_1+b_3)+c^2x^2(b_1a^2k^4+b_3)+2\sqrt{a^2k^4+c^2x^2}b_2a^2k^4}{k^2x^3\sqrt{a^2k^4+c^2x^2}}dx\right), \tag{48}$$

where b_1, b_2, b_3, b_4 are constants determined by the given initial conditions. Thus we have obtained Riemann extension of Minkowski metric in Rindler coordinates and have obtained solution of its geodesic equations.

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