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Method of Successive Approximations for Solving the Multi-Pantograph Delay Equations

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Abstract

In this paper, the successive approximations method is applied to solve multi-pantograph equations. The multi-pantograph equation is a kind of delay differential equations (DDEs). By this method, good approximate solutions can be obtained with only a few iteration. In addition, this method can be applied easily to solve neutral functional-differential equation with proportional delays. In this sense, to illustrate the method, some examples are provided. The results show that the method is very effective.

Keywords: *Multi-pantograph equation, Neutral functional-differential equation, Successive approximations method, Picard-Lindelof method, Initial value problems.*

1 Introduction

Many real-life phenomena in physics, engineering, biology, and etc. can be modeled by an initial value problem (IVP) of the type

$$\begin{cases} \dfrac{dy}{dt} = F(t, y(t)), \ t \geqslant t_0, \\ \\ y(t_0) = y_0. \end{cases} \tag{1}$$

In order to make the model more consistent with the real phenomena, it is sometimes necessary to modify the right-hand side of (1) to include also the dependence of the derivative y' on past values of the state variable y. The most general form of such models is given by [5]

$$\begin{cases} \dfrac{dy}{dt} = G(t, y_\alpha(t)),\ t \geqslant t_0, \\ \\ y_\alpha(t_0) = y(t_0 + \theta) = \Phi(\theta),\ \theta \in [-r, 0], \end{cases} \tag{2}$$

where $\Phi(\theta)$ represents initial value. The multi-pantograph equation is a kind of delay differential equations in the following form

$$\begin{cases} y'(t) = a(t)y(t) + b_1(t)y(p_1 t) + b_2(t)y(p_2 t) + \cdots b_l(t)y(p_l t) + f(t), \\ \\ y(0) = y_0,\ 0 < t \leqslant T, \end{cases} \tag{3}$$

where f is a given function and $0 < p_1 < p_2 < \cdots < p_l \leqslant 1$. Many powerful methods such as the variational iteration method (VIM) [2, 6, 7], the homotopy perturbation method (HPM) [1], and the Taylor polynomials method [4] (for further see references therein) have been presented to solve multi-pantograph equations. In this paper, the successive approximations method is applied to solve multi-pantograph delay equations and neutral functional-differential equations.

This paper is organized as follows. In section 2, the analysis of the method is introduced. Subsequently, in section 3, some examples are provided. Finally, in section 4, conclusion is presented.

2 Successive Approximations Method

As we know, it is almost impossible to obtain the analytic solution of an arbitrary differential equation. Hence, numerical methods are usually used to obtain information about the exact solution. First order differential equations can be solved by the well-known successive approximations method (Picard-Lindelof method) [3]. In this method the approximate solution for solving (1) is defined as

$$y_{k+1} = y_0 + \int_0^t F(y_k, x)dx,\ k \in \mathbb{N}. \tag{4}$$

By continuing this process, when $k \longrightarrow \infty$, the exact solution is obtained. In practice, the exact solution is approximated for sufficient large k by y_k. For solving multi-pantograph equation (i.e. Eq (3)), this method can be applied as

$$y_{k+1} = y_0 + \int_0^t \left[a(x)y_k(x) + \sum_{i=1}^{l} b_i(x)y_k(p_i x) + f(x) \right] dx,\ k \in \mathbb{N}. \tag{5}$$

Besides, the successive approximations method can be applied easily to solve neutral equation with proportional delays defined as [1, 2]

$$(y(t)+a(t)y(p_m t))^{(m)} = \beta y(t) + \sum_{l=0}^{m-1} b_l(t) y^{(l)}(p_l t) + f(t),\ t \geqslant 0, \tag{6}$$

with initial conditions

$$y^{(j)}(0) = \lambda_j,\ j = 0, 1, \cdots, m-1. \tag{7}$$

In this way, the following well-known identity is applied

$$\int_0^t \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_{n-1}} g(x_n) dx_n \cdots dx_2 dx_1 = \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} g(x) dx. \tag{8}$$

Integrating both sides of (6) yields

$$y(t) = -a(t)y(p_m t) + \sum_{j=0}^{m-1} \lambda_j t^j + \int_0^t \int_0^{x_1} \cdots \int_0^{x_{n-1}} \left[\beta y(x_n) + \sum_{l=0}^{m-1} b_l(x_n) y^{(l)}(p_l x_n) + f(x_n) \right] dx_n \cdots dx_2 dx_1. \tag{9}$$

The identity (8) simplifies the equation (9) as

$$y(t) = -a(t)y(p_m t) + \sum_{j=0}^{m-1} \lambda_j t^j + \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} \left[\beta y(x) + \sum_{l=0}^{m-1} b_l(x) y^{(l)}(p_l x) + f(x) \right] dx. \tag{10}$$

Consequently, the recursive relation is obtained as

$$y_{k+1}(t) = -a(t)y_k(p_m t) + \sum_{j=0}^{m-1} \lambda_j t^j + \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} \left[\beta y_k(x) + \sum_{l=0}^{m-1} b_l(x) y_k^{(l)}(p_l x) + f(x) \right] dx. \tag{11}$$

3 Illustrative Examples

In this section, in order to illustrate the method, we consider four examples [2, 4].

Example 1. Consider the following multi-pantograph equation

$$\begin{cases} y'(t) = -y(t) + \mu_1(t)y(\frac{t}{2}) + \mu_2(t)y(\frac{t}{4}), \\ \\ y(0) = 1, \ 0 \leqslant t \leqslant 1, \end{cases} \tag{12}$$

where $\mu_1(t) = -e^{-0.5t}\sin(0.5t)$, $\mu_2(t) = -2e^{-0.75t}\cos(0.5t)\sin(0.25t)$. The exact solution is $y(t) = e^{-t}\cos(t)$. In figure 1, the error functions $|y(t) - y_{10}(t)|$ and $|y(t) - y_{10}(t)|/|y(t)|$ are plotted. Better approximations can be obtained using more iterations.

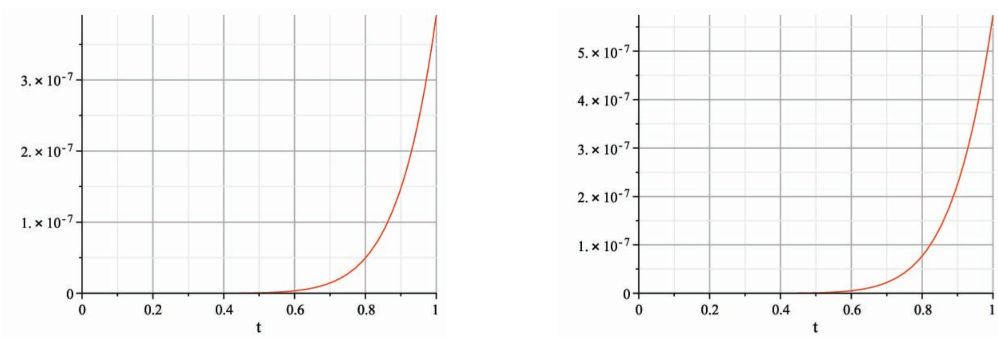

Figure 1: Plot of the $|y(t) - y_{10}(t)|$ (left) and $|y(t) - y_{10}(t)|/|y(t)|$ (right).

Example 2. Consider the following pantograph equation

$$\begin{cases} y'(t) = \frac{1}{2}e^{\frac{t}{2}}y(\frac{t}{2}) + \frac{1}{2}y(t), \\ \\ y(0) = 1, \ 0 \leqslant t \leqslant 1. \end{cases} \tag{13}$$

The exact solution is $y(t) = e^{t}$. In figure 2, the error functions $|y(t) - y_{10}(t)|$ and $|y(t) - y_{10}(t)|/|y(t)|$ are plotted.

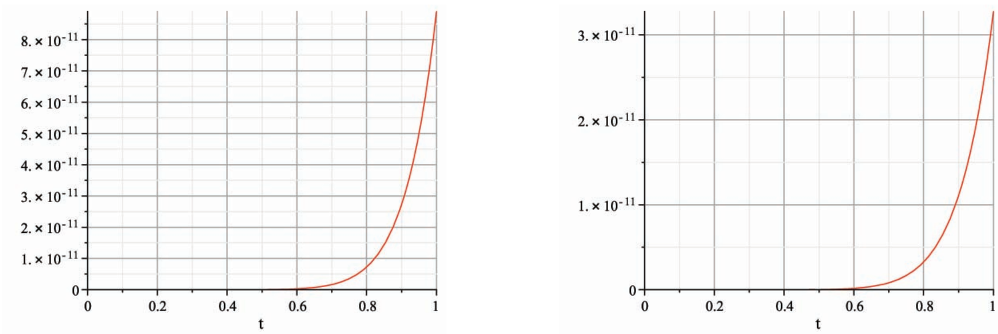

Figure 2: Plot of the $|y(t) - y_{10}(t)|$ (left) and $|y(t) - y_{10}(t)|/|y(t)|$ (right).

Example 3. Consider the following neutral functional-differential equation

$$\begin{cases} y'(t) = -y(t) + 0.1y(0.8t) + 0.5y'(0.8t) + (0.32t - 0.5)e^{-0.8t} + e^{-t}, \\ \\ y(0) = 0, \;\; 0 \leqslant t \leqslant 1. \end{cases} \tag{14}$$

The exact solution is $y(t) = te^{-t}$. In figure 3, the error functions $|y(t) - y_{10}(t)|$ and $|y(t) - y_{10}(t)|/|y(t)|$ are plotted.

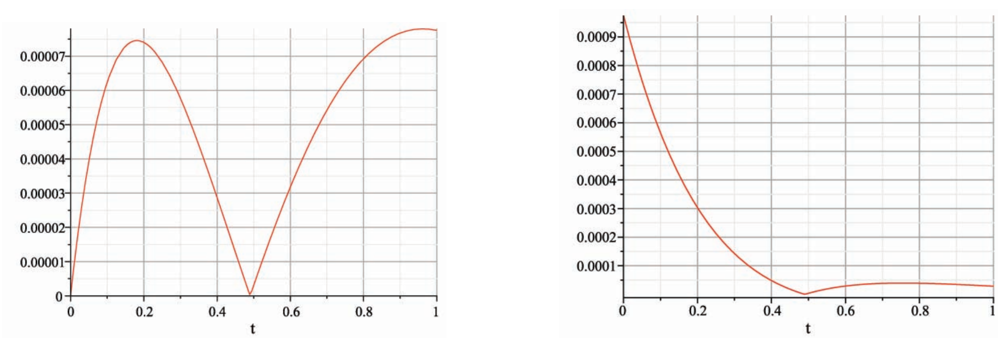

Figure 3: Plot of the $|y(t) - y_{10}(t)|$ (left) and $|y(t) - y_{10}(t)|/|y(t)|$ (right).

Example 4. Consider the following neutral functional-differential equation

$$\begin{cases} y''(t) = y'(\frac{1}{2}t) - \frac{1}{2}ty''(\frac{1}{2}t) + 2, \\ \\ y(0) = 1, \;\; y'(0) = 0, \;\; 0 \leqslant t \leqslant 1. \end{cases} \tag{15}$$

The exact solution is $y(t) = t^2 + 1$. By using (11), with only two iterations, the exact solution is obtained.

4 Conclusion

In this paper, we applied successive approximations method to solve multi-pantograph and neutral functional-differential equations and obtain high approximate solutions with a few iteration. It is concluded from figures that the successive approximations method is an accurate and efficient method to solve multi-pantograph equations and neutral functional-differential equations.

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