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Periodic Solution of Nonlinear System of Integro-Differential Equations Depending on the Gamma Distribution

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Abstract

In this paper we investigate the periodic solution of nonlinear system of integro- differential equations depending on the gamma distribution by using the numerical analytic method for investigate periodic solutions of ordinary differential equation which given by Samoilenko A. M. .These investigations lend us to the improving and extending the above method. Also we expand the results gained by Butris R. N. to change the periodic system of nonlinear integro-differential equations to periodic system of nonlinear integro-differential equations depending the on gamma distribution.

Keywords: *Numerical-analytic methods existence of periodic solutions, nonlinear integro-differential equations, subjected, depending on the gamma distribution.*

1 Introduction

They are many subjects in physics and technology using mathematical methods that depends on the linear and nonlinear integro-differential equations, and it become clear that the existence of periodic solutions and it is algorithm structure from more important problems, to present time where many of studies and researches [3, 4, 5, 8, 9] dedicates for treatment the autonomous and non-autonomous periodic systems and specially with integro-differential equations.

Numerical-analytic method [2,3,4,6,8] owing to the great possibilities of exploiting computers are becoming versatile means of the finding and approximate construction of periodic solutions of integro-differential equations. Samoilenko [7] assumes the numerical-analytic method to study the periodic solutions for ordinary differential equations and it is algorithm structure and this method include uniformly sequences of periodic functions and the results of that study is using of the periodic solutions on wide range in the difference of new processes industry and technology as in the studies[2,6,8,9].

Butris R. N. [1] has been used the numerical-analytic method of periodic solution for ordinary differential equations which were introduced by Samoilenko A. M. [7] to study the periodic solution of the system, nonlinear integro-differential equation which has the form:

$$\frac{dx}{dt} = f\left(t, x, \int_t^{t+T} g\big(s, x(s)\big)ds\right) \qquad \cdots (1.1)$$

Where $x \in D \subset R^n, D$ is a closed and bounded domain. The vectors functions $f(t, x, y)$ and $g(t, x)$ are continuous functions in t, x, y and periodic in t of period T.

In this study we have employed the numerical-analytic method of Samoilenko A.M. [7] to investigate the existence and approximation of periodic solution for nonlinear system of integro-differential equations which depending on the gamma distribution. The study of such integro-differential equations leads to improving and extending Samoilenko A.M. [7] method.

Thus, the integro-differential equtions which depending on the gamma distribution that we have introduced in this study, becomes more general and detailed than those introduced by Butris R. N. [1]. The study is considered a theoretical one, however, the results that we have got, may several applications in field physical as well as mathematical problems.

In this paper, we consider the periodic integro-differential equations which depending on the gamma distribution:

$$\frac{dx}{dt} = f\left(t, \gamma(t,\alpha), x, \int_t^{t+T} g(s, \gamma(s,\alpha), x(s))ds\right) \quad \cdots (1.2)$$

where $x \in D \subset R^n$, D is a closed bounded domain.

The vector functions $f(t, \gamma(t,\alpha), x)$ and $g(t, \gamma(t,\alpha), x)$ are defined on the domain:

$$(t, \gamma(t,\alpha), x) \in R^1 \times [0,T] \times D \times D_1$$

$$= (-\infty, \infty) \times [0,T] \times D \times D_1 \quad \cdots (1.3)$$

Continuous in the totality of variables periodic in t of period T and satisfies the inequalities:

$$|f(t, \gamma(t,\alpha), x, y)| \le MM_\alpha \quad , \quad |g(t, \gamma(t,\alpha), x)| \le MM_\alpha \quad \cdots (1.4)$$

$$|f(t, \gamma(t,\alpha), x_1, y_1) - f(t, \gamma(t,\alpha), x_2, y_2)| \le |M_\alpha|(R|x_1 - x_2| + L|y_1 - y_2|) \quad \cdots (1.5)$$

$$|g(t, \gamma(t,\alpha), x_1) - g(t, \gamma(t,\alpha), x_2)| \le Q|M_\alpha||x_1 - x_2| \quad \cdots (1.6)$$

for all $t \in R^1$ and $x, x_1, x_2 \in D$, where $M = (M_1, M_2, \cdots, M_n)$is a positive constant vectors and gamma distribution is defined by the following equation:

$$\left.\begin{array}{l} \gamma(t,\alpha) = \dfrac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)} \quad , \quad \alpha > 0 \, , \\ \text{where} \qquad T < \left(\dfrac{\Gamma(\alpha+1)}{(\alpha+1)}\right)^{\frac{1}{\alpha}} . \end{array}\right\} \quad \cdots (1.7)$$

We define the non-empty sets as follows:

$$\left.\begin{array}{r} D_{\gamma f} = D - MM_\alpha \dfrac{T}{2} \\ D_{\gamma g} = D - MM_\alpha \dfrac{T^\alpha e^{-T}}{\Gamma(\alpha)} Q \end{array}\right\} \cdots (1.8)$$

Furthermore, we suppose that the greatest eigen value λ_{max} of the matrix

$\Lambda = [M_\alpha(K+LQ)\frac{T}{2}]$ does not exceed unity, i.e.

$$\lambda_{max}(\Lambda) < 1 \qquad \cdots (1.9)$$

Lemma 1.1 [7]: *Let $f(t)$ be a continuous vector function defined on the interval $[0,T]$, then*

$$\left|\int_0^t (f(s) - \frac{1}{T}\int_0^T f(s)ds)ds\right| \le \alpha(t) \max_{t\in[0,T]}|f(t)|$$

Where $\alpha(t) = 2t(1-\frac{t}{T})$. (For the proof see [7]).

By using Lemma 1.1, we can state and prove the following Lemma.

Lemma 1.2: *Suppose that the function of gamma distribution $\gamma(t,\alpha)$is continuous on the interval $[0,T]$. Then*

$$\left|\int_0^t (\gamma(s,\alpha) - \frac{1}{T}\int_0^T \gamma(s,\alpha)ds)ds\right| \le M_\alpha \alpha(t)$$

is hold for all values of α .

Proof: Taking

$$\left|\int_0^t (\gamma(s,\alpha) - \frac{1}{T}\int_0^T \gamma(s,\alpha)ds)ds\right| \le (1-\frac{t}{T})\int_t^T |\lambda(s,\alpha)|ds + \frac{t}{T}\int_t^T |\lambda(s,\alpha)|\, ds =$$

$$= (1-\frac{t}{T})\int_0^t \frac{T^{\alpha-1}e^{-t}}{\Gamma(\alpha)}ds + \frac{t}{T}\int_t^T \frac{T^{\alpha-1}e^{-t}}{\Gamma(\alpha)}ds \le$$

$$\le \frac{T^{\alpha-1}e^{-t}}{\Gamma(\alpha)}[(1-\frac{t}{T})t + \frac{t}{T}(T-t)] =$$

$$= \alpha(t)\ M_\alpha$$

So that

$$\left|\int_0^t (\gamma(s,\alpha) - \frac{1}{T}\int_0^T \gamma(s,\alpha)ds)ds\right| \le \alpha(t)M_\alpha \cdots (1.10)$$

For all $t \in [0,T]$ and $\alpha(t) \le \frac{T}{2}$.

2 Approximate Solution

The investigation of periodic approximate solution of the problem (1.2) is formulated by the following theorem.

Theorem 2.1: *If the problem (1.2) satisfy the inequalities (1.4),(1.5),(1.6) and the conditions (1.7), (1.8) has a periodic solution* $x = x(t, \gamma(t, \alpha), x_0)$ *, then the sequence of functions*

$$x_{m+1}(t, \gamma(t, \alpha), x_0) = x_0 + \int_0^t [f(s, \gamma(s, \alpha), x_m(s, \gamma(s, \alpha), x_0), \int_s^{s+T} g(\tau, \gamma(\tau, \alpha), , x_m(\tau, \gamma(\tau, \alpha), x_0) d\tau) - \frac{1}{T} \int_0^T (f(s, \gamma(s, \alpha), x_m(s, \gamma(s, \alpha), x_0), , \int_s^{s+\tau} g(\tau, \gamma(\tau, \alpha), x_m(\tau, \gamma(\tau, \alpha), x_0) d\tau) ds] ds \cdots (2.1)$$

With

$$x_0(t, \gamma(t, \alpha), x_0) = x_0 \, , \qquad m = 0,1,2, \cdots$$

is periodic in t of period T, and uniformly convergent as $m \to \infty$ in the domain

$$(t, \gamma(t, \alpha), x_0) \in R^1 \times [0, T] \times D_{\gamma f} \cdots (2.2)$$

To the limit function $x^0(t, \gamma(t, \alpha), x_0)$ which is defined on the domain (2.2), periodic in t of period T and satisfying the system of integral equations.

$$x(t, \gamma(t, \alpha), x_0) = x_0 + \int_0^t [f(s, \gamma(s, \alpha), x(s, \gamma(s, \alpha), x_0), \int_s^{s+T} g(\tau, \gamma(\tau, \alpha), , x(\tau, \gamma(\tau, \alpha), x_0) d\tau) - \frac{1}{T} \int_0^T (f(s, \gamma(s, \alpha), x(s, \gamma(s, \alpha), x_0), , \int_s^{s+\tau} g(\tau, \gamma(\tau, \alpha), x(\tau, \gamma(\tau, \alpha), x_0) d\tau) ds] ds \cdots (2.3)$$

Which is a unique solution of the problem (1.2) provided that:

$$|x^0(t, \gamma(t, \alpha), x_0) - x_0| \le M \, M_\alpha \, \alpha(t) \cdots (2.4)$$

And

$$|x^0(t, \gamma(t, \alpha), x_0) - x_m(t, \gamma(t, \alpha), x_0)| \le \Lambda^m (E - \Lambda)^{-1} M \, M_\alpha \, \alpha(t) \ \cdots (2.5)$$

for all $m \ge 1$ and $t \in R^1$, where E is the identity matrix.

Proof: Consider the sequence of functions $x_1(t,\gamma(t,\alpha),x_0), x_2(t,\gamma(t,\alpha),x_0),\cdots$, $x_m(t,\gamma(t,\alpha),x_0),\cdots$ defined be recurrence relation (2.1). Each of these sequence of functions is defined and continuous on the domain (1.3) and periodic in t of period T.

Now, by Lemma 1.1, 1.2 and using (2.1), when $m=0$, we get:

$$|x_1(t,\gamma(t,\alpha),x_0)-x_0|$$

$$\leq (1-\frac{t}{T})\int_0^t |f(s,\gamma(s,\alpha),x_0,\int_s^{s+T} g(\tau,\gamma(\tau,\alpha),x_0)\,d\tau)|ds +$$

$$+\frac{t}{T}\int_t^T |f(s,\gamma(s,\alpha),x_0,\int_s^{s+T} g(\tau,\gamma(\tau,\alpha),x_0)\,d\tau)|ds \leq$$

$$\leq (1-\frac{t}{T})\int_0^t M\,M_\alpha\,ds+\frac{t}{T}\int_t^T M\,M_\alpha\,ds$$

$$= M\,M_\alpha\,[(1-\frac{t}{T})t+\frac{t}{T}(T-t)]$$

$$= 2t(1-\frac{t}{T})\;M\,M_\alpha$$

$$= \alpha(t)\,M\,M_\alpha$$

So that

$|x_1(t,\gamma(t,\alpha),x_0)-x_0| \leq M\,M_\alpha\alpha(t)\cdots(2.6)$

i.e. $x_1(t,\gamma(t,\alpha),x_0)\in D$, for all $t\in R^1, x_0\in D_{\gamma f}$.

Also from (2.6), we have:

$$|y_1(t,\gamma(t,\alpha),x_0)-y_0(t,\gamma(t,\alpha),x_0)| = \left|\int_t^{t+T} g(s,\gamma(s,\alpha),x_1(s,\gamma(s,\alpha),x_0)ds - ,\int_t^{t+T} g(s,\gamma(s,\alpha),x_0)ds\right| \leq$$

$$\leq \int_t^{t+T} Q\,|\gamma(s,\alpha)||x_1(s,\gamma(s,\alpha),x_0)-x_0|ds \leq$$

$$\leq \int_{t}^{t+T} Q \; \frac{T^{\alpha-1}e^{-T}}{\Gamma(\alpha)} \; M \, M_{\alpha} \, \alpha(s) \, ds$$

$$= Q \; \frac{T^{\alpha}e^{-T}}{\Gamma(\alpha)} \; M \, M_{\alpha} \frac{T}{2}$$

So that

$$|y_1(t,\gamma(t,\alpha),x_0) - y_0(t,\gamma(t,\alpha),x_0)| \leq Q \; \frac{T^{\alpha}e^{-T}}{\Gamma(\alpha)} \; M \, M_{\alpha} \frac{T}{2} \cdots (2.7)$$

i.e. $y_1(t,\gamma(t,\alpha),x_0) \in D_1$, for all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

Thus by mathematical induction, we find that:

$$|x_m(t,\gamma(t,\alpha),x_0) - x_0| \leq M \, M_{\alpha}\alpha(t) \cdots (2.8)$$

For all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

i.e. $x_m(t,\gamma(t,\alpha),x_0) \in D$, for all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

And from (2.8), we get:

$$|y_m(t,\gamma(t,\alpha),x_0) - y_0(t,\gamma(t,\alpha),x_0)| \leq M \, M_{\alpha} Q \; \frac{T^{\alpha}e^{-T}}{\Gamma(\alpha)} \frac{T}{2}$$

For all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

i.e. $y_m(t,\gamma(t,\alpha),x_0) \in D_1$, for all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

Where $y_m(t,\gamma(t,\alpha),x_0) = \int_t^{t+T} g(s,\gamma(s,\alpha),x_m(s,\gamma(s,\alpha),x_0)ds$,

$m = 0,1,2,\cdots$.

We claim that the sequence of functions (2.1) is uniformly convergent on the domain (2.2).

By using the Lemmas 1.1, 1.2 and putting $m = 1$ in (2.1), we have:

$$|x_2(t,\gamma(t,\alpha),x_0) - x_1(t,\gamma(t,\alpha),x_0)| = |x_0 + \int_0^t [f(s,\gamma(s,\alpha),x_1(s,\gamma(s,\alpha),x_0),$$

$$,\int_s^{s+T} g(\tau,\gamma(\tau,\alpha),x_1(\tau,\gamma(\tau,\alpha),x_0))d\tau) - \frac{1}{T}\int_0^T f(s,\gamma(s,\alpha),x_1(s,\gamma(s,\alpha),x_0),$$

$$,\int_{s}^{s+T} g(\tau,\gamma(\tau,\alpha),x_1(\tau,\gamma(\tau,\alpha),x_0))d\tau)\,ds]ds - x_0 - \int_0^t [f(s,\gamma(s,\alpha),x_0,$$

$$,\int_{s}^{s+T} g(\tau,\gamma(\tau,\alpha),x_0)d\tau) - \frac{1}{T}\int_0^T f(s,\gamma(s,\alpha),x_0,\int_{s}^{s+T} g(\tau,\gamma(\tau,\alpha),x_0)d\tau)\,ds]ds| \le$$

$$\le (1-\frac{t}{T})\int_0^t |\gamma(s,\alpha)|(K+LQT)|x_1(s,\gamma(s,\alpha),x_0)-x_0|ds +$$

$$+\frac{t}{T}\int_t^T |\gamma(s,\alpha)|\,(K+LQT)|x_1(s,\gamma(s,\alpha),x_0)-x_0|ds \le$$

$$\le M_\alpha\, M\, M_\alpha \frac{T}{2}(K+LQT)\alpha(t)$$

$$|x_2(t,\gamma(t,\alpha),x_0)-x_1(t,\gamma(t,\alpha),x_0)| \le M\, {M_\alpha}^2 \frac{T}{2}(K+LQT)\alpha(t) \cdots (2.9)$$

Suppose that the following inequality is true

$$|x_m(t,\gamma(t,\alpha),x_0)-x_{m-1}(t,\gamma(t,\alpha),x_0)| \le M\, {M_\alpha}^m [\frac{T}{2}(K+LQT)]^{m-1}\alpha(t) \cdots (2.10)$$

For all $m \ge 1$.

Now, we shall prove the following:

$$|x_{m+1}(t,\gamma(t,\alpha),x_0)-x_m(t,\gamma(t,\alpha),x_0)| \le$$

$$\le (1-\frac{t}{T})\int_0^t M_\alpha\ (K+LQT)|x_m(s,\gamma(s,\alpha),x_0)-x_{m-1}(s,\gamma(t,\alpha),x_0)|ds +$$

$$+\frac{t}{T}\int_t^T M_\alpha\,(K+LQT)|x_m(s,\gamma(s,\alpha),x_0)-x_{m-1}(s,\gamma(t,\alpha),x_0)|ds \le$$

$$\le (1-\frac{t}{T})\int_0^t M_\alpha\,(K+LQT)M\,{M_\alpha}^m\left[(K+LQ)\frac{T}{2}\right]^{m-1}\alpha(s)ds +$$

$$+\frac{t}{T}\int_{t}^{T} M_{\alpha}\,(K+LQT)M\,{M_{\alpha}}^{m}\left[(K+LQ)\frac{T}{2}\right]^{m-1}\alpha(s)ds =$$

$$= \;{M_{\alpha}}^{m+1}\left[(K+LQ)\frac{T}{2}\right]^{m-1} M\,\frac{T}{2}\,\alpha(t) =$$

$$= M\,{M_{\alpha}}^{m+1}\left[(K+LQ)\frac{T}{2}\right]^{m}\alpha(t) =$$

$$= \;M\,M_{\alpha}\left[M_{\alpha}(K+LQ)\frac{T}{2}\right]^{m}\alpha(t) =$$

And hence

$$|x_{m+1}(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)|$$

$$\leq MM_{\alpha}\left[M_{\alpha}(K+LQ)\frac{T}{2}\right]^{m}\alpha(t)\cdots(2.11)$$

For all $m \geq 0$.

From (2.11) we conclude that for any$k \geq 1$, we have the inequality

$$|x_{m+k}(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)| \leq \sum_{i=0}^{k-1}\Lambda^{m+i}\,M\,M_{\alpha}\;\alpha(t)$$

Such that

$$|x_{m+k}(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)| \leq$$

$$\leq \sum_{i=0}^{\infty}|x_{m+1+i}(t,\gamma(t,\alpha),x_0) - x_{m+i}(t,\gamma(t,\alpha),x_0)| \leq$$

$$\leq \sum_{i=0}^{\infty} MM_{\alpha}\alpha(t)\,\Lambda^{m+1+i} =$$

$$\leq M\,M_{\alpha}\,\alpha(t)\,\Lambda^{m}\sum_{i=0}^{\infty}\Lambda^{i+1} =$$

$$\leq M\,M_{\alpha}\,\alpha(t)\,\Lambda^{m}\,(\mathrm{E}-\Lambda)^{-1} =$$

So that

$|x_{m+k}(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)|$

$\leq \Lambda^m (\mathrm{E}-\Lambda)^{-1} M\, M_\alpha\, \alpha(t)$ $\cdots$ (2.12)

for all $k \geq 1$, where E is identity matrix.

from (2.12) and the condition (1.9), we find that:

$$\lim_{m\to\infty} \Lambda^m = 0 \quad \cdots (2.13)$$

Relations (2.12) and (2.13) prove the uniform convergence of the sequence of functions (2.1) on the domain (2.2).

Let

$$\lim_{m\to\infty} x_m(t,\gamma(t,\alpha),x_0) = x^0(t,\gamma(t,\alpha),x_0) \cdots (2.14)$$

Since the sequence of functions (2.2) is periodic in t of period T, then the limiting function $x^0(t,\gamma(t,\alpha),x_0)$ is also periodic in t of period T.

Moreover, by Lemmas 1.1, 1.2 and inequality (2.12) the inequalities (2.4) and (2.5) are holds.

Finally, we have to show that $x(t,\gamma(t,\alpha),x_0)$ is a unique solution of the system (1.1).Assume that $r(t,\gamma(t,\alpha),x_0)$ is another solution of the system (1.1),

i.e.

$$r(t,\gamma(t,\alpha),x_0) = x_0 + \int_0^t [f(s,\gamma(s,\alpha),r(s,\gamma(s,\alpha),x_0),\int_s^{s+T} g(\tau,\gamma(\tau,\alpha),$$

$$,r(\tau,\gamma(\tau,\alpha),x_0)d\tau) - \frac{1}{T}\int_0^T (f(s,\gamma(s,\alpha),r(s,\gamma(s,\alpha),x_0),$$

$$,\int_s^{s+\tau} g(\tau,\gamma(\tau,\alpha),r(\tau,\gamma(\tau,\alpha),x_0)d\tau)ds]ds \cdots (2.15)$$

Now, we prove that $x(t,\gamma(t,\alpha),x_0) = r(t,\gamma(t,\alpha),x_0)$ for all $x_0 \in D_{\gamma f}$ and to do this, we need to drive the following inequality:

$$|r(t,\gamma(t,\alpha),x_0) - x(t,\gamma(t,\alpha),x_0)| \leq \Lambda^m (\mathrm{E}-\Lambda)^{-1} M^*\, M_\alpha\, \alpha(t) \quad \cdots (2.16)$$

where$M^* = \max_{x_0\in D_{\gamma f}} |f(t,r(t,\gamma(t,\alpha),x_0)\,,x(t,\gamma(t,\alpha),x_0),$

$$,\int_{t}^{t+T} g(s,\gamma(s,\alpha),x(s,\gamma(s,\alpha),x_0))\,ds|$$

Suppose that (2.15) is true for $m = k$, i.e.
$|r(t,\gamma(t,\alpha),x_0) - x(t,\gamma(t,\alpha),x_0)| \leq \Lambda^k\,(\mathrm{E}-\Lambda)^{-1}M^*\,M_\alpha\,\alpha(t)$

Then
$|r(t,\gamma(t,\alpha),x_0) - x(t,\gamma(t,\alpha),x_0)| \leq$

$$\leq (1-\frac{t}{T})\int_{0}^{t} M_\alpha\,(K+LQT)|r(s,\gamma(s,\alpha),x_0) - x(s,\gamma(s,\alpha),x_0)|ds +$$

$$+\frac{t}{T}\int_{t}^{T} M_\alpha\,(K+LQT)|r(s,\gamma(s,\alpha),x_0) - x(s,\gamma(s,\alpha),x_0)|ds \leq$$

$$\leq (1-\frac{t}{T})\int_{0}^{t} M_\alpha\,(K+LQT)\Lambda^k(E-\Lambda)^{-1}M^*M_\alpha\alpha(s)ds +$$

$$+\frac{t}{T}\int_{t}^{T} M_\alpha\,(K+LQT)\Lambda^k(E-\Lambda)^{-1}M^*M_\alpha\alpha(s)ds =$$

$$= \Lambda^{k+1}(E-\Lambda)^{-1}M^*M_\alpha\alpha(t)$$

By induction, inequality (2.16) is true for $m = 0,1,2,\cdots$.

Thus from (2.14) and (2.16), we have:

$$\lim_{m\to\infty}|r(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)| = 0$$

And hence
$$\lim_{m\to\infty} x_m(t,\gamma(t,\alpha),x_0) = r(t,\gamma(t,\alpha),x_0)$$

By the relation (2.14), we get:

$$x(t,\gamma(t,\alpha),x_0) = r(t,\gamma(t,\alpha),x_0)$$

i.e. $x(t,\gamma(t,\alpha),x_0)$ is a unique solution of (1.1) on the domain (1.2). ∎

3 Existence of Solution

The problem of existence of a periodic solution of period T of the system (1.1) is uniquely connected with the existence of zeros of the function $\Delta(0,\gamma(0,\alpha),x_0)$ which has the form:

$$\Delta(0,\gamma(0,\alpha),x_0) = \frac{1}{T}\int_0^T [f(t,\gamma(t,\alpha),x^0(t,\gamma(t,\alpha),x_0),$$

$$,\int_t^{t+T} g(s,\gamma(s,\alpha),x^0(s,\gamma(s,\alpha),x_0))ds]dt \qquad \cdots (3.1)$$

Where $x^0(t,\gamma(t,\alpha),x_0)$ is the limiting function of the sequence of functions (2.1).

The function (3.1) can find only approximately, say by computing the following functions:

$$\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_0) = \frac{1}{T}\int_0^T [f(t,\gamma(t,\alpha),x_m(t,\gamma(t,\alpha),x_0),$$

$$,\int_t^{t+T} g(s,\gamma(s,\alpha),x_m(s,\gamma(s,\alpha),x_0))ds]dt \qquad \cdots (3.2)$$

and $m = 0,1,2,\cdots$.
Now, we prove the following theorem.

Theorem3.1: *Let all assumptions and conditions of theorem 2.1 were given. Then the inequality:*

$$|\Delta(0,\gamma(0,\alpha),x_0) - \Delta_m(0,\gamma(0,\alpha),x_0)| \le \Lambda^{m+1}(E-\Lambda)^{-1} M\, M_\alpha \cdots (3.3)$$

will be satisfied for all $m \ge 0, x_0 \in D_{\gamma f}$.

Proof: By the relations (3.1) and (3.2), the estimate

$$|\Delta(0,\gamma(0,\alpha),x_0) - \Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_0)| \le$$

$$\le \frac{1}{T}\int_0^T |\gamma(t,\alpha)|\,(K+LQT)|x^0(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)|dt \le$$

$$\le \frac{1}{T}\int_0^T M_\alpha\,(K+LQT)\Lambda^m(E-\Lambda)^{-1}M\, M_\alpha\, \alpha(t)dt =$$

$$= \Lambda^{m+1}(E-\Lambda)^{-1}M\,M_{\alpha}$$

Thus the inequality (3.2) is hold for all $m \geq 0$.
Next, we prove the following theorem taking into account at the inequality (3.3) will be satisfied for all $m \geq 0$.

Theorem 3.2: *If the system (1.1) satisfies the following condition*

(i)The sequence of functions (3.2) has an isolated singular point $x_0 = x^0$, $\Delta_m(0,\gamma(0,\alpha),x^0) \equiv 0$, for all $t \in R^1$.

(ii)The index of this point is nonzero;
(iii)There exists a closed convex domain D_γ^* belonging to domain $D_{\gamma f}$ and possessing a unique singular point x^0 such that on it is boundary $\Gamma_{D_\gamma^*}$ the following inequality is holds

$$inf\,\|\Delta_m(t,\gamma(t,\alpha),x_0)\| \leq \|\Lambda^m(E-\Lambda)^{-1}M\,M_\alpha\| \cdots \qquad (3.4)$$

when $x \in \Gamma_{D_\gamma^*}$ for all $m \geq 0$. Then the system (1.1) has a periodic solution $x = x(t,\gamma(t,\alpha),x_0)$ for which $x(0,\gamma(0,\alpha),x_0)$ belongs to the domain D_γ^* .

Proof: By using the inequality (3.1) we can prove the theorem 7.1[7].

Remark 3.1. [7]: When $R^n = R^1$, i.e. when x_0 is a scalar, the existence of solution can be strengthens by giving up the requirement that the singular point shout be isolated, thus we have

Theorem3.3: *Let the system of nonlinear integro-differential equations (1.1) are defined on the interval $[a,b]$. Suppose that for $m \geq 0$, the function $\Delta_m(0,\gamma(0,\alpha),x_0)$ defined according to formula (3.2) satisfies the inequalities:*

$$\left.\begin{aligned} \min_{a+h\leq x_0\leq b-h}\|\Delta_{\mathrm{m}}(t,\gamma(t,\alpha),x_0)\| \leq -\sigma_m \quad ; \\ \max_{a+h\leq x_0\leq b-h}\|\Delta_{\mathrm{m}}(t,\gamma(t,\alpha),x_0)\| \geq \sigma_m \quad . \end{aligned}\right\} \cdots (3.5)$$

Then the system (1.1) has a periodic solution $x = x(t,\gamma(t,\alpha)x_0)$ for which

$x_0 \in [a+h, b-h]$, where $h = \|M\,M_\alpha\|\frac{T}{2}$ and $\sigma_m = \|\Lambda^{m+1}(E-\Lambda)^{-1}M\,M_\alpha\|$.

Proof: Let x_1 and x_2 be any two points on the interval $[a,b]$ such that:

$$\left.\begin{aligned} \Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_1) = \min_{a+h\leq x_0\leq b-h}\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_0) \;; \\ \Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_2) = \max_{a+h\leq x_0\leq b-h}\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_0) \;. \end{aligned}\right\} \cdots (3.6)$$

Taking into account inequalities (3.3) and (3.5), we have

$$\left.\begin{array}{l}\Delta(0,\gamma(0,\alpha),x_1)=\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_1)+[\Delta(0,\gamma(0,\alpha),x_1)-\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_1)]\\ \Delta(0,\gamma(0,\alpha),x_2)=\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_2)+[\Delta(0,\gamma(0,\alpha),x_2)-\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_2)]\end{array}\right\} \quad \cdots (3.7)$$

It follows from the inequalities (3.7) and the continuity of the function $\Delta(0,\gamma(0,\alpha),x_0)$, that there exist an isolated singular point $x^0, x^0 \in [x_1,x_2]$, such that $\Delta(0,\gamma(0,\alpha),x_0)\equiv 0$, this means that the system (1.1) has a periodic solution $x=x(t,\gamma(t,\alpha),x_0)$ for which $x_0\in[a+h,b-h]$. ■.

Theorem3.4: *If the function $\Delta(0,\gamma(0,\alpha),x_0)$ is defined by*
$\Delta\colon D_{\gamma f}\to R^n$,

$$\Delta(0,\gamma(0,\alpha),x_0)=\frac{1}{T}\int_0^T[f(t,\gamma(t,\alpha),x^0(t,\gamma(t,\alpha),x_0),$$

$$,\int_t^{t+T} g(s,\gamma(s,\alpha),x^0(s,\gamma(s,\alpha),x_0))ds]dt \quad \cdots (3.7)$$

Where $x^0(t,\gamma(t,\alpha),x_0)$ is a limit of the sequence of functions (2.1). Then the following inequalities are holds

$$|\Delta(0,\gamma(0,\alpha),x_0)|\le \mathrm{M}\ \mathrm{M}_\alpha \cdots (3.8)$$

And

$$|\Delta(0,\gamma(0,\alpha),x_0{}^1)-\Delta(0,\gamma(0,\alpha),x_0{}^2)|\le\frac{2}{T}\Lambda(E-\Lambda)^{-1}M_\alpha \cdots (3.9)$$

For all $x_0, x_0{}^1,\ x_0{}^2\in D_{\gamma f}$.

Proof: From the properties function $x^0(t,\gamma(t,\alpha),x_0)$ established theorem 2.1; it follows that function $\Delta(0,\gamma(0,\alpha),x_0)$ is continuous and bounded by $M\ M_\alpha$.

By using (3.7), we get:

$$|\Delta(0,\gamma(0,\alpha),x_0{}^1)-\Delta(0,\gamma(0,\alpha),x_0{}^2)|=\left|\frac{1}{T}\int_0^T[f(t,\gamma(t,\alpha),x^0(t,\gamma(t,\alpha),x_0{}^1),\right.$$

$$,\int_t^{t+T} g(s,\gamma(s,\alpha),x^0(s,\gamma(s,\alpha),x_0{}^1))ds]dt-\frac{1}{T}\int_0^T[f(t,\gamma(t,\alpha),x^0(t,\gamma(t,\alpha),x_0{}^2),$$

$$,\int_t^{t+T} g(s,\gamma(s,\alpha),x^0(s,\gamma(s,\alpha),x_0{}^2))ds]dt\le$$

$$\leq \frac{1}{T}\int_{0}^{T} M_\alpha\ (K+LQT)|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|dt \leq$$

$$\leq M_\alpha(K+LQT)\frac{T}{2}.\frac{2}{T}|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|dt =$$

$$= \frac{2}{T}\Lambda|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|$$

And hence

$$|\Delta(0,\gamma(0,\alpha),x_0{}^1)-\Delta(0,\gamma(0,\alpha),x_0{}^2)| \leq$$

$$\leq \frac{2}{T}\ \Lambda\ |x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|M_\alpha \cdots (3.10)$$

Where $x_0{}^1(t,\gamma(t,\alpha),x_0)$ and $x_0{}^2(t,\gamma(t,\alpha),x_0)$ are the solutions of the integral equation:

$$x(t,\gamma(t,\alpha),x_0{}^k) = x_0{}^k + \int_0^t [f(s,\gamma(s,\alpha),x(s,\gamma(s,\alpha),x_0{}^k),\int_s^{s+T} g(\tau,\gamma(\tau,\alpha),$$

$$,x(\tau,\gamma(\tau,\alpha),x_0{}^k)d\tau) - \frac{1}{T}\int_0^T (f(s,\gamma(s,\alpha),x(s,\gamma(s,\alpha),x_0{}^k),$$

$$,\int_s^{s+\tau} g(\tau,\gamma(\tau,\alpha),x(\tau,\gamma(\tau,\alpha),x_0)d\tau)ds]ds \cdots (3.11)$$

With

$$x_0{}^k(t,\gamma(t,\alpha),x_0) = x_0{}^k\ , \qquad k = 1,2.$$

From (3.11), we have:

$$|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)| \leq |x_0{}^1-x_0{}^2| +$$

$$+(1-\frac{t}{T})\int_0^t M_\alpha\ (K+LQT)|x^0(s,\gamma(s,\alpha),x_0{}^1)-x^0(s,\gamma(s,\alpha),x_0{}^2)|ds +$$

$$+\frac{t}{T}\int_t^T M_\alpha\,(K+LQT)|x^0(s,\gamma(s,\alpha),x_0{}^1)-x^0(s,\gamma(s,\alpha),x_0{}^2)|ds \leq$$

$$\leq |x_0{}^1-x_0{}^2| + M_\alpha(K+LQT)|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|\alpha(t) \leq$$

$$\leq |x_0{}^1-x_0{}^2| + \Lambda|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|$$

Thus

$$|x^0(t,\gamma(t,\alpha),{x_0}^1) - x^0(t,\gamma(t,\alpha),{x_0}^2)| \leq (E-\Lambda)^{-1}|{x_0}^1 - {x_0}^2| \cdots (3.12)$$

Using the inequality (3.12) in (3.10), we get (3.9).

Remark 3.2. [4]: The theorem 3.4 ensure the stability solution of the system (1.1) when there is a slight change on the point $x_0$accompanied with noticeable change in the function $\Delta(0,\gamma(0,\alpha),x_0)$.

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