Gen. Math. Notes, Vol. 10, No. 2, June 2012, pp. 36-42
ISSN 2219-7184; Copyright © ICSRS Publication, 2012
www.i-csrs.org
Available free online at http://www.geman.in

Spacelike Biharmonic New Type *B*-Slant Helices According to Bishop Frame in the Lorentzian Heisenberg Group H^3

Talat Körpınar and Essin Turhan

Fırat University, Department of Mathematics 23119, Elazığ, Turkey
E-mails: talatkorpinar@gmail.com, essin.turhan@gmail.com

(Received: 20-5-12 / Accepted: 12-6-12)

Abstract

In this paper, we study biharmonic spacelike new type B-slant helices according to Bishop frame in the Lorentzian Heisenberg group H^3. We give necessary and sufficient conditions for new type B-slant helices to be biharmonic. We characterize these curves in the Lorentzian Heisenberg group H^3. Additionally, we illustrate our results.

Keywords: *Bienergy, Bishop frame, Lorentzian Heisenberg group.*

1 Introduction

Jiang derived the first and the second variation formula for the bienergy in [7,8], showing that the Euler--Lagrange equation associated to E_2 is

$$\tau_2(f) = -\mathrm{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df,\tau(f))df$$
$$= 0,$$

where J^f is the Jacobi operator of f . The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic.

Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.
This study is organised as follows: Firstly, we give necessary and sufficient conditions for new type $\mathbf{B}$-slant helices to be biharmonic. We characterize this curves in the Lorentzian Heisenberg group $\mathbb{H}^3$. Secondly, we study biharmonic spacelike new type $\mathbf{B}$-slant helices according to Bishop frame in the Lorentzian Heisenberg group $\mathbb{H}^3$. Finally, we illustrate our results.

2 The Lorentzian Heisenberg Group $\mathbb{H}^3$

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x}, y + \bar{y}, z + \bar{z} - \bar{x}y + x\bar{y}).$$

The identity of the group is (0,0,0) and the inverse of (x, y, z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on H^3 is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \qquad (1)$$

The characterising properties of this algebra are the following commutation relations, [13]:

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1,\ g(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \qquad (2)$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$\{\mathbf{e}_k, k = 1,2,3\}$.

3 Spacelike Biharmonic New Type $\mathbb{B}-$Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $\mathbb{H}^3$

Let $\gamma : I \to \mathbb{H}^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group $\mathbb{H}^3$ parametrized by arc length. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}^3$ along γ defined as follows:
$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{b}$ is chosen so that $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= \kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{n}, \end{aligned} \qquad (1)$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} &g(\mathbf{t},\mathbf{t}) = 1, g(\mathbf{n},\mathbf{n}) = -1, g(\mathbf{b},\mathbf{b}) = 1, \\ &g(\mathbf{t},\mathbf{n}) = g(\mathbf{t},\mathbf{b}) = g(\mathbf{n},\mathbf{b}) = 0. \end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= k_2\mathbf{t}, \end{aligned} \qquad (2)$$

where

$$\begin{aligned} &g(\mathbf{t},\mathbf{t}) = 1, g(\mathbf{m}_1,\mathbf{m}_1) = -1, g(\mathbf{m}_2,\mathbf{m}_2) = 1, \\ &g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{t},\mathbf{m}_2) = g(\mathbf{m}_1,\mathbf{m}_2) = 0. \end{aligned}$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures.
Also, $\tau(s) = \psi'(s)$ and $\kappa(s) = \sqrt{\left|k_2^2 - k_1^2\right|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\sinh\psi(s), \\ k_2 &= \kappa(s)\cosh\psi(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\mathbf{t} = t^1\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3,$$

$$\mathbf{m}_1 = m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3,5 \qquad (3)$$
$$\mathbf{m}_2 = m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3.$$

Theorem 3.1. $\gamma : I \to \mathbb{H}^3$ *is a spacelike biharmonic curve with Bishop frame if and only if*

$$k_1^2 - k_2^2 = \text{constant} = C \neq 0,$$

$$k_1'' + \left[k_1^2 - k_2^2\right]k_1 = -k_1\left[1 + \left(m_2^1\right)^2\right] + k_2 m_1^1 m_2^1, \qquad (4)$$

$$k_2'' + \left[k_1^2 - k_2^2\right]k_2 = -k_1 m_1^1 m_2^1 - k_2\left[-1 + \left(m_1^1\right)^2\right].$$

To separate a spacelike new type slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as spacelike new type $\mathcal{B}$-slant helix.

Theorem 3.2. *Let* $\gamma : I \to \mathbb{H}^3$ *be a unit speed biharmonic spacelike new type* $\mathcal{B}-$ *slant helix with non-zero curvatures. Then the equation of biharmonic spacelike new type* $\mathcal{B}-$ *slant helix are*

$$\mathbf{x}(s) = \frac{1}{\mathcal{C}_0}\cos\mathcal{Q}\cosh[\mathcal{C}_0 s + \mathcal{C}_1] + \mathcal{C}_2,$$
$$\mathbf{y}(s) = \frac{1}{\mathcal{C}_0}\cos\mathcal{Q}\sinh[\mathcal{C}_0 s + \mathcal{C}_1] + \mathcal{C}_3, \qquad (5)$$
$$\mathbf{z}(s) = \sin\mathcal{Q}s - \frac{\mathcal{C}_2}{\mathcal{C}_0}\cos\mathcal{Q}\sinh[\mathcal{C}_0 s + \mathcal{C}_1]$$
$$-\frac{1}{4\mathcal{C}_0}\cos^2\mathcal{Q}(2[\mathcal{C}_0 s + \mathcal{C}_1] + \sinh 2[\mathcal{C}_0 s + \mathcal{C}_1]) + \mathcal{C}_4,$$

where $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ *are constants of integration and*

$$\mathcal{C}_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos\mathcal{Q}} - \sin\mathcal{Q}.$$

Proof. The vector $\mathbf{m}_2$ is a unit spacelike vector, we reach

$$\mathbf{m}_2 = \cos\mathcal{Q}\mathbf{e}_1 + \sin\mathcal{Q}\cosh\mathcal{A}(s)\mathbf{e}_2 + \sin\mathcal{Q}\sinh\mathcal{A}(s)\mathbf{e}_3. \qquad (8)$$

On the other hand, using Bishop formulas Eq.(4) and Eq.(1), we have

$$\mathbf{m}_1 = \sinh\mathcal{A}(s)\mathbf{e}_2 + \cosh\mathcal{A}(s)\mathbf{e}_3. \qquad (9)$$

It is apparent that

$$\mathbf{t} = \sin \mathrm{Q}\mathbf{e}_1 + \cos \mathrm{Q} \cosh \mathrm{A}(s)\mathbf{e}_2 + \cos \mathrm{Q} \sinh \mathrm{A}(s)\mathbf{e}_3. \quad (10)$$

A straightforward computation shows that

$$\nabla_{\mathbf{t}} \mathbf{t} = (t_1^{'})\mathbf{e}_1 + (t_2^{'} + t_1 t_3)\mathbf{e}_2 + (t_3^{'} + t_1 t_2)\mathbf{e}_3. \quad (11)$$

Therefore, we use Bishop formulas Eq.(4) and above equation we get

$$\mathrm{A}(s) = [\frac{\sqrt{k_2^2 - k_1^2}}{\cos \mathrm{Q}} - \sin \mathrm{Q}]s + \mathrm{C}_1, \quad (12)$$

where C_1 is a constant of integration.

From Eq.(10), we get

$$\mathbf{t} = (\cos \mathrm{Q} \sinh[\mathrm{C}_0 s + \mathrm{C}_1], \cos \mathrm{Q} \cosh[\mathrm{C}_0 s + \mathrm{C}_1], \sin \mathrm{Q} - x \cos \mathrm{Q} \cosh[\mathrm{C}_0 s + \mathrm{C}_1]), \quad (13)$$

where , $$\mathrm{C}_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos \mathrm{Q}} - \sin \mathrm{Q}.$$

Therefore, by Eq(13) and taking into account Eq.(12), we obtain the system Eq.(12). This completes the proof.

Corollary 3.3. *Let $\gamma : I \to \mathrm{H}^3$ be a unit speed biharmonic spacelike new type $\mathrm{B}-$slant helix with non-zero Bishop curvatures. Then the equation of γ is*

$$\begin{aligned}\gamma(s) = {} & [\sin \mathrm{Q} s - \frac{\mathrm{C}_2}{\mathrm{C}_0} \cos \mathrm{Q} \sinh[\mathrm{C}_0 s + \mathrm{C}_1] \\ & - \frac{1}{4\mathrm{C}_0} \cos^2 \mathrm{Q}(2[\mathrm{C}_0 s + \mathrm{C}_1] + \sinh 2[\mathrm{C}_0 s + \mathrm{C}_1]) + \mathrm{C}_4 \\ & + [\frac{1}{\mathrm{C}_0} \cos \mathrm{Q} \cosh[\mathrm{C}_0 s + \mathrm{C}_1] + \mathrm{C}_2][\frac{1}{\mathrm{C}_0} \cos \mathrm{Q} \sinh[\mathrm{C}_0 s + \mathrm{C}_1] + \mathrm{C}_3]]\mathbf{e}_1 \\ & + [\frac{1}{\mathrm{C}_0} \cos \mathrm{Q} \sinh[\mathrm{C}_0 s + \mathrm{C}_1] + \mathrm{C}_3]\mathbf{e}_2 + [\frac{1}{\mathrm{C}_0} \cos \mathrm{Q} \cosh[\mathrm{C}_0 s + \mathrm{C}_1] + \mathrm{C}_2]\mathbf{e}_3,\end{aligned}$$

where $\mathrm{C}_0, \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$ are constants of integration and

$$\mathrm{C}_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos \mathrm{Q}} - \sin \mathrm{Q}.$$

If we use Mathematica in above system, we get:

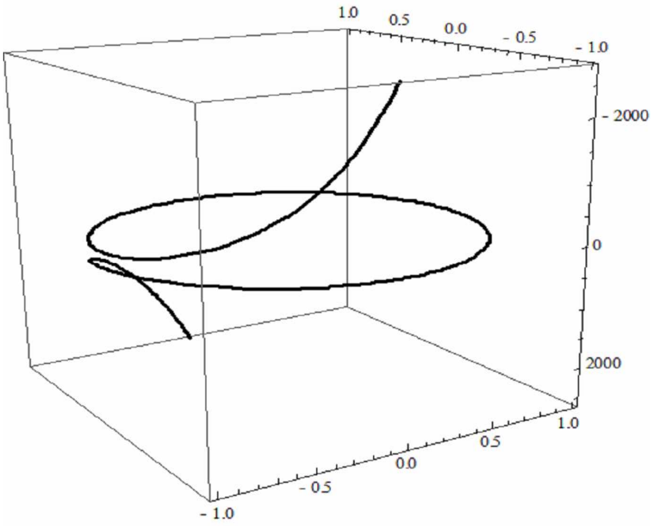

Fig.1.

References

[1] L.R. Bishop, There is More Than One Way to Frame a Curve, *Amer. Math. Monthly*, 82(3) (1975), 246-251.

[2] B. Bukcu, M.K. Karacan, Bishop frame of the spacelike curve with a spacelike binormal in Minkowski 3 space, *Selçuk Journal of Applied Mathematics*, 11(1) (2010), 15-25.

[3] J. Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, 86 (1964), 109-160.

[4] R.T. Farouki and C.A. Neff, Algebraic properties of plane offset curves, *Comput. Aided Geom. Design*, 7 (1990), 101-127.

[5] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, CRC Press, (1998).

[6] G.Y. Jiang, 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. Ser.*, A 7(2) (1986), 130-144.

[7] K. Onda, Lorentz Ricci solitons on 3-dimensional lie groups, *Geom Dedicata*, 147(1) (2010), 313-322.

[8] T. Körpınar and E. Turhan, On characterization of B-canal surfaces in terms of biharmonic B-slant helices according to Bishop frame in Heisenberg group $Heis^3$, *J. Math. Anal. Appl.*, 382(2011), 57-65.

[9] E. Loubeau and C. Oniciuc, On the biharmonic and harmonic indices of the Hopf map, *Transactions of the American Mathematical Society*, 359(11) (2007), 5239-5256.

[10] W. Lü, Rationality of the offsets to algebraic curves and surfaces, *Appl. Math. (A Journal of Chinese Universities)*, 9(Ser.B) (1994), 265-278.

[11] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, (1983).

[12] M. Peternell and H. Pottmann, Computing rational parametrizations of canal surfaces, *J. Symb. Comput.*, 23(1997), 255-266.

[13]S. Rahmani, Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois, *Journal of Geometry and Physics*, 9(1992), 295-302.

[14]U. Shani and D.H. Ballard, Splines as embeddings for generalized cylinders, *Comput. Vision Graphics Image Process*, 27(1984), 129-156.

[15]E. Turhan and T. Körpınar, On characterization of timelike horizontal biharmonic curves in the Lorentzian Heisenberg group Heis^3, *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, 65a(2010), 641-648.

[16]E. Turhan and T. Körpınar, On characterization canal surfaces around timelike horizontal biharmonic curves in Lorentzian Heisenberg group Heis^3, *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, 66a(2011), 441-449.