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Alternating Direction Implicit Formulation of the Differential Quadrature Method for Solving the Unsteady State Two-Dimensional Convection-Diffusion Equation

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Abstract

In this paper, we propose alternating direction implicit formulation of the differential quadrature method (ADI-DQM) for calculating the numerical solutions of the two-dimensional unsteady state convection-diffusion equation with appropriate initial and boundary conditions. The results confirm that this method has a high accuracy, good convergence and less workload compared with the other numerical methods.

Keywords: *Differential quadrature method, Convection-diffusion, ADI, Accuracy*.

1 Introduction

Consider the unsteady state two -dimensional convection-diffusion equation

$$\frac{\partial u}{\partial t}(x,y,t) + \beta_x \frac{\partial u}{\partial x}(x,y,t) + \beta_y \frac{\partial u}{\partial y}(x,y,t)$$

$$(x,y,t) \in [0,L] \times [0,L] \times [0,T], \tag{1.1a}$$

with the initial condition

$$u(x,y,t) = \emptyset(x,y) \tag{1.1b}$$

and the boundary conditions

$$u(x,0,t) = f_0(x,t), \quad u(x,L,t) = f_1(x,t), \;\; t \geq 0\,,$$

$$u(0,y,t) = g_0(y,t), \quad u(L,y,t) = g_1(y,t), \;\; t \geq 0, \tag{1.1c}$$

where β_x and β_y are arbitrary constants and represent convection coefficients, α_x and α_y are arbitrary functions and represent diffusion coefficients, , u is a transported variable, $\emptyset$, f_0, f_1, g_0 and g_1 are the known functions and T is the optimal time. Convection-diffusion equation is a parabolic partial differential equation combining the diffusion equation and the advection equation, which always attracts the attention of many researchers its importance to academics**.** Processes involving a combination of convection and diffusion are found in physical and engineering problems. These problems arise in petroleum reservoir simulation, subsurface contaminant remediation, and many other applications [1-7,13]. Many researchers use the Equation (1.1a) and mentioned in [4, 5, 6, 13]. We compare the numerical results of DAI-DQM for solving convection-diffusion problem (1.1) with the results of other numerical methods such as the differential quadrature method (DQM) , the finite difference method (FDM) [5] and the radial basis function based meshless method (RBFBMM) [4].
The purpose of this paper is to introduce and apply our newly developed of DQM that is known as the alternating direction implicit formulation of the differential quadrature method for solving unsteady state two-dimensional convection-diffusion equation. The results that we obtain from using ADI-DQM will be saved and compared to prove the efficiency of the method in accuracy and stability.

2 Differential Quadrature Method

The differential quadrature is a numerical technique used to solve the initial and boundary value problems. This method was proposed by Bellman in the early 70s [2]. The essence of the method is that, the partial (ordinary) derivatives of a function with respect to variables in governing equation are approximated by a weighted linear sum of function values at all discrete points in that direction ((here, let $h = \Delta x = \Delta y$ denote the step size of spatial space and Δt is the step size with respect to time)), then the equation can be transformed into a set of ordinary differential equations or algebraic equations. According to the DQM, the r^{th} -order partial derivatives $\frac{\partial^r u}{\partial x^r}$ of a function $u(x,y)$ at a point (x_i, y_j) and the s^{th} -order partial derivatives $\frac{\partial^s u}{\partial y^s}$ of a function $u(x,y)$ at a point (x_i, y_j) , can be approximated by the same formula given in [11],as:

$$\left.\frac{\partial^r u}{\partial x^r}\right|_{x=x_i} = \sum_{k=1}^{N} A_{ik}^{(r)}\, u(x_k, y)\ , \qquad i = 1,2,\dots,N \tag{2.1}$$

$$\left.\frac{\partial^s u}{\partial y^s}\right|_{y=y_j} = \sum_{l=1}^{M} B_{jl}^{(s)} u(x, y_l)\ , \qquad j = 1,2,\dots,M \tag{2.2}$$

where $A_{ik}^{(r)}$, $B_{jl}^{(s)}$ are the respective weighting coefficients for the r^{th} -order and s^{th}-order derivatives with respect to x and y respectively. Bellman et al. [2] proposed two approaches to compute the weighting coefficients $A_{ik}^{(r)}$, $B_{jl}^{(s)}$. To improve Bellman's approaches in computing the weighting coefficients, many attempts have been made by researchers. Quan and Chang [8, 9] introduce one of the most valuable attempts. After that, Shu's [11] introduced a general approach, which was inspired from Bellman's approach, was made available in the literature. Shu's [11] give Shu's recurrence formulation for higher order derivatives as,

$$A_{ik}^{(r)} = r\left(A_{ii}^{(r-1)} A_{ik}^{(1)} - \frac{A_{ik}^{(r-1)}}{(x_i - x_k)}\right)\ ,\ \ k, i = 1, \dots, N, \quad 2 \le r \le N-1\ ,\ \ i \neq k \tag{2.3}$$

and

$$A_{ii}^{(r)} = -\sum_{k=1}^{N} A_{ik}^{(r)}\ ,1 \le r \le N-1\ ,\ \ i \neq k\ , \quad i = 1,2,\dots,N \tag{2.4}$$

where $A_{ik}^{(1)}$ are the weighting coefficients of the first order derivative given below

$$A_{ik}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_k) M^{(1)}(x_k)} \qquad for \quad i \neq k$$

where

$$M(x) = (x - x_1)(x - x_2) \dots (x - x_N)\ \ and\ \ M^{(1)}(x_i) = \prod_{j=1}^{N} (x_i - x_j)\ \ i, k \neq j$$

The same formulas can be obtained for weighting coefficients of the high order derivatives with respect to y. By using equations (2.1) and (2.2), we can approximate the partial derivatives of the convection-diffusion equation (1.1) to obtain the system of ordinary differential equations as:

$$\left.\frac{\partial u}{\partial t}\right|_{ij}^{n} + \sum_{k=1}^{N} \beta_x A_{ik}^{(1)} u_{kj}^{n} + \sum_{l=1}^{M} \beta_y B_{jl}^{(1)}\, u_{il}^{n} = \sum_{k=1}^{N} \alpha_x A_{ik}^{(2)} u_{kj}^{n} + \sum_{l=1}^{M} \alpha_y\, B_{jl}^{(2)} u_{il}^{n} \tag{2.5}$$

Approximating the first-order derivatives with respect to the temporal variable by using the forward differences and then arrangement the terms equation (2.5) ,we obtain the system of algebraic equations as:

$$\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\Delta t}+\sum_{k=1}^{N}\left(\beta_x A_{ik}^{(1)}-\alpha_x A_{ik}^{(2)}\right)u_{kj}^{n}+\sum_{l=1}^{M}(\beta_y B_{jl}^{(1)}-\alpha_y\, B_{jl}^{(2)})u_{il}^{n}=0 \qquad (2.6)$$

3 Alternating Direction Formulation of the DQM

Peaceman and Rachford [7] introduced the alternating direction implicit technique in the mid-50s for solving the system of algebraic equations, which results from finite difference discretization of partial differential equations (PDEs). From iterative method's perspective, ADI method can be considered as a special relaxation method, where a big system is simplified into a number of smaller systems such that each of them can be solved efficiently and the solution of the whole system is got from the solutions of the sub-systems in an iterative method. Using alternating direction implicit method into equation (2.6), we get the following two systems of algebraic equations in the form:

$$\frac{u_{ij}^{n+\frac{1}{2}}-u_{ij}^{n}}{\frac{\Delta t}{2}}+\sum_{k=1}^{N}\left(\beta_x A_{ik}^{(1)}-\alpha_x A_{ik}^{(2)}\right)u_{kj}^{n+\frac{1}{2}}+\sum_{l=1}^{M}(\beta_y B_{jl}^{(1)}-\alpha_y\, B_{jl}^{(2)})u_{il}^{n}=0 \qquad (3.1)$$

$$\frac{u_{ij}^{n+1}-u_{ij}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}+\sum_{k=1}^{N}\left(\beta_x A_{ik}^{(1)}-\alpha_x A_{ik}^{(2)}\right)u_{kj}^{n+\frac{1}{2}}+\sum_{l=1}^{M}(\beta_y B_{jl}^{(1)}-\alpha_y\, B_{jl}^{(2)})u_{il}^{n+1}=0 \qquad (3.2)$$

Formula (3.1) is used to compute function values at all interval mesh points along rows and is known as a horizontal traverse or $x-$sweep. While, Formula (3.2) is used to compute function values at all interval mesh points along columns and is known as a vertical traverse or $y-$sweep.

4 Numerical Experiments and Discussion

In this section, we apply ADI-DQM on three test problems to demonstrate the efficiency of the ADI-DQM. Other researchers also consider these problems.

Problem 1. (Akman [1])

We consider convection-diffusion (1.1) with $\alpha_x=\alpha_y=1 \quad \beta_x=\beta_y=0$, $L=1$ and initial condition in the following form:

$$u(x,y,0)=sin(\pi x)\sin(2\pi y)\,, \quad 0\le x,y\le 1 \qquad (4.1)$$

The exact solution is given by

$$u(x,y,t)=\ e^{-5\pi^2 t}\sin(\pi x)\sin(2\pi y)\ , \qquad 0\le x,y\le 1\,,t>0 \qquad (4.2)$$

The boundary conditions can be obtained easily from (4.2) by using$x,y=0,1$. In this problem, we put $\Delta t=0.0001$ and use equally spaced grid points. In Table 1 we show the errors obtained in solving problem 1 with the ADI-DQM and DQM at $t=0.01$ and $(x,y)\in\ [0,1]$ for different values of h. In Fig. (1) we show the exact and approximate solutions of the problem 1 . The results confirm that ADI-DQM has a high accuracy, good convergence compared with DQM.

Table 1. Errors obtained for problem 1 with $t = 0.01$

h	$Max|error|$ *of DQM*	$Max|error|$ *of ADI-DQM*
0.2	6.729439E-03	1.994167E-05
0.111	9.071963E-03	1.337680E-05
0.09	2.272577E-03	1.031041E-05

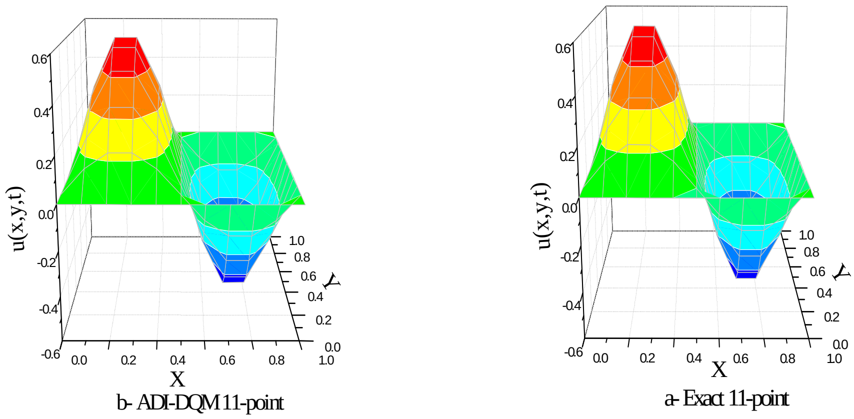

Fig. 1 Exact and approximate solutions of the problem 1 with, t=0.01 and $\Delta t = 0.0001$

Problem 2 . (Akman [1])

We consider convection-diffusion equation (1.1) with $\beta_x = \beta_y = 0$, $\alpha_x = \frac{1}{4}(1-y^2)$, $\alpha_y = \frac{1}{4}(1-y^2)$, $L = 0.9$ and initial condition in the following form:

$$u(x,y,0) = (1-x^2)(1-y^2) \quad , \quad 0 \le x,y \le 0.9 \tag{4.3}$$

The exact solution is given by

$$u(x,y,t) = (1-x^2)(1-y^2)\, e^{-t} \quad , \quad 0 \le x,y \le 0.9, t > 0 \tag{4.4}$$

The boundary conditions can be obtained easily from (4.4) by using $x, y = 0, 0.9$. In this problem we put $\Delta t = 0.0001$ and use equally spaced grid points. In Table 2 we show the errors obtained in solving problem 2 with the ADI-DQM and DQM at $t = 0.1$ and $(x,y) \in [0, 0.9]$ for different values of h. In Figs. (2) we show the exact and approximate solutions of the problem 2. The results confirm that ADI-DQM has a high accuracy, good convergence compare with DQM

Table 2. Errors obtained for problem 2 with $t = 0.1$

h	$Max|error|$ *of DQM*	$Max|error|$ *of ADI-DQM*
0.18	1.636210E-03	2.316217E-04
0.11	3.174482E-03	4.667084E-04
0.08	3.142371E-03	5.668763E-04

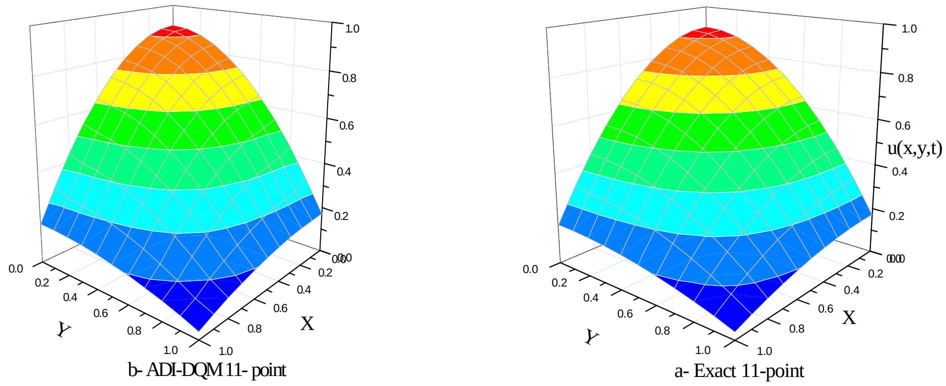

Fig. 2 Exact and approximate solutions of the problem 2 with, $t = 0.1$ and $\Delta t = 0.0001$

Problem 3. (Dehghan and Mohebbi [5])

We consider convection-diffusion equation (1.1) with, $\beta_x = \beta_y = -1$, $\alpha_x = \alpha_y = 0.01,\ 0.1$, $L = 1$ and initial condition in the following form:

$$u(x, y, 0) = a\left(\mathrm{e}^{(-c_x x)} + \mathrm{e}^{(-c_y y)}\right), \quad 0 \le x, y \le 1 \tag{4.5}$$

In which $c_x = \frac{-\beta_x \mp \sqrt{\beta_x^2 + 4b\alpha_x}}{2\alpha_x} > 0$, $c_y = \frac{-\beta_y \mp \sqrt{\beta_y^2 + 4b\alpha_y}}{2\alpha_y} > 0$

The exact solution is given with

$$u(x, y, t) = a\,\mathrm{e}^{(\mathrm{bt})}(\mathrm{e}^{(-c_x x)} + \mathrm{e}^{(-c_y y)}) \qquad 0 \le x, y \le 1, \mathrm{t} > 0 \tag{4.6}$$

The boundary conditions can be obtained easily from (4.6) by using$x, y = 0, 1$. In this problem we put $a = 1, b = 0.1$, $\Delta t = 0.001$ and use equally spaced grid points. In Table 3 we show the errors obtained in solving problem 3 with the ADI-DQM and DQM at $t = 0.1, \alpha_x = \alpha_y = 0.01$ and $(x, y) \in [0, 1]$ for different values of h. In Figs. (3) and (4) we show the exact and approximate solutions for $\alpha_x = \alpha_y = 0.01$ and $\alpha_x = \alpha_y = 0.1$ respectively. In Table 4 we show the errors obtained in solving problem 3 with the present method $t = 0.1, \alpha_x = \alpha_y = 0.1$ and $(x, y) \in [0, 1]$ for different values of h. The results confirm that ADI-DQM has a high accuracy, good convergence compare with DQM.

Table 3. Errors obtained for problem 3 with $t = 0.1, \alpha_x = \alpha_y = 0.01$

h	$Max\|error\|$ of DQM	$Max\|error\|$ of ADI-DQM
0.2	4.693330E-26	2.644524E -26
0.111	3.834508E-15	2.720141E-15
0.09	5.667731E-13	4.110175E -13

Table 4 . Errors obtained for problem 3 with $t = 0.1, \alpha_x = \alpha_y = 0.1$

h	$Max\|error\|$ of DQM	$Max\|error\|$ of ADI-DQM
0.2	1.617776E-06	1.055500E-06
0.111	9.042375E-06	6.623465E-06
0.09	1.471308E-05	1.131288E-05

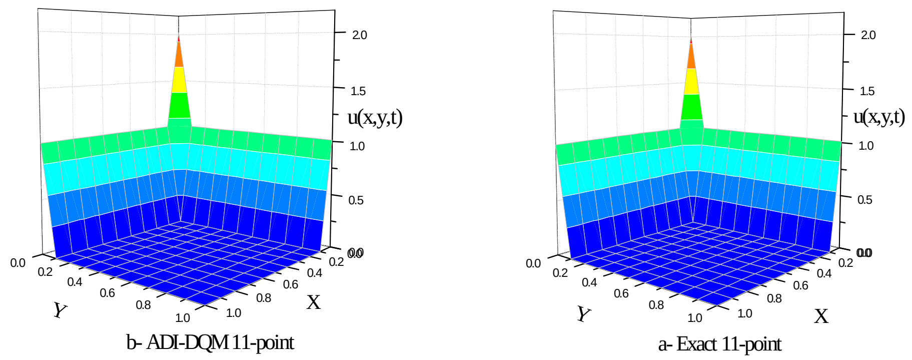

Fig. 3 Exact and approximate solution of the problem 3 with $\alpha_x = \alpha_y = 0.01$

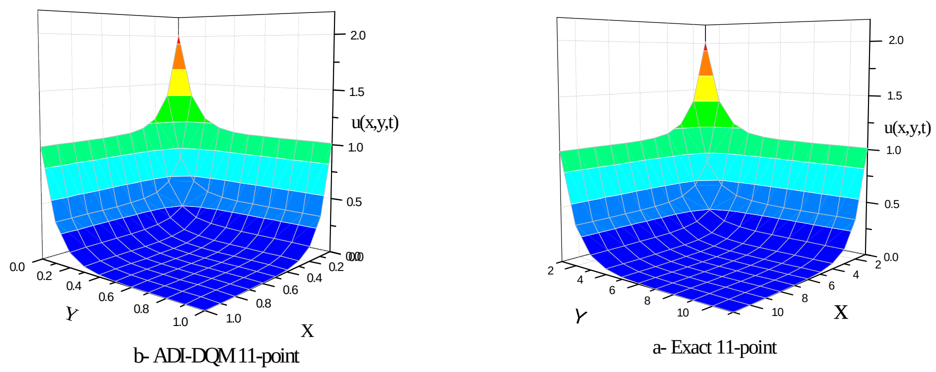

Fig. 4 Exact and approximate solutions of the problem 3 with $\alpha_x = \alpha_y = 0.1$

From Figs. 3 and 4, we note if case $\alpha_x = \alpha_y = 0.01$ has a sharp discontinuity near the left boundary while if case $\alpha_x = \alpha_y = 0.1$ is not able to capture a sharp discontinuity near the left boundary.

5 Error Analysis and Stability of DQM

We can resolve another mission of the truncation error in the differential quadrature method. Depending on the DQM is identical to Lagrange polynomial interpolation of order $N-1$, Chen [3] has presented new formulas for the analysis of truncation error distribution of derivative in this method. The truncation error of the first-order derivative approximation by the DQ method at the grid point x_i is given as;

$$\varepsilon^{(1)}(x_i) \leq \frac{K_1 M^{(1)}(x_i)}{N!} = K_1 e^{(1)}(x_i) \tag{5.1}$$

where $K_1 = Max\{|T^{(N)}(\xi)|\}$,ξ is unknown function of variable x, and $e^{(1)}(x_i)$ denotes the error distributions of the first- order derivative. For the truncation error of the second-order derivative approximation by DQM is given as,

$$\left|\varepsilon^{(2)}(x_i)\right| \le 2K_2\left(1+\left|A_{ii}^{(1)}\right|\right)\frac{M^{(1)}(x_i)}{N!} = K_2 e^{(2)}(x_i) \tag{5.2}$$

where $K_2 = Max\{(|T^{(N)}(\xi)|, |\xi_x T^{(N+1)}(\xi)|)\}$, $e^{(2)}(x_i)$ denotes the error distributions of the second- order derivative. While the stability, from equation (2.5) we obtained the systems of ordinary differential equations in the form:

$$[A]\{u\} = \{b\} - \{s\} \tag{5.3}$$

where $[A]$ is the coefficient matrix containing the weighting coefficients, the dimension of the matrix $[A]$ is $(N-2)(M-2)\ by\ (N-2)(M-2)$,$\{u\}$ is a vector of unknown functional values at all the interior points ,$\{b\}$ is a vector still containing discretized time derivatives of u and $\{s\}$ vector contains known values of u at the boundary grid points. The stability analysis of this equation is based on the eigenvalue distribution of the DQ discretization matrix $[A]$ .If $[A]$ has eigenvalues λ_i and corresponding eigenvector ξ_i, (i=1,2,…,K) K being the size of the matrix $[A]$, the similarity transformation reduces the system(5.3) of the from[1] .

$$\frac{d\{U\}}{dt} = [D]\{U\} + \{S\} \tag{5.4}$$

where $[D] = [P]^{-1}[A][P]$, $\{U\} = [P]^{-1}\{u\}$ and $\{S\} = -[P]^{-1}\{s\}$
Since $[D]$ is a diagonal matrix Equation (5.4) is an uncoupled set of ordinary differential equations and [P] is a nonsingular matrix containing the eigenvectors as columns. Considering the i^{th}equation of (5.4)

$$\frac{dU_i}{dt} = \lambda_i U_i + S_i \tag{5.5}$$

This system has the solution

$$\{u\} = [P]\{U\} = \sum_{i=1}^{K} U_i\xi_i = \sum_{i=1}^{K}\left[U_i(0)e^{\lambda_i t} + \frac{S_i}{\lambda_i}(e^{\lambda_i t-1)}\right]\xi_i$$

and this solution is stable as $t \to \infty$ if

$$Re(\lambda_i) < 0 \qquad , \qquad i = 1,2,\dots,K \tag{5.6}$$

where $Re(\lambda_i)$ denotes the real part of λ_i.This is the stability condition for the system (5.3).
We explain the stability condition (5.6) for problems 1,2 and 3 with using 4×4 grid points. the eigenvalues of the matrix [A] are;

For problem 1, $\lambda_1 = -18$, $\lambda_2 = -36$, $\lambda_3 = -36$ and $\lambda_4 = -54$.
For problem 2, $\lambda_1 = -4.041$, $\lambda_2 = -8.268$, $\lambda_3 = -8.61$ and $\lambda_4 = -13.519$
. For problem 3, $\lambda_1 = -3.6$, $\lambda_2 = -3.6$, $\lambda_3 = -3.6$ and $\lambda_4 = -3.6$.

This means the stability condition (5.6) is satisfied.

Zong and Lam (2002)[14] have shown that too large numbers of grid points may lead to instability. We conclude from the above discussion that accuracy requires large number of grid points, but stability requires the opposite. The accuracy and stability of the numerical solutions depend on the choice of grid points selected. Here we use equally spaced types, which are introduced by Shu and Richards (1992) [10], Shu et al (2001) [12],

6 Comparison with Other Schemes

We compare the numerical results of ADI-DQM for the problem 3 with the results of other numerical methods such as DQM, High-order compact boundary value method HOCBVM [5] and Radial basis function based meshless RBFBMM [4]. Tables 5 and 6 show the number of grid points and maximum absolute error in the numerical solutions resulted from using ADI-DQM with other methods. The error measurements resulted from ADI-DQM is more accurate than the methods DQM, HOCBVM and RBFBMM. Moreover, the number of grid points by using ADI-DQM is less than the other methods HOCBVM and RBFBMM.

Table 5. Comparison of the numerical results of the problem 3 for different methods at $t = 0.1$ and $\alpha_x = \alpha_y = 0.01$

Method	*Number of grid points*	$Max\|error\|$
ADI-DQM	11×11	4.110175E-13
DQM	11×11	5.667731E-13
HOCBVM [5]	100×100	9.4696E-04
RBFBMM. [4]	41×41	4.97E-02

Table 6. Comparison of the numerical results of the problem 3 for different method at $t = 0.1$ and $\alpha_x = \alpha_y = 0.1$

Method	*Number of grid points*	$Max\|error\|$
ADI-DQM	11×11	1.131288E-05
DQM	11×11	1.471307E-05
HOCBVM [5]	25×25	2.4170E-05
RBFBMM.[4]	41×41	4.25E-04

7 Conclusions

In this work, we employed the ADI-DQM to solve the unsteady state two-dimensional convection–diffusion equation. The numerical results show that the ADI-DQM has higher accuracy and good convergence as well as a less computation workload by using few grid points. The results show that ADI-DQM has a good potential for solving convection-diffusion problems. Moreover, the efficiency of the method was proved in accuracy and stability.

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