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Subordination and Superordination Properties of p-Valent Functions Involving Certain Fractional Calculus Operator

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Abstract

In this paper, we study different applications of the differential subordination and superordination of analytic functions in the open unit disc associated with the fractional differintegral operator $U_{0,z}^{\alpha,\beta,\gamma}$. *Sandwich-type result involving this operator is also derived.*

Keywords: *Analytic function, p-valent function, fractional differintegral operator, differential subordination and superordination.*

1 Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z : z \in C \text{ and } |z| < 1\}$ and $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form

$$f(z) = a + a_p z^k + a_{p+1} z^{p+1} + \ldots \quad \left(a \in \mathbb{C},\ p \in \mathbb{N} = \{1, 2, \ldots\}\right).$$

Let A_p denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k \quad (p\in\mathbb{N},\ z\in U), \tag{1.1}$$

which are analytic in the open unit disk U, and set $A\equiv A_1$.

Let f and F be members of $H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0)=0$ and $|w(z)|<1\ (z\in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z)\prec F(z)$. In particular, if F is univalent, then $f(z)\prec F(z)$ if and only if $f(0)=F(0)$ and $f(U)\subset F(U)$(see [2]).

Suppose that p and h are two functions in U, let

$$\phi(r,s,t;z):C^3\times U\to C\ .$$

If p and $\phi\left(p(z),zp'(z),z^2p''(z);z\right)$ are univalent in U. If p is analytic in U and satisfies the first order differential superordination

$$h(z)\prec\phi\left(p(z),zp'(z),z^2p''(z);z\right)\ (z\in U), \tag{1.2}$$

then p is called a solution of the differential superordination (1.2).

The univalent function q is called a subordinant solutions of (1.2) if $q\prec p$ for all p satisfying (1.2). A subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all subordinant q of (1.2) is said to be the best subordinant. (see the monograph by Miller and Mocanu [10], and [11]).

Recently, Miller and Mocanu [11] obtained sufficient conditions on the functions h,q and ϕ for which the following implication holds:

$$h(z)\prec\phi\left(p(z),zp'(z),z^2p''(z);z\right)\ \to\ q(z)\prec p(z)$$

Using these results, the second author considered certain classes of first-order differential superordinations [6], as well as superordination-preserving integral operators [5]. Ali et al. [1], using the results from [6], obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z), \tag{1.3}$$

where q_1 and q_2 are given univalent normalized functions in U.

Very recently, Shanmugam et al. [22–24] obtained the such called sandwich results for certain classes of analytic functions. Further subordination results can be found in [13, 21, 27 and 28].

we recall the definitions of the fractional derivative and integral operators introduced and studied by Saigo (cf. [17] and [19], see also [20]).

Definition1 *let* $\alpha > 0$ and $\beta, \gamma \in R$, *then the generalized fractional integral operator* $\mathrm{I}_{0,z}^{\alpha,\beta,\gamma}$ *of order* α *of a function* $f(z)$ *is defined by*

$$\mathrm{I}_{0,z}^{\alpha,\beta,\gamma} f(z) = \frac{z^{-\alpha-\beta,}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{t}{z}\right) f(t)dt \,, \tag{1.4}$$

where the function $f(z)$ *is analytic in a simply-connected region of the* z *- plane containing the origin and the multiplicity of* $(z-t)^{\alpha\text{-}1}$ *is removed by requiring* $\log(z\text{ -}t)$ *to be real when* $(z-t) > 0$ *provided further that*

$$f(z) = \mathrm{O}(|z|^{\varepsilon}),\ z \to 0 \ \ for \ \varepsilon > \max(0, \beta-\gamma) - 1. \tag{1.5}$$

Definition 2 *let* $0 \le \alpha < 1$ *and* $\beta, \gamma \in R$, *then the generalized fractional derivative operator* $J_{0,z}^{\alpha,\beta,\gamma}$ *of order* α *of a function* $f(z)$ *defined by*

$$\begin{aligned} J_{0,z}^{\alpha,\beta,\gamma} f(z) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz}\left[z^{\alpha-\beta} \int_0^z (z-t)^{-\alpha} {}_2F_1\left(\beta-\alpha, 1-\gamma; 1-\alpha; 1-\frac{t}{z}\right) f(t)dt \right], \\ &= \frac{d^n}{dz^n} J_{0,z}^{\alpha-n,\beta,\gamma} f(z) \qquad (n \le \alpha < n+1; n \in N), \end{aligned} \tag{1.6}$$

where the function $f(z)$ *is analytic in a simply-connected region of the* z *- plane containing the origin, with the order as given in (1.5) and multiplicity of* $(z-t)^{\alpha}$ *is removed by requiring* $\log(z-t)$ *to be real when* $(z-t) > 0$.

Not that

$$I_{0,z}^{\alpha,-\alpha,\gamma} \ f(z) = D_z^{-\alpha} \ f(z), (\alpha > 0) \tag{1.7}$$

$$J_{0,z}^{\alpha,\alpha,\gamma} \ f(z) = D_z^{\alpha} \ f(z), (0 \le \alpha < 1), \tag{1.8}$$

where $D_z^{-\alpha} f(z)$ *and* $D_z^{\alpha} f(z)$ *are respectively the well known Riemann-Liouvill fractional integral and derivative operators (cf. [14] and [15], see also [25]).*

Definition 3 *For real number* $\alpha(\text{-}\infty < \alpha < 1)$ *and* $\beta(\text{-}\infty < \beta < 1)$ *and a positive real*

number γ, *the fractional operator* $U_{0,z}^{\alpha,\beta,\gamma} : A_p \to A_p$ *for the function* $f(z)$ *given by (1.1) is defined in terms of* $J_{0,z}^{\alpha,\beta,\gamma}$ *and* $\mathrm{I}_{0,z}^{\alpha,\beta,\gamma}$ *by (see [12] and [9])*

$$U_{0,z}^{\alpha,\beta,\gamma} f(z) = z^{p} + \sum_{k=p+1}^{\infty} \frac{(1+p)_{k-p}(1+p+\gamma-\beta)_{k-p}}{(1+p-\beta)_{k-p}(1+p+\gamma-\alpha)_{k-p}} a_k z^{k}, \quad (1.9)$$

which for $f(z) \neq 0$ *may be written as*

$$U_{0,z}^{\alpha,\beta,\gamma} f(z) = \begin{cases} \dfrac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} J_{0,z}^{\alpha,\beta,\gamma} f(z); & 0 \leq \alpha \leq 1 \\ \dfrac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} \mathrm{I}_{0,z}^{-\alpha,\beta,\gamma} f(z); & -\infty \leq \alpha < 0 \end{cases} \quad (1.10)$$

where $J_{0,z}^{\alpha,\beta,\gamma} f(z)$ *and* $\mathrm{I}_{0,z}^{-\alpha,\beta,\gamma} f(z)$ *are, respectively the fractional derivative of* f *of order* α *if* $0 \leq \alpha < 1$ *and the fractional integral of* f *of order* $-\alpha$ *if* $-\infty \leq \alpha < 0$.

It is easily verified (see Choi [8]) from (1.9) that

$$\left(p-\beta\right) U_{0,z}^{\alpha+1,\beta+1,\gamma+1} f(z) + \beta U_{0,z}^{\alpha,\beta,\gamma} f(z) = z\left(U_{0,z}^{\alpha,\beta,\gamma} f(z)\right)'. \quad (1.11)$$

Note that

$$U_{0,z}^{\alpha,\alpha,\gamma} f(z) = \Omega_z^{(\alpha,p)} f\left(z\right)\left(-\infty < \alpha < 1\right), \quad (1.12)$$

The fractional differintegral operator $\Omega_z^{(\alpha,p)} f\left(z\right)$ *for* $\left(-\infty < \alpha < p+1\right)$ *is studied by Patel and Mishra [16], and the fractional differential operator* $\Omega_z^{(\alpha,p)}$ *with* $0 \leq \alpha < 1$ *was investigated by Srivastava and Aouf [26]. We, further observe that* $\Omega_z^{(\alpha,1)} = \Omega_z^{\alpha}$ *is the operator introduced and studied by Owa and Srivastava [15].*
It is interesting to observe that

$$U_{0,z}^{0,0,\gamma} f(z) = f(z) \quad (1.13)$$

$$U_{0,z}^{1,1,\gamma} f(z) = \frac{z}{p} f'(z) \quad (1.14)$$

To prove our results, we need the following definitions and lemmas.

Definition 4([10]) *Denote by* Q *the set of all functions* $q(z)$ *that are analytic and injective on* $\overline{U}/E(q)$ *where* $E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\}$,

and are such that $q'(\zeta) \neq 0$ *for* $\zeta \in \partial U / E(q)$. *Further let the subclass of* Q *for which* $q(0) = a$ *be denoted by* $Q(a)$, $Q(0) \equiv Q_0$ *and* $Q(1) \equiv Q_1$.

Lemma 1([10]) *Let* $q(z)$ *be univalent function in the unit disc* U *and let* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ *and suppose that*

i) Q *is a starlike function in* U ,
ii) $\mathrm{Re}\, zh'(z)/Q(z) > 0,\ z \in U$.
If p *is analytic in* U *with* $p(0) = q(0),\ p(U) \subseteq D$ *and*

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{1.15}$$

then $p(z) \prec q(z)$, *and* q *is the best dominant of (1.15).*

Lemma 2([23]) *Let* $q(z)$ *be a convex univalent function in* U *and let* $\alpha \in \mathbb{C},\ \eta \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ *with*

$$\Re\left\{1 + \frac{z\,q''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $g(z)$ *is analytic in* U *and*

$$\sigma g(z) + \eta z g'(z) \prec \sigma q'(z) + \eta z q'(z) \ ,$$

then $g(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 3([7]) *Let* $q(z)$ *be univalent function in the unit disc* U *and let* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that*

i) $\mathrm{Re}\,\theta(q(z))/\phi(q(z)) > 0 \ \ z \in U$,
ii) $h(z) = zq'(z)\varphi(q(z))$ *is starlike in* U .

If $p \in H[q(0),1] \cap Q$ *with* $p(U) \subseteq D$, $\theta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent* U , *and*

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{1.16}$$

then $q(z) \prec p(z)$, *and* q *is the best dominant of (1.16).*

Lemma 4([11]) *Let* $q(z)$ *be convex function in* U *and let* $\gamma \in \mathbb{C}$, *with* $\mathrm{Re}\,\gamma > 0$. *If* $p \in H[q(0),1] \cap Q$ *and* $p(z) + \gamma z p'(z)$ *is univalent in* U , *then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z), \tag{1.17}$$

Implies $q(z) \prec p(z)$, *and* q *is the best dominant of (1.17).*

Lemma 5 ([18]) *The function* $q(z) = (1-z)^{-2ab}$ *is univalent in* U *if and only if* $|2ab - 1| \le 1$ *or* $|2ab + 1| \le 1$.

2 Subordination Results for Analytic Functions

Theorem 1 *Let* $q(z)$ *be a univalent function in* U *, with* $q(0)=1$*, and suppose that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0;-p(p-\beta)\operatorname{Re}\frac{1}{\lambda}\right\},\ z\in U, \tag{2.1}$$

Where $-\infty<\alpha<1; -\infty<\beta<1;\ \gamma\in\mathbb{R}^{+};\lambda\in\mathbb{C}^{*}=\mathbb{C}\setminus\{0\}$ *and* $p\in\mathbb{N}$.
If $f\in A_p$ *satisfies the subordination*

$$\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^{p}}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^{p}}\right)\prec q(z)+\frac{\lambda zq'(z)}{p(p-\beta)}, \tag{2.2}$$

then

$$\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^{p}}\prec q(z),$$

and the function q *is the best dominant of (2.2).*

Proof. If we consider the analytic function

$$h(z)=\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^{p}},$$

by differentiating logarithmically with respect to z , we deduce that

$$\frac{zh'(z)}{h(z)}=\frac{z\left(U_{0,z}^{\alpha,\beta,\gamma}f(z)\right)'}{U_{0,z}^{\alpha,\beta,\gamma}f(z)}-p\ . \tag{2.3}$$

From (2.3), by using the identity (1.11), a simple computation shows that

$$\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^{p}}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^{p}}\right)=h(z)+\frac{\lambda zh'(z)}{p(p-\beta)},$$

hence the subordination (2.2) is equivalent to

$$h(z)+\frac{\lambda zh'(z)}{p(p-\beta)}\prec q(z)+\frac{\lambda zq'(z)}{p(p-\beta)}.$$

Combining the last relation together with Lemma 2 for the special case $\eta=\lambda/p(p-\beta)$ and $\sigma=1$, we obtain our result.

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 1, where $-1\le B<A\le 1$, the condition (2.1) becomes

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\}>\max\left\{0;-p(p-\beta)\operatorname{Re}\frac{1}{\lambda}\right\},\ z\in U. \tag{2.4}$$

It is easy to check that the function $\phi(\zeta)=\dfrac{1-\zeta}{1+\zeta}$, $|\zeta|<|B|$, is convex in U and since

$\phi(\overline{\zeta})=\overline{\phi(\zeta)}$ for all $|\zeta|<|B|$, it follows that the image $\varphi(U)$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\Re\frac{1-Bz}{1+Bz};z\in U\right\}=\frac{1-|B|}{1+|B|}>0. \tag{2.5}$$

Then, the inequality (2.4) is equivalent to

$$p(p-\beta)\operatorname{Re}\frac{1}{\lambda}\geq\frac{1-|B|}{1+|B|},$$

hence we obtain the following result:

Corollary 1 *Let* $-\infty<\alpha<1;-\infty<\beta<1;\ \gamma\in\mathbb{R}^{+};\ \lambda\in\mathbb{C}^{*};\ p\in\mathbb{N}$ *and* $-1\leq B<A\leq 1$ *with*

$$\max\left\{0;-p(p-\beta)\operatorname{Re}\frac{1}{\lambda}\right\}\leq\frac{1-|B|}{1+|B|}.$$

If $f\in A_p$ *satisfies the subordination*

$$\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^{p}}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^{p}}\right)\prec\frac{1+Az}{1+Bz}+\frac{\lambda(A-B)z}{p(p-\beta)(1+Bz)^{2}}, \tag{2.6}$$

then

$$\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^{p}}\prec\frac{1+Az}{1+Bz},$$

and the function $1+Az/1+Bz$ *is the best dominant of (2.6).*

For $p=1$, $A=1$ *and* $B=-1$, *the above corollary reduces to:*

Corollary 2 *Let* $-\infty<\alpha<1;-\infty<\beta<1;\ \gamma\in\mathbb{R}^{+};\ \lambda\in\mathbb{C}^{*}$ *with*

$$(1-\beta)\operatorname{Re}\frac{1}{\lambda}\geq 0.$$

If $f\in A_p$ *satisfies the subordination*

$$\lambda\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z}\right)+(1-\lambda)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z}\right)\prec\frac{1+z}{1-z}+\frac{2\lambda z}{(1-\beta)(1+z)^{2}}, \tag{2.7}$$

then

$$\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^{p}}\prec\frac{1+z}{1-z},$$

and the function $1+z/1-z$ *is the best dominant of (2.7).*

Theorem 2 *Let* $q(z)$ *be a univalent function in* U *, with* $q(0)=1$ *and* $q(z)\neq 0$ *for all* $z\in U$ *. Let* $\delta,\mu\in\mathbb{C}^*$ *and* $\nu,\eta\in\mathbb{C}$ *with* $\nu+\eta\neq 0$*. Let* $f\in A_p$ *and suppose that* f *and* q *satisfy the conditions:*

$$\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\neq 0,\quad z\in U$$

$\left(-\infty<\alpha<1;-\infty<\beta<1;\ \gamma\in\mathbb{R}^+;\ p\in\mathbb{N}\right)$, *(2.8)*

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0,\ z\in U. \tag{2.9}$$

If

$$1+\delta\mu\left[\frac{\nu z\left(U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)\right)'+\eta z\left(U_{0,z}^{\alpha,\beta,\gamma}f(z)\right)'}{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}-p\right]\prec 1+\delta\frac{zq'(z)}{q(z)}, \tag{2.10}$$

then

$$\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q(z),$$

and the function q *is the best dominant of (2.10) . (the power is the principal one).*

Proof. *Let*

$$h(z)=\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\right]^{\mu},\ z\in U. \tag{2.11}$$

According to (2.8) the function h *is analytic in* U *. and differentiating (2.11) logarithmically with respect to* z *we get*

$$\frac{zh'(z)}{h(z)}=\mu\left[\frac{\nu z\left(U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)\right)'+\eta z\left(U_{0,z}^{\alpha,\beta,\gamma}f(z)\right)'}{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}-p\right]. \tag{2.12}$$

In order to prove our result we will use Lemma 1. Considering in this lemma

$$\theta(w)=1 \text{ and } \phi(w)=\frac{\delta}{w},$$

Then θ *is analytic in* $\mathbb{C}$ *and* $\phi(w)\neq 0$ *is analytic in* $\mathbb{C}^*$*. Also, if we let*

$$Q(z)=zq'(z)=\varphi(q(z))=\delta\frac{zq'(z)}{q(z)},$$

and

$$g(z)=\theta(q(z))+Q(z)=1+\delta\frac{zq'(z)}{q(z)},$$

then, since $Q(0)=1$ *and* $Q'(o)\neq 0$, *the assumption (2.9) yields that* Q *is a starlike function in* U *. From (2.9) we also have*

$$\Re\frac{zq'(z)}{Q(z)}=\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0,\ z\in U,$$

and then, by using Lemma 1 we deduce that the subordination (2.10) implies $h(z)\prec q(z)$ *and the function* q *is the best dominant of (2.10).*

Taking $\nu=0,\ \eta=\delta=1$ *and* $q(z)=\dfrac{1+Az}{1+Bz}$ *in Theorem 2 , it is easy to check that the assumption (2.9) holds whenever* $-1\le A<B\le 1$, *hence we obtain the next results.*

Corollary 3 *Let* $-\infty<\alpha<1; -\infty<\beta<1;\ \gamma\in\mathbb{R}^+;\ \mu\in\mathbb{C}^*;\ p\in\mathbb{N}$ *and* $-1\le A<B\le 1$. *Let* $f\in A_p$ *and suppose that*

$$\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\neq 0,\qquad z\in U.$$

If

$$1+\mu\left[\frac{z\left(U_{0,z}^{\alpha,\beta,\gamma}f(z)\right)'}{U_{0,z}^{\alpha,\beta,\gamma}f(z)}-p\right]\prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)},\tag{2.13}$$

then

$$\left[\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right]^{\mu}\prec\frac{1+Az}{1+Bz},$$

and the function $1+Az/1+Bz$ *is the best dominant of (2.13). (the power is the principal one).*

Remarks

1) Putting $\nu=0,\ \eta=p=1,\ \alpha=\beta=0,\ \delta=1/ab\ \left(a,b\in\mathbb{C}^*\right),\ \mu=a$, and $q(z)=(1-z)^{-2ab}$ in Theorem 2, then combining this together with Lemma 5 we obtain the corresponding result due to Obradović et al. [13, Theorem 1], see also Aouf and Bulboacă [3, Corollary 3.3].
2) For $\nu=0,\ \eta=p=1,\ \alpha=\beta=0,\ \delta=1/b\ \left(b\in\mathbb{C}^*\right),\ \mu=1$, and $q(z)=(1-z)^{-2ab}$, Theorem 2 reduces to the recent result of Srivastava and Lashin [27].
3) Putting $\nu=0,\ \eta=p=\delta=1,\ \alpha=\beta=0$, and $q(z)=(1+Bz)^{\mu(A-B)/B}$ $(-1\le B<A\le 1,\ B\neq 0)$ in Theorem 2, and using Lemma 5 we get the corresponding result due to Aouf and Bulboacă [3, Corollary 3.4].
4) Putting $\nu=0,\ \eta=p=1,\ \alpha=\beta=0$,

$\delta = e^{i\lambda}/ab\cos\lambda\left(a,b\in\mathbb{C}^*;|\lambda|<\pi/2\right), \mu = a$ and $q(z) = (1-z)^{-2a\cos\lambda e^{-i\lambda}}$ in Theorem 2, we obtain the corresponding result due to Aouf et al. [4, Theorem 1], see also Aouf and Bulboacă [3, Corollary 3.5].

Theorem 3 *Let $q(z)$ be a univalent function in U, with $q(0)=1$. Let $\lambda,\mu\in\mathbb{C}^*$ and $\nu,\eta,\delta,\Omega\in\mathbb{C}$ with $\nu+\eta\neq 0$. Let $f\in A_p$ and suppose that f and q satisfy the conditions:*

$$\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\neq 0,\quad z\in U$$

$\left(-\infty<\alpha<1;-\infty<\beta<1;\ \gamma\in\mathbb{R}^+;\ p\in\mathbb{N}\right)$ *(2.14)*

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0;-\operatorname{Re}\frac{\delta}{\lambda}\right\},\ z\in U, \tag{2.15}$$

If

$$\psi(z)=\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\right]^{\mu}$$

$$\times\left[\delta+\mu\lambda\left(\frac{\nu z\left(U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)\right)'+\eta z\left(U_{0,z}^{\alpha,\beta,\gamma}f(z)\right)'}{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}-p\right)\right]+\Omega, \tag{2.16}$$

and

$$\psi(z)\prec\delta q(z)+\lambda zq'(z)+\Omega, \tag{2.17}$$

then

$$\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q(z),$$

and the function q is the best dominant of (2.17) (all the power are the principal ones).

Proof. Let $h(z)$ be defined by (2.11), the we have from (2.12)

$$zh'(z)=\mu h(z)\left[\frac{\nu z\left(U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)\right)'+\eta z\left(U_{0,z}^{\alpha,\beta,\gamma}f(z)\right)'}{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}-p\right].$$

Let us consider the following functions:

$$\theta(w)=\delta w+\Omega,\ \text{and}\ \ \phi(w)=\lambda, w\in\mathbb{C},$$

$$Q(z) = zq'(z) = \varphi(q(z)) = \lambda \frac{zq'(z)}{q(z)},\ z \in U\ ,$$

and

$$g(z) = \theta(q(z)) + Q(z) = \delta q(z) + \lambda zq'(z) + \Omega,\ z \in U\,.,$$

From the assumption (3.15) we see that Q is starlike in U and, that

$$\Re \frac{zq'(z)}{Q(z)} = \Re \left\{ \frac{\delta}{\lambda} + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0,\ z \in U\,,$$

thus, by applying Lemma 1 the proof is completed.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 3, where $-1 \le B < A \le 1$, and according to (2.5),

the condition (2.15) becomes

$$\max\left\{0; -\operatorname{Re}\frac{\delta}{\lambda}\right\} \le \frac{1-|B|}{1+|B|}.$$

Hence, for the special case $\nu = \lambda = 0,\ \eta = 0$, we obtain the following result:

Corollary 4 *Let* $-1 \le B < A \le 1$, $\mu \in \mathbb{C}^*$ *and* $\delta \in \mathbb{C}$ *with*

$$\max\{0; -\operatorname{Re}\delta\} \le \frac{1-|B|}{1+|B|}.$$

Let $f \in A_p$ *and suppose that*

$$\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p} \ne 0, \qquad z \in U\ \left(-\infty < \alpha < 1; -\infty < \beta < 1;\ \gamma \in \mathbb{R}^+;\ p \in \mathbb{N}\right),$$

$$\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} f(z)}{z^p}\right)^{\mu} \left[\delta + \mu\left(\frac{z\left(U_{0,z}^{\alpha,\beta,\gamma} f(z)\right)'}{U_{0,z}^{\alpha,\beta,\gamma} f(z)} - p\right)\right] + \Omega \prec \delta\frac{1+Az}{1+Bz} + \Omega + \frac{(A-B)z}{(1+Bz)^2}, \quad (2.18)$$

then

$$\left[\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right]^{\mu} \prec \frac{1+Az}{1+Bz},$$

and the function $1+Az/1+Bz$ *is the best dominant of (2.18) (all the powers are the principal ones).*

Remark Taking $\nu = 0,\ \eta = \lambda = p = 1,\ \alpha = \beta = 0$ and $q(z) = \dfrac{1+z}{1-z}$ in Theorem 3 we obtain the corresponding result due to Aouf and Bulboacă [3, Corollary 3.7].

3 Superordination and Sandwich Results

Theorem 4 *Let* $q(z)$ *be convex function in* U, *with* $q(0)=1$. *Let* $-\infty<\alpha<1$, $-\infty<\beta<1$, $\gamma\in\mathbb{R}^+, p\in\mathbb{N}$ *and* $\lambda\in\mathbb{C}^*$ *with* $(p-\beta)\operatorname{Re}\lambda>0$. *Let* $f\in A_p$ *and suppose that* $U_{0,z}^{\alpha,\beta,\gamma}f(z)/z^p\in H[q(0),1]\cap Q$. *If the function*

$$\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right),$$

is univalent in U, *and*

$$q(z)+\frac{\lambda zq'(z)}{p(p-\beta)}\prec\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right), \tag{3.1}$$

then

$$q(z)\prec\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p},$$

and q *is the best subordinate of (3.1).*

Proof. Let us define the function g by

$$g(z)=\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p},\quad z\in U.$$

From the assumption of the theorem, the function g is analytic in U, by differentiating logarithmically with respect to z the function g, we deduce that

$$\frac{zg'(z)}{g(z)}=\frac{z\left(U_{0,z}^{\alpha,\beta,\gamma}f(z)\right)'}{U_{0,z}^{\alpha,\beta,\gamma}f(z)}-p. \tag{3.2}$$

After some computations, and using the identity (1.11), from (3.2) we get

$$g(z)+\frac{\lambda zg'(z)}{p(p-\beta)}=\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right)$$

and now, by using Lemma 4 we get the desired result.

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 4 , where $-1\le B<A\le 1$, hence we obtain the next results.

Corollary 5 *Let* $q(z)$ *be convex function in* U, *with* $q(0)=1$. *Let* $-\infty<\alpha<1$, $-\infty<\beta<1$, $\gamma\in\mathbb{R}^+, p\in\mathbb{N}$ *and* $\lambda\in\mathbb{C}^*$ *with* $(p-\beta)\operatorname{Re}\lambda>0$. *Let* $f\in A_p$ *and suppose that* $U_{0,z}^{\alpha,\beta,\gamma}f(z)/z^p\in H[q(0),1]$. *If the function*

$$\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f\ (z)}{z^{p}}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f\ (z)}{z^{p}}\right),$$

is univalent in U *, and*

$$\frac{1+Az}{1+Bz}+\frac{\lambda(A-B)z}{p(p-\beta)(1+Bz)^{2}}\prec\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f\ (z)}{z^{p}}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f\ (z)}{z^{p}}\right), \qquad (3.3)$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{U_{0,z}^{\alpha,\beta,\gamma}f\ (z)}{z^{p}},$$

and $1+Az/1+Bz$ *is the best subordinate of (3.3).*

Using arguments similar to those of the proof of Theorem 3, and then by applying Lemma 3 we obtain the following result.

Theorem 5 *Let* $q(z)$ *be convex function in* U *, with* $q(0)=1$. *Let* $\lambda,\mu\in\mathbb{C}^{*}$ *and* $\nu,\eta,\delta,\Omega\in\mathbb{C}$ *with* $\nu+\eta\neq 0$ $\operatorname{Re}(\delta/\lambda)>0$. *Let* $f\in A_{p}$ *and suppose that* f *satisfies the conditions:*

$$\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f\ (z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f\ (z)}{(\nu+\eta)z^{p}}\neq 0, \quad z\in U$$

$\left(-\infty<\alpha<1; -\infty<\beta<1;\ \gamma\in\mathbb{R}^{+};\ p\in\mathbb{N}\right)$,

and

$$\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f\ (z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f\ (z)}{(\nu+\eta)z^{p}}\right]^{\mu}\in H[q(0),1]\cap Q$$

If the function ψ *given by (2.16) is univalent in* U *, and*

$$\delta q(z)+\lambda zq'(z)+\Omega\prec\psi(z), \qquad (3.4)$$

then

$$q(z)\prec\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f\ (z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f\ (z)}{(\nu+\eta)z^{p}}\right]^{\mu},$$

and the function q *is the best subordinate of (3.4). (all the power are the principal ones).*

Combining Theorem 1 with Theorem 4 and Theorem 3 with Theorem 5, we obtain, respectively, the following two sandwich results:

Theorem 6 *Let* q_1 *and* q_2 *be two convex function in* U *, with* $q_1(0)=q_2(0)=1$. *Let* $-\infty<\alpha<1$, $-\infty<\beta<1$, $\gamma\in\mathbb{R}^{+}, p\in\mathbb{N}$ *and* $\lambda\in\mathbb{C}^{*}$ *with* $(p-\beta)\operatorname{Re}\lambda>0$. *Let* $f\in A_{p}$ *and suppose that* $U_{0,z}^{\alpha,\beta,\gamma}f\ (z)/z^{p}\in H[q(0),1]\cap Q$. *If the function*

$$\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right),$$

is univalent in U *, and*

$$q_1(z)+\frac{\lambda z q_1'(z)}{p(p-\beta)}\prec\frac{\lambda}{p}\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right)\prec q_2(z)+\frac{\lambda z q_2'(z)}{p(p-\beta)}, \tag{3.5}$$

then

$$q_1(z)\prec\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinate and the best dominant of (3.5).*

Theorem 7 *Let* q_1 *and* q_2 *be two convex function in* U *, with* $q_1(0)=q_2(0)=1$. *Let* $\lambda,\mu\in\mathbb{C}^*$ *and* $\nu,\eta,\delta,\Omega\in\mathbb{C}$ *with* $\nu+\eta\neq 0$ $\operatorname{Re}(\delta/\lambda)>0$. *Let* $f\in A_p$ *and suppose that* f *satisfies the conditions:*

$$\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\neq 0,\quad z\in U$$

$$\left(-\infty<\alpha<1;-\infty<\beta<1;\ \gamma\in\mathbb{R}^+;\ p\in\mathbb{N}\right),$$

and

$$\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\right]^{\mu}\in H[q(0),1]\cap Q$$

If the function ψ *given by (2.16) is univalent in* U*, and*

$$\delta q_1(z)+\lambda z q_1'(z)+\Omega\prec\psi(z)\prec\delta q_2(z)+\lambda z q_2'(z)+\Omega, \tag{3.6}$$

then

$$q_1(z)\prec\left[\frac{\nu U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)+\eta U_{0,z}^{\alpha,\beta,\gamma}f(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinate and the best dominant of (3.6).*
(all the power are the principal ones).

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