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Bc-Separation Axioms In Topological Spaces

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Abstract

In this paper, we introduce and investigate some weak separation axioms by using the notions of Bc-open sets and the Bc-closure operator.

Keywords: *Bc-open, b-open.*

1 Introduction

Throughout this paper, (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 1.1 *[1] A subset A of a space X is said to be b-open if $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$. The family of all b-open subsets of a topological space (X, τ) is denoted by $BO(X, \tau)$ or (Briefly. $BO(X)$).*

Definition 1.2 *[2] A subset A of a space X is called Bc-open if for each $x \in A \subseteq BO(X)$, there exists a closed set F such that $x \in F \subseteq A$. The family of all Bc-open subsets of a topological space (X, τ) is denoted by $BcO(X, \tau)$ or (Briefly. $BcO(X)$).*

Definition 1.3 *[2] For any subset A in the space X, the Bc-closure of A, denoted by $BcCl(A)$, is defined by the intersection of all Bc-closed sets containing A.*

2 Bc-g.Closed Sets

In this section, we define and study some properties of Bc-g.closed sets.

Definition 2.1 *A subset A of X is said to be a Bc-generalized closed (briefly, Bc-g.closed) set if $BcCl(A) \subseteq U$ whenever $A \subseteq U$ and U is a Bc-open set in (X, τ). A subset A of X is Bc-g.open if its complement $X \setminus A$ is Bc-g.closed in X.*

It is clear that every Bc-closed set is Bc-g.closed. But the converse is not true in general as it is shown in the following example.

Example 2.2 *Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Now, if we let $A = \{a\}$, since the only Bc-open supersets of A are $\{a, c\}$ and X, then A is Bc-g.closed. But it is easy to see that A is not Bc-closed.*

Proposition 2.3 *If A is Bc-open and Bc-g.closed then A is Bc-closed.*

Proof. Suppose that A is Bc-open and Bc-g.closed. Since A is Bc-open and $A \subseteq A$, we have $BcCl(A) \subseteq A$, also $A \subseteq BcCl(A)$, therefore $BcCl(A) = A$. That is A is Bc-closed.

Proposition 2.4 *The intersection of a Bc-g.closed set and a Bc-closed set is always Bc-g.closed.*

Proof. Let A be Bc-g.closed and F be Bc-closed. Assume that U is Bc-open set such that $A \cap F \subseteq U$, set $G = X \setminus F$. Then $A \subseteq U \cup G$, since G is Bc-open, then $U \cup G$ is Bc-open and since A is Bc-g.closed, then $BcCl(A) \subseteq U \cup G$. Now by Theorem 3.17 (10) [2], $BcCl(A \cap F) \subseteq BcCl(A) \cap BcCl(F) = BcCl(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup (G \cap F) = (U \cap F) \cup \phi \subseteq U$.

Proposition 2.5 *If a subset A of X is Bc-g.closed and $A \subseteq B \subseteq BcCl(A)$, then B is a Bc-g.closed set in X.*

Proof. Let A be a Bc-g.closed set such that $A \subseteq B \subseteq BcCl(A)$. Let U be a Bc-open set of X such that $B \subseteq U$. Since A is Bc-g.closed, we have $BcCl(A) \subseteq U$. Now $BcCl(A) \subseteq BcCl(B) \subseteq BcCl[BcCl(A)] = BcCl(A) \subseteq U$. That is $BcCl(B) \subseteq U$, where U is Bc-open. Therefore B is a Bc-g.closed set in X.

Proposition 2.6 *For each $x \in X$, $\{x\}$ is Bc-closed or $X \setminus \{x\}$ is Bc-g.closed in (X, τ).*

Proof. Suppose that $\{x\}$ is not Bc-closed, then $X \setminus \{x\}$ is not Bc-open. Let U be any Bc-open set such that $X \setminus \{x\} \subseteq U$, implies $U = X$. Therefore $BcCl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is Bc-g.closed.

Proposition 2.7 *A subset A of X is Bc-g.closed if and only if $BcCl(\{x\}) \cap A \neq \phi$, holds for every $x \in BcCl(A)$.*

Proof. Let U be a Bc-open set such that $A \subseteq U$ and let $x \in BcCl(A)$. By assumption, there exists a point $z \in BcCl(\{x\})$ and $z \in A \subseteq U$. Then, $U \cap \{x\} \neq \phi$, hence $x \in U$, this implies $BcCl(A) \subseteq U$. Therefore A is Bc-g.closed.
Conversely, suppose that $x \in BcCl(A)$ such that $BcCl(\{x\}) \cap A = \phi$. Since, $BcCl(\{x\})$ is Bc-closed. Therefore, $X \setminus BcCl(\{x\})$ is a Bc-open set in X. Since $A \subseteq X \setminus (BcCl(\{x\}))$ and A is Bc-g.closed implies that $BcCl(A) \subseteq X \setminus BcCl(\{x\})$ holds, and hence $x \notin BcCl(A)$. This is a contradiction. Therefore $BcCl(\{x\}) \cap A \neq \phi$.

Proposition 2.8 *A set A of a space X is Bc-g.closed if and only if $BcCl(A) \setminus A$ does not contain any non-empty Bc-closed set.*

Proof. **Necessity.** Suppose that A is a Bc-g.closed set in X. We prove the result by contradiction. Let F be a Bc-closed set such that $F \subseteq BcCl(A) \setminus A$ and $F \neq \phi$. Then $F \subseteq X \setminus A$ which implies $A \subseteq X \setminus F$. Since A is Bc-g.closed and $X \setminus F$ is Bc-open, therefore $BcCl(A) \subseteq X \setminus F$, that is $F \subseteq X \setminus BcCl(A)$. Hence $F \subseteq BcCl(A) \cap (X \setminus BcCl(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $BcCl(A) \setminus A$ does not contain any non-empty Bc-closed set in X.
Sufficiency. Let $A \subseteq U$, where U is Bc-open in X. If $BcCl(A)$ is not contained in U, then $BcCl(A) \cap X \setminus U \neq \phi$. Now, since $BcCl(A) \cap X \setminus U \subseteq BcCl(A) \setminus A$ and $BcCl(A) \cap X \setminus U$ is a non-empty Bc-closed set, then we obtain a contradication and therefore A is Bc-g.closed.

Proposition 2.9 *If A is a Bc-g.closed set of a space X, then the following are equivalent:*

1. *A is Bc-closed.*
2. *$BcCl(A) \setminus A$ is Bc-closed.*

Proof. (1) $\Rightarrow$ (2). If A is a Bc-g.closed set which is also Bc-closed, then, $BcCl(A) \setminus A = \phi$, which is Bc-closed.
(2) $\Rightarrow$ (1). Let $BcCl(A) \setminus A$ be a Bc-closed set and A be Bc-g.closed. Then by Proposition 2.8, $BcCl(A) \setminus A$ does not contain any non-empty Bc-closed subset. Since $BcCl(A) \setminus A$ is Bc-closed and $BcCl(A) \setminus A = \phi$, this shows that A is Bc-closed.

Proposition 2.10 *For a space (X, τ), the following are equivalent:*

1. *Every subset of X is Bc-g.closed.*

2. *$BcO(X, \tau) = BcC(X, \tau)$.*

Proof. (1) $\Rightarrow$ (2). Let $U \in BcO(X, \tau)$. Then by hypothesis, U is Bc-g.closed which implies that $BcCl(U) \subseteq U$, so, $BcCl(U) = U$, therefore $U \in BcC(X, \tau)$. Also let $V \in BcC(X, \tau)$. Then $X \setminus V \in BcO(X, \tau)$, hence by hypothesis $X \setminus V$ is Bc-g.closed and then $X \setminus V \in BcC(X, \tau)$, thus $V \in BcO(X, \tau)$ according to the above we have $BcO(X, \tau) = BcC(X, \tau)$.
(2) $\Rightarrow$ (1). If A is a subset of a space X such that $A \subseteq U$ where $U \in BcO(X, \tau)$, then $U \in BcC(X, \tau)$ and therefore $BcCl(U) \subseteq U$ which shows that A is Bc-g.closed.

3 Bc-T_k $(k = 0, 1, 2)$

In this section, some new types of separation axioms are defined and studied in topological spaces namely, Bc-T_k for $k = 0, \frac{1}{2}, 1, 2$ and Bc-D_k for $k = 0, 1, 2$, and also some properties of these spaces are explained.

The following definitions are introduced via Bc-open sets.

Definition 3.1 *A topological space (X, τ) is said to be:*

1. *Bc-T_0 if for each pair of distinct points x, y in X, there exists a Bc-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.*

2. *Bc-T_1 if for each pair of distinct points x, y in X, there exist two Bc-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*

3. *Bc-T_2 if for each distinct points x, y in X, there exist two disjoint Bc-open sets U and V containing x and y respectively.*

4. *Bc-$T_{\frac{1}{2}}$ if every Bc-g.closed set is Bc-closed.*

Proposition 3.2 *A topological space (X, τ) is Bc-T_0 if and only if for each pair of distinct points x, y of X, $BcCl(\{x\}) \neq BcCl(\{y\})$.*

Proof. **Necessity.** Let (X, τ) be a Bc-T_0 space and x, y be any two distinct points of X. There exists a Bc-open set U containing x or y, say x but not y. Then $X \setminus U$ is a Bc-closed set which does not contain x but contains y. Since $BcCl(\{y\})$ is the smallest Bc-closed set containing y, $BcCl(\{y\}) \subseteq X \setminus U$ and therefore $x \notin BcCl(\{y\})$. Consequently $BcCl(\{x\}) \neq BcCl(\{y\})$.
Sufficiency. Suppose that $x, y \in X$, $x \neq y$ and $BcCl(\{x\}) \neq BcCl(\{y\})$. Let

z be a point of X such that $z \in BcCl(\{x\})$ but $z \notin BcCl(\{y\})$. We claim that $x \notin BcCl(\{y\})$. For, if $x \in BcCl(\{y\})$ then $BcCl(\{x\}) \subseteq BcCl(\{y\})$. This contradicts the fact that $z \notin BcCl(\{y\})$. Consequently x belongs to the Bc-open set $X \setminus BcCl(\{y\})$ to which y does not belong.

Proposition 3.3 *The following statements are equivalent for a topological space (X, τ) :*

1. *(X, τ) is Bc-$T_{\frac{1}{2}}$.*

2. *Each singleton $\{x\}$ of X is either Bc-closed or Bc-open.*

Proof. (1) $\Rightarrow$ (2). Suppose $\{x\}$ is not Bc-closed. Then by Proposition 2.6, $X \setminus \{x\}$ is Bc-g.closed. Now since (X, τ) is Bc-$T_{\frac{1}{2}}$, $X \setminus \{x\}$ is Bc-closed, that is $\{x\}$ is Bc-open.
(2) $\Rightarrow$ (1). Let A be any Bc-g.closed set in (X, τ) and $x \in BcCl(A)$. By (2), we have $\{x\}$ is Bc-closed or Bc-open. If $\{x\}$ is Bc-closed then $x \notin A$ will imply $x \in BcCl(A) \setminus A$, which is not possible by Proposition 2.8. Hence $x \in A$. Therefore, $BcCl(A) = A$, that is A is Bc-closed. So, (X, τ) is Bc-$T_{\frac{1}{2}}$. **On the other hand,** if $\{x\}$ is Bc-open then as $x \in BcCl(A)$, $\{x\} \cap A \neq \phi$. Hence $x \in A$. So A is Bc-closed.

Proposition 3.4 *A topological space (X, τ) is Bc-T_1 if and only if the singletons are Bc-closed sets.*

Proof. Let (X, τ) be Bc-T_1 and x any point of X. Suppose $y \in X \setminus \{x\}$, then $x \neq y$ and so there exists a Bc-open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subseteq X \setminus \{x\}$, that is $X \setminus \{x\} = \cup\{U : y \in X \setminus \{x\}\}$ which is Bc-open.
Conversely, suppose $\{p\}$ is Bc-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X \setminus \{x\}$. Hence $X \setminus \{x\}$ is a Bc-open set contains y but not x. Similarly $X \setminus \{y\}$ is a Bc-open set contains x but not y. Accordingly X is a Bc-T_1 space.

Proposition 3.5 *The following statements are equivalent for a topological space (X, τ) :*

1. *X is Bc-T_2.*

2. *Let $x \in X$. For each $y \neq x$, there exists a Bc-open set U containing x such that $y \notin BcCl(U)$.*

3. *For each $x \in X$, $\cap\{BcCl(U) : U \in BcO(X)$ and $x \in U\} = \{x\}$.*

Proof. (1) $\Rightarrow$ (2). Since X is Bc-T_2, there exist disjoint Bc-open sets U and V containing x and y respectively. So, $U \subseteq X \setminus V$. Therefore, $BcCl(U) \subseteq X \setminus V$. So $y \notin BcCl(U)$.
(2) $\Rightarrow$ (3). If possible for some $y \neq x$, we have $y \in BcCl(U)$ for every Bc-open set U containing x, which then contradicts (2).
(3) $\Rightarrow$ (1). Let $x, y \in X$ and $x \neq y$. Then there exists a Bc-open set U containing x such that $y \notin BcCl(U)$. Let $V = X \setminus BcCl(U)$, then $y \in V$ and $x \in U$ and also $U \cap V = \phi$.

Proposition 3.6 *Let (X, τ) be a topological space, then the following statements are hold:*

1. *Every Bc-T_2 space is Bc-T_1.*
2. *Every Bc-T_1 space is Bc-$T_{\frac{1}{2}}$.*
3. *Every Bc-$T_{\frac{1}{2}}$ space is Bc-T_0.*

Proof.

1. The proof is straightforward from the definitions.
2. The proof is obvious by Proposition 3.4.
3. Let x and y be any two distinct points of X. By Proposition 3.3, the singleton set $\{x\}$ is Bc-closed or Bc-open.
   (a) If $\{x\}$ is Bc-closed, then $X \setminus \{x\}$ is Bc-open. So $y \in X \setminus \{x\}$ and $x \notin X \setminus \{x\}$. Therefore, we have X is Bc-T_0.
   (b) If $\{x\}$ is Bc-open. Then $x \in \{x\}$ and $y \notin \{x\}$. Therefore, we have X is Bc-T_0.

Definition 3.7 *A subset A of a topological space X is called a BcDifference set (briefly, BcD-set) if there are $U, V \in BcO(X, \tau)$ such that $U \neq X$ and $A = U \setminus V$.*

It is true that every Bc-open set U different from X is a BcD-set if $A = U$ and $V = \phi$. So, we can observe the following.

Remark 3.8 *Every proper Bc-open set is a BcD-set.*

Now we define another set of separation axioms called Bc-D_k, for $k = 0, 1, 2$, by using the BcD-sets.

Definition 3.9 *A topological space (X, τ) is said to be:*

1. *Bc-D_0 if for any pair of distinct points x and y of X there exists a BcD-set of X containing x but not y or a BcD-set of X containing y but not x.*

2. *Bc-D_1 if for any pair of distinct points x and y of X there exists a BcD-set of X containing x but not y and a BcD-set of X containing y but not x.*

3. *Bc-D_2 if for any pair of distinct points x and y of X there exist disjoint BcD-set G and E of X containing x and y, respectively.*

Remark 3.10 *For a topological space (X, τ), the following properties hold:*

1. *If (X, τ) is Bc-T_k, then it is Bc-D_k, for $k = 0, 1, 2$.*

2. *If (X, τ) is Bc-D_k, then it is Bc-D_{k-1}, for $k = 1, 2$.*

Proof. Obvious.

Proposition 3.11 *A space X is Bc-D_0 if and only if it is Bc-T_0.*

Proof. Suppose that X is Bc-D_0. Then for each distinct pair $x, y \in X$, at least one of x, y, say x, belongs to a BcD-set G but $y \notin G$. Let $G = U_1 \setminus U_2$ where $U_1 \neq X$ and $U_1, U_2 \in BcO(X, \tau)$. Then $x \in U_1$, and for $y \notin G$ we have two cases: (a) $y \notin U_1$, (b) $y \in U_1$ and $y \in U_2$.
In case (a), $x \in U_1$ but $y \notin U_1$.
In case (b), $y \in U_2$ but $x \notin U_2$.
Thus in both the cases, we obtain that X is Bc-T_0.
Conversely, if X is Bc-T_0, by Remark 3.10 (1), X is Bc-D_0.

Proposition 3.12 *A space X is Bc-D_1 if and only if it is Bc-D_2.*

Proof. **Necessity.** Let $x, y \in X$, $x \neq y$. Then there exist BcD-sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where U_1, U_2, U_3 and U_4 are Bc-open sets in X. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.
(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases:
(a) $y \notin U_1$. Since $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$, and since $y \in U_3 \setminus U_4$ we have $y \in U_3 \setminus (U_1 \cup U_4)$. Therefore $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \phi$.
(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2$, and $y \in U_2$. Therefore $(U_1 \setminus U_2) \cap U_2 = \phi$.
(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4$ and $x \in U_4$. Hence $(U_3 \setminus U_4) \cap U_4 = \phi$. Therefore X is Bc-D_2.
sufficiency. Follows from Remark 3.10 (2).

Corollary 3.13 *If (X, τ) is Bc-D_1, then it is Bc-T_0.*

Proof. Follows from Remark 3.10 (2) and Proposition 3.11.

Definition 3.14 *A point $x \in X$ which has only X as the Bc-neighbourhood is called a Bc-neat point.*

Proposition 3.15 *For a Bc-T_0 topological space (X, τ) the following are equivalent:*

1. *(X, τ) is Bc-D_1.*

2. *(X, τ) has no Bc-neat point.*

Proof. (1) $\Rightarrow$ (2). Since (X, τ) is Bc-D_1, then each point x of X is contained in a BcD-set $A = U \setminus V$ and thus in U. By definition $U \neq X$. This implies that x is not a Bc-neat point.
(2) $\Rightarrow$ (1). If X is Bc-T_0, then for each distinct pair of points $x, y \in X$, at least one of them, x (say) has a Bc-neighbourhood U containing x and not y. Thus U which is different from X is a BcD-set. If X has no Bc-neat point, then y is not a Bc-neat point. This means that there exists a Bc-neighbourhood V of y such that $V \neq X$. Thus $y \in V \setminus U$ but not x and $V \setminus U$ is a BcD-set. Hence X is Bc-D_1.

Corollary 3.16 *A Bc-T_0 space X is not Bc-D_1 if and only if there is a unique Bc-neat point in X.*

Proof. We only prove the uniqueness of the Bc-neat point. If x and y are two Bc-neat points in X, then since X is Bc-T_0, at least one of x and y, say x, has a Bc-neighbourhood U containing x but not y. Hence $U \neq X$. Therefore x is not a Bc-neat point which is a contradiction.

Definition 3.17 *A topological space (X, τ), is said to be Bc-symmetric if for x and y in X, $x \in BcCl(\{y\})$ implies $y \in BcCl(\{x\})$.*

Proposition 3.18 *If (X, τ) is a topological space, then the following are equivalent:*

1. *(X, τ) is a Bc-symmetric space.*

2. *$\{x\}$ is Bc-g.closed, for each $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Assume that $\{x\} \subseteq U \in BcO(X)$, but $BcCl(\{x\}) \nsubseteq U$. Then $BcCl(\{x\}) \cap X \setminus U \neq \phi$. Now, we take $y \in BcCl(\{x\}) \cap X \setminus U$, then by hypothesis $x \in BcCl(\{y\}) \subseteq X \setminus U$ and $x \notin U$, which is a contradiction. Therefore $\{x\}$ is Bc-g.closed, for each $x \in X$.
(2) $\Rightarrow$ (1). Assume that $x \in BcCl(\{y\})$, but $y \notin BcCl(\{x\})$. Then $\{y\} \subseteq X \setminus BcCl(\{x\})$ and hence $BcCl(\{y\}) \subseteq X \setminus BcCl(\{x\})$. Therefore $x \in X \setminus BcCl(\{x\})$, which is a contradiction and hence $y \in BcCl(\{x\})$.

Corollary 3.19 *If a topological space (X, τ) is a Bc-T_1 space, then it is Bc-symmetric.*

Proof. In a Bc-T_1 space, every singleton is Bc-closed (Proposition 3.4) and therefore is Bc-g.closed. Then by Proposition 3.18, (X, τ) is Bc-symmetric.

Corollary 3.20 *For a topological space (X, τ), the following statements are equivalent:*

1. *(X, τ) is Bc-symmetric and Bc-T_0.*
2. *(X, τ) is Bc-T_1.*

Proof. By Corollary 3.19 and Proposition 3.6, it suffices to prove only (1) $\Rightarrow$ (2).
Let $x \neq y$ and as (X, τ) is Bc-T_0, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in BcO(X)$. Then $x \notin BcCl(\{y\})$ and hence $y \notin BcCl(\{x\})$. There exists a Bc-open set V such that $y \in V \subseteq X \setminus \{x\}$ and thus (X, τ) is a Bc-T_1 space.

Proposition 3.21 *If (X, τ) is a Bc-symmetric space, then the following statements are equivalent:*

1. *(X, τ) is a Bc-T_0 space.*
2. *(X, τ) is a Bc-$T_{\frac{1}{2}}$ space.*
3. *(X, τ) is a Bc-T_1 space.*

Proof. (1) $\Leftrightarrow$ (3). Obvious from Corollary 3.20.
(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). Directly from Proposition 3.6.

Corollary 3.22 *For a Bc-symmetric space (X, τ), the following are equivalent:*

1. *(X, τ) is Bc-T_0.*
2. *(X, τ) is Bc-D_1.*

3. (X, τ) *is Bc-*T_1.

Proof. (1) $\Rightarrow$ (3). Follows from Corollary 3.20.
(3) $\Rightarrow$ (2) $\Rightarrow$ (1). Follows from Remark 3.10 and Corollary 3.13.

Definition 3.23 *Let A be a subset of a topological space* (X, τ). *The Bc-kernel of A, denoted by* $Bcker(A)$ *is defined to be the set*

$$Bcker(A) = \cap\{U \in BcO(X) \colon A \subseteq U\}.$$

Proposition 3.24 *Let* (X, τ) *be a topological space and* $x \in X$. *Then* $y \in Bcker(\{x\})$ *if and only if* $x \in BcCl(\{y\})$.

Proof. Suppose that $y \notin Bcker(\{x\})$. Then there exists a Bc-open set V containing x such that $y \notin V$. Therefore, we have $x \notin BcCl(\{y\})$. The proof of the converse case can be done similarly.

Proposition 3.25 *Let* (X, τ) *be a topological space and A be a subset of X. Then,* $Bcker(A) = \{x \in X \colon BcCl(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in Bcker(A)$ and suppose $BcCl(\{x\}) \cap A = \phi$. Hence $x \notin X \setminus BcCl(\{x\})$ which is a Bc-open set containing A. This is impossible, since $x \in Bcker(A)$. Consequently, $BcCl(\{x\}) \cap A \neq \phi$. Next, let $x \in X$ such that $BcCl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin Bcker(A)$. Then, there exists a Bc-open set V containing A and $x \notin V$. Let $y \in BcCl(\{x\}) \cap A$. Hence, V is a Bc-neighbourhood of y which does not contain x. By this contradiction $x \in Bcker(A)$ and the claim.

Proposition 3.26 *The following properties hold for the subsets* A, B *of a topological space* (X, τ) *:*

1. $A \subseteq Bcker(A)$.

2. $A \subseteq B$ *implies that* $Bcker(A) \subseteq Bcker(B)$.

3. *If A is Bc-open in* (X, τ), *then* $A = Bcker(A)$.

4. $Bcker(Bcker(A)) = Bcker(A)$.

Proof. (1), (2) and (3) are immediate consequences of Definition 3.23. To prove (4), first observe that by (1) and (2), we have $Bcker(A) \subseteq Bcker(Bcker(A))$. If $x \notin Bcker(A)$, then there exists $U \in BcO(X, \tau)$ such that $A \subseteq U$ and $x \notin U$. Hence $Bcker(A) \subseteq U$, and so we have $x \notin Bcker(Bcker(A))$. Thus $Bcker(Bcker(A)) = Bcker(A)$.

Proposition 3.27 *If a singleton $\{x\}$ is a BcD-set of (X,τ), then $Bcker(\{x\}) \neq X$.*

Proof. Since $\{x\}$ is a BcD-set of (X,τ), then there exist two subsets $U_1, U_2 \in BcO(X,\tau)$ such that $\{x\} = U_1 \setminus U_2$, $\{x\} \subseteq U_1$ and $U_1 \neq X$. Thus, we have that $Bcker(\{x\}) \subseteq U_1 \neq X$ and so $Bcker(\{x\}) \neq X$.

Proposition 3.28 *For a Bc-$T_{\frac{1}{2}}$ topological space (X,τ) with at least two points, (X,τ) is a Bc-D_1 space if and only if $Bcker(\{x\}) \neq X$ holds for every point $x \in X$.*

Proof. **Necessity.** Let $x \in X$. For a point $y \neq x$, there exists a BcD-set U such that $x \in U$ and $y \notin U$. Say $U = U_1 \setminus U_2$, where $U_i \in BcO(X,\tau)$ for each $i \in \{1,2\}$ and $U_1 \neq X$. Thus, for the point x, we have a Bc-open set U_1 such that $\{x\} \subseteq U_1$ and $U_1 \neq X$. Hence, $Bcker(\{x\}) \neq X$.
Sufficiency. Let x and y be a pair of distinct points of X. We prove that there exist BcD-sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$. Using Proposition 3.3, we can take the subsets A and B for the following four cases for two points x and y.
Case1. $\{x\}$ is Bc-open and $\{y\}$ is Bc-closed in (X,τ). Since $Bcker(\{y\}) \neq X$, then there exists a Bc-open set V such that $y \in V$ and $V \neq X$. Put $A = \{x\}$ and $B = \{y\}$. Since $B = V \setminus (X \setminus \{y\})$, then V is a Bc-open set with $V \neq X$ and $X \setminus \{y\}$ is Bc-open, and B is a required BcD-set containing y such that $x \notin B$. Obviously, A is a required BcD-set containing x such that $y \notin A$.
Case 2. $\{x\}$ is Bc-closed and $\{y\}$ is Bc-open in (X,τ). The proof is similar to Case 1.
Case 3. $\{x\}$ and $\{y\}$ are Bc-open in (X,τ). Put $A = \{x\}$ and $B = \{y\}$.
Case 4. $\{x\}$ and $\{y\}$ are Bc-closed in (X,τ). Put $A = X\setminus\{y\}$ and $B = X\setminus\{x\}$.
For each case of the above, the subsets A and B are the required BcD-sets. Therefore, (X,τ) is a Bc-D_1 space.

Definition 3.29 *A function $f : X \to Y$ is called a Bc-open function if the image of every Bc-open set in X is a Bc-open set in Y.*

Proposition 3.30 *Suppose that $f : X \to Y$ is Bc-open and surjective. If X is Bc-T_k, then Y is Bc-T_k, for $k = 0, 1, 2$.*

Proof. We prove only the case for Bc-T_1 space the others are similarly.
Let X be a Bc-T_1 space and let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is surjective, so there exist distinct points x_1, x_2 of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is a Bc-T_1 space, there exist Bc-open sets G and H such that $x_1 \in G$ but $x_2 \notin G$ and $x_2 \in H$ but $x_1 \notin H$. Since f is a Bc-open function, $f(G)$ and $f(H)$ are Bc-open sets of Y such that $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$, and $y_2 = f(x_2) \in f(H)$ but $y_1 = f(x_1) \notin f(H)$. Hence Y is a Bc-T_1 space.

4 Bc-R_k $(k = 0, 1)$

In this section, new classes of topological spaces called Bc-R_0 and Bc-R_1 spaces are introduced.

Definition 4.1 *A topological space (X, τ), is said to be Bc-R_0 if U is a Bc-open set and $x \in U$ then $BcCl(\{x\}) \subseteq U$.*

Proposition 4.2 *For a topological space (X, τ), the following properties are equivalent:*

1. *(X, τ) is Bc-R_0.*
2. *For any $F \in BcC(X)$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in BcO(X)$.*
3. *For any $F \in BcC(X)$, $x \notin F$ implies $F \cap BcCl(\{x\}) = \phi$.*
4. *For any distinct points x and y of X, either $BcCl(\{x\}) = BcCl(\{y\})$ or $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$.*

Proof. (1) $\Rightarrow$ (2). Let $F \in BcC(X)$ and $x \notin F$. Then by (1), $BcCl(\{x\}) \subseteq X \setminus F$. Set $U = X \setminus BcCl(\{x\})$, then U is a Bc-open set such that $F \subseteq U$ and $x \notin U$.
(2) $\Rightarrow$ (3). Let $F \in BcC(X)$ and $x \notin F$. There exists $U \in BcO(X)$ such that $F \subseteq U$ and $x \notin U$. Since $U \in BcO(X)$, $U \cap BcCl(\{x\}) = \phi$ and $F \cap BcCl(\{x\}) = \phi$.
(3) $\Rightarrow$ (4). Suppose that $BcCl(\{x\}) \neq BcCl(\{y\})$ for distinct points $x, y \in X$. There exists $z \in BcCl(\{x\})$ such that $z \notin BcCl(\{y\})$ (or $z \in BcCl(\{y\})$ such that $z \notin BcCl(\{x\})$). There exists $V \in BcO(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin BcCl(\{y\})$. By (3), we obtain $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$.
(4) $\Rightarrow$ (1). let $V \in BcO(X)$ and $x \in V$. For each $y \notin V$, $x \neq y$ and $x \notin BcCl(\{y\})$. This shows that $BcCl(\{x\}) \neq BcCl(\{y\})$. By (4), $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$ for each $y \in X \setminus V$ and hence $BcCl(\{x\}) \cap (\bigcup_{y \in X \setminus V} BcCl(\{y\})) = \phi$. On other hand, since $V \in BcO(X)$ and $y \in X \setminus V$, we have $BcCl(\{y\}) \subseteq X \setminus V$ and hence $X \setminus V = \bigcup_{y \in X \setminus V} BcCl(\{y\})$. Therefore, we obtain $(X \setminus V) \cap BcCl(\{x\}) = \phi$ and $BcCl(\{x\}) \subseteq V$. This shows that (X, τ) is a Bc-R_0 space.

Proposition 4.3 *A topological space (X, τ) is Bc-T_1 if and only if (X, τ) is a Bc-T_0 and a Bc-R_0 space.*

Proof. **Necessity.** Let U be any Bc-open set of (X, τ) and $x \in U$. Then by Proposition 3.4, we have $BcCl(\{x\}) \subseteq U$ and so by Proposition 3.6, it is clear that X is a Bc-T_0 and a Bc-R_0 space.

Sufficiency. Let x and y be any distinct points of X. Since X is Bc-T_0, there exists a Bc-open set U such that $x \in U$ and $y \notin U$. As $x \in U$ implies that $BcCl(\{x\}) \subseteq U$. Since $y \notin U$, so $y \notin BcCl(\{x\})$. Hence $y \in V = X \setminus BcCl(\{x\})$ and it is clear that $x \notin V$. Hence it follows that there exist Bc-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$. This implies that X is Bc-T_1.

Proposition 4.4 *For a topological space (X, τ), the following properties are equivalent:*

1. *(X, τ) is Bc-R_0.*
2. *$x \in BcCl(\{y\})$ if and only if $y \in BcCl(\{x\})$, for any points x and y in X.*

Proof. (1) $\Rightarrow$ (2). Assume that X is Bc-R_0. Let $x \in BcCl(\{y\})$ and V be any Bc-open set such that $y \in V$. Now by hypothesis, $x \in V$. Therefore, every Bc-open set which contain y contains x. Hence $y \in BcCl(\{x\})$.
(2) $\Rightarrow$ (1). Let U be a Bc-open set and $x \in U$. If $y \notin U$, then $x \notin BcCl(\{y\})$ and hence $y \notin BcCl(\{x\})$. This implies that $BcCl(\{x\}) \subseteq U$. Hence (X, τ) is Bc-R_0.

From Definition 3.17 and Proposition 4.4, the notions of Bc-symmetric and Bc-R_0 are equivalent.

Proposition 4.5 *The following statements are equivalent for any points x and y in a topological space (X, τ) :*

1. *$Bcker(\{x\}) \neq Bcker(\{y\})$.*
2. *$BcCl(\{x\}) \neq BcCl(\{y\})$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $Bcker(\{x\}) \neq Bcker(\{y\})$, then there exists a point z in X such that $z \in Bcker(\{x\})$ and $z \notin Bcker(\{y\})$. From $z \in Bcker(\{x\})$ it follows that $\{x\} \cap BcCl(\{z\}) \neq \phi$ which implies $x \in BcCl(\{z\})$. By $z \notin Bcker(\{y\})$, we have $\{y\} \cap BcCl(\{z\}) = \phi$. Since $x \in BcCl(\{z\})$, $BcCl(\{x\}) \subseteq BcCl(\{z\})$ and $\{y\} \cap BcCl(\{x\}) = \phi$. Therefore, it follows that $BcCl(\{x\}) \neq BcCl(\{y\})$. Now $Bcker(\{x\}) \neq Bcker(\{y\})$ implies that $BcCl(\{x\}) \neq BcCl(\{y\})$.
(2) $\Rightarrow$ (1). Suppose that $BcCl(\{x\}) \neq BcCl(\{y\})$. Then there exists a point z in X such that $z \in BcCl(\{x\})$ and $z \notin BcCl(\{y\})$. Then, there exists a Bc-open set containing z and therefore x but not y, namely, $y \notin Bcker(\{x\})$ and thus $Bcker(\{x\}) \neq Bcker(\{y\})$.

Proposition 4.6 *Let (X, τ) be a topological space. Then $\cap\{BcCl(\{x\}) : x \in X\} = \phi$ if and only if $Bcker(\{x\}) \neq X$ for every $x \in X$.*

Proof. **Necessity.** Suppose that $\cap\{BcCl(\{x\}) : x \in X\} = \phi$. Assume that there is a point y in X such that $Bcker(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every Bc-open set V containing y and hence $y \in BcCl(\{x\})$ for any $x \in X$. This implies that $y \in \cap\{BcCl(\{x\}) : x \in X\}$. But this is a contradiction.
Sufficiency. Assume that $Bcker(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap\{BcCl(\{x\}) : x \in X\}$, then every Bc-open set containing y must contain every point of X. This implies that the space X is the unique Bc-open set containing y. Hence $Bcker(\{y\}) = X$ which is a contradiction. Therefore, $\cap\{BcCl(\{x\}) : x \in X\} = \phi$.

Proposition 4.7 *A topological space (X, τ) is Bc-R_0 if and only if for every x and y in X, $BcCl(\{x\}) \neq BcCl(\{y\})$ implies $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$.*

Proof. **Necessity.** Suppose that (X, τ) is Bc-R_0 and $x, y \in X$ such that $BcCl(\{x\}) \neq BcCl(\{y\})$. Then, there exists $z \in BcCl(\{x\})$ such that $z \notin BcCl(\{y\})$ (or $z \in BcCl(\{y\})$ such that $z \notin BcCl(\{x\})$). There exists $V \in BcO(X)$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin BcCl(\{y\})$. Thus $x \in [X \setminus BcCl(\{y\})] \in BcO(X)$, which implies $BcCl(\{x\}) \subseteq [X \setminus BcCl(\{y\})]$ and $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$.
Sufficiency. Let $V \in BcO(X)$ and let $x \in V$. We still show that $BcCl(\{x\}) \subseteq V$. Let $y \notin V$, that is $y \in X \setminus V$. Then $x \neq y$ and $x \notin BcCl(\{y\})$. This shows that $BcCl(\{x\}) \neq BcCl(\{y\})$. By assumption, $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$. Hence $y \notin BcCl(\{x\})$ and therefore $BcCl(\{x\}) \subseteq V$.

Proposition 4.8 *A topological space (X, τ) is Bc-R_0 if and only if for any points x and y in X, $Bcker(\{x\}) \neq Bcker(\{y\})$ implies $Bcker(\{x\}) \cap Bcker(\{y\}) = \phi$.*

Proof. Suppose that (X, τ) is a Bc-R_0 space. Thus by Proposition 4.5, for any points x and y in X if $Bcker(\{x\}) \neq Bcker(\{y\})$ then $BcCl(\{x\}) \neq BcCl(\{y\})$. Now we prove that $Bcker(\{x\}) \cap Bcker(\{y\}) = \phi$. Assume that $z \in Bcker(\{x\}) \cap Bcker(\{y\})$. By $z \in Bcker(\{x\})$ and Proposition 3.24, it follows that $x \in BcCl(\{z\})$. Since $x \in BcCl(\{x\})$, by Proposition 4.2, $BcCl(\{x\}) = BcCl(\{z\})$. Similarly, we have $BcCl(\{y\}) = BcCl(\{z\}) = BcCl(\{x\})$. This is a contradiction. Therefore, we have $Bcker(\{x\}) \cap Bcker(\{y\}) = \phi$.
Conversely, let (X, τ) be a topological space such that for any points x and y in X, $Bcker(\{x\}) \neq Bcker(\{y\})$ implies $Bcker(\{x\}) \cap Bcker(\{y\}) = \phi$. If $BcCl(\{x\}) \neq BcCl(\{y\})$, then by Proposition 4.5, $Bcker(\{x\}) \neq Bcker(\{y\})$. Hence, $Bcker(\{x\}) \cap Bcker(\{y\}) = \phi$ which implies $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$. Because $z \in BcCl(\{x\})$ implies that $x \in Bcker(\{z\})$ and therefore $Bcker(\{x\}) \cap Bcker(\{z\}) \neq \phi$. By hypothesis, we have $Bcker(\{x\}) = Bcker(\{z\})$. Then

$z \in BcCl(\{x\}) \cap BcCl(\{y\})$ implies that $Bcker(\{x\}) = Bcker(\{z\}) = Bcker(\{y\})$. This is a contradiction. Therefore, $BcCl(\{x\}) \cap BcCl(\{y\}) = \phi$ and by Proposition 4.2, (X, τ) is a Bc-R_0 space.

Proposition 4.9 *For a topological space (X, τ), the following properties are equivalent:*

1. *(X, τ) is a Bc-R_0 space.*
2. *For any non-empty set A and $G \in BcO(X)$ such that $A \cap G \neq \phi$, there exists $F \in BcC(X)$ such that $A \cap F \neq \phi$ and $F \subseteq G$.*
3. *For any $G \in BcO(X)$, we have $G = \cup\{F \in BcC(X)\colon F \subseteq G\}$.*
4. *For any $F \in BcC(X)$, we have $F = \cap\{G \in BcO(X)\colon F \subseteq G\}$.*
5. *For every $x \in X$, $BcCl(\{x\}) \subseteq Bcker(\{x\})$.*

Proof. (1) $\Rightarrow$ (2). Let A be a non-empty subset of X and $G \in BcO(X)$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in BcO(X)$, $BcCl(\{x\}) \subseteq G$. Set $F = BcCl(\{x\})$, then $F \in BcC(X)$, $F \subseteq G$ and $A \cap F \neq \phi$.
(2) $\Rightarrow$ (3). Let $G \in BcO(X)$, then $G \supseteq \cup\{F \in BcC(X)\colon F \subseteq G\}$. Let x be any point of G. There exists $F \in BcC(X)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in BcC(X)\colon F \subseteq G\}$ and hence $G = \cup\{F \in BcC(X)\colon F \subseteq G\}$.
(3) $\Rightarrow$ (4). Obvious.
(4) $\Rightarrow$ (5). Let x be any point of X and $y \notin Bcker(\{x\})$. There exists $V \in BcO(X)$ such that $x \in V$ and $y \notin V$, hence $BcCl(\{y\}) \cap V = \phi$. By (4), $(\cap\{G \in BcO(X)\colon BcCl(\{y\}) \subseteq G\}) \cap V = \phi$ and there exists $G \in BcO(X)$ such that $x \notin G$ and $BcCl(\{y\}) \subseteq G$. Therefore $BcCl(\{x\}) \cap G = \phi$ and $y \notin BcCl(\{x\})$. Consequently, we obtain $BcCl(\{x\}) \subseteq Bcker(\{x\})$.
(5) $\Rightarrow$ (1). Let $G \in BcO(X)$ and $x \in G$. Let $y \in Bcker(\{x\})$, then $x \in BcCl(\{y\})$ and $y \in G$. This implies that $Bcker(\{x\}) \subseteq G$. Therefore, we obtain $x \in BcCl(\{x\}) \subseteq Bcker(\{x\}) \subseteq G$. This shows that (X, τ) is a Bc-R_0 space.

Corollary 4.10 *For a topological space (X, τ), the following properties are equivalent:*

1. *(X, τ) is a Bc-R_0 space.*
2. *$BcCl(\{x\}) = Bcker(\{x\})$ for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Suppose that (X, τ) is a Bc-R_0 space. By Proposition 4.9, $BcCl(\{x\}) \subseteq Bcker(\{x\})$ for each $x \in X$. Let $y \in Bcker(\{x\})$, then $x \in BcCl(\{y\})$ and by Proposition 4.2, $BcCl(\{x\}) = BcCl(\{y\})$. Therefore, $y \in BcCl(\{x\})$ and hence $Bcker(\{x\}) \subseteq BcCl(\{x\})$. This shows that $BcCl(\{x\}) = Bcker(\{x\})$.
(2) $\Rightarrow$ (1). Follows from Proposition 4.9.

Proposition 4.11 *For a topological space (X, τ), the following properties are equivalent:*

1. *(X, τ) is a Bc-R_0 space.*
2. *If F is Bc-closed, then $F = Bcker(F)$.*
3. *If F is Bc-closed and $x \in F$, then $Bcker(\{x\}) \subseteq F$.*
4. *If $x \in X$, then $Bcker(\{x\}) \subseteq BcCl(\{x\})$.*

Proof. (1) $\Rightarrow$ (2). Let F be a Bc-closed and $x \notin F$. Thus $(X \setminus F)$ is a Bc-open set containing x. Since (X, τ) is Bc-R_0, $BcCl(\{x\}) \subseteq (X \setminus F)$. Thus $BcCl(\{x\}) \cap F = \phi$ and by Proposition 3.25, $x \notin Bcker(F)$. Therefore $Bcker(F) = F$.
(2) $\Rightarrow$ (3). In general, $A \subseteq B$ implies $Bcker(A) \subseteq Bcker(B)$. Therefore, it follows from (2), that $Bcker(\{x\}) \subseteq Bcker(F) = F$.
(3) $\Rightarrow$ (4). Since $x \in BcCl(\{x\})$ and $BcCl(\{x\})$ is Bc-closed, by (3), $Bcker(\{x\}) \subseteq BcCl(\{x\})$.
(4) $\Rightarrow$ (1). We show the implication by using Proposition 4.4. Let $x \in BcCl(\{y\})$. Then by Proposition 3.24, $y \in Bcker(\{x\})$. Since $x \in BcCl(\{x\})$ and $BcCl(\{x\})$ is Bc-closed, by (4), we obtain $y \in Bcker(\{x\}) \subseteq BcCl(\{x\})$. Therefore $x \in BcCl(\{y\})$ implies $y \in BcCl(\{x\})$. The converse is obvious and (X, τ) is Bc-R_0.

Definition 4.12 *A topological space (X, τ), is said to be Bc-R_1 if for x, y in X with $BcCl(\{x\}) \neq BcCl(\{y\})$, there exist disjoint Bc-open sets U and V such that $BcCl(\{x\}) \subseteq U$ and $BcCl(\{y\}) \subseteq V$.*

Proposition 4.13 *A topological space (X, τ) is Bc-R_1 if it is Bc-T_2.*

Proof. Let x and y be any points of X such that $BcCl(\{x\}) \neq BcCl(\{y\})$. By Proposition 3.6 (1), every Bc-T_2 space is Bc-T_1. Therefore, by Proposition 3.4, $BcCl(\{x\}) = \{x\}$, $BcCl(\{y\}) = \{y\}$ and hence $\{x\} \neq \{y\}$. Since (X, τ) is Bc-T_2, there exist disjoint Bc-open sets U and V such that $BcCl(\{x\}) = \{x\} \subseteq U$ and $BcCl(\{y\}) = \{y\} \subseteq V$. This shows that (X, τ) is Bc-R_1.

Proposition 4.14 *For a topological space (X, τ), the following are equivalent:*

1. (X, τ) *is Bc-*T_2.
2. (X, τ) *is Bc-*R_1 *and Bc-*T_1.
3. (X, τ) *is Bc-*R_1 *and Bc-*T_0.

Proof. Straightforward.

Proposition 4.15 *For a topological space (X, τ), the following statements are equivalent:*

1. (X, τ) *is Bc-*R_1.
2. *If $x, y \in X$ such that $BcCl(\{x\}) \neq BcCl(\{y\})$, then there exist Bc-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.*

Proof. Obvious.

Proposition 4.16 *If (X, τ) is Bc-R_1, then (X, τ) is Bc-R_0.*

Proof. Let U be Bc-open such that $x \in U$. If $y \notin U$, since $x \notin BcCl(\{y\})$, we have $BcCl(\{x\}) \neq BcCl(\{y\})$. So, there exists a Bc-open set V such that $BcCl(\{y\}) \subseteq V$ and $x \notin V$, which implies $y \notin BcCl(\{x\})$. Hence $BcCl(\{x\}) \subseteq U$. Therefore, (X, τ) is Bc-R_0.

Corollary 4.17 *A topological space (X, τ) is Bc-R_1 if and only if for $x, y \in X$, $Bcker(\{x\}) \neq Bcker(\{y\})$, there exist disjoint Bc-open sets U and V such that $BcCl(\{x\}) \subseteq U$ and $BcCl(\{y\}) \subseteq V$.*

Proof. Follows from Proposition 4.5.

Proposition 4.18 *A topological space (X, τ) is Bc-R_1 if and only if $x \in X \setminus BcCl(\{y\})$ implies that x and y have disjoint Bc-open neighbourhoods.*

Proof. **Necessity.** Let $x \in X \setminus BcCl(\{y\})$. Then $BcCl(\{x\}) \neq BcCl(\{y\})$, so, x and y have disjoint Bc-open neighbourhoods.
Sufficiency. First, we show that (X, τ) is Bc-R_0. Let U be a Bc-open set and $x \in U$. Suppose that $y \notin U$. Then, $BcCl(\{y\}) \cap U = \phi$ and $x \notin BcCl(\{y\})$. There exist Bc-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \phi$. Hence, $BcCl(\{x\}) \subseteq BcCl(U_x)$ and $BcCl(\{x\}) \cap U_y \subseteq BcCl(U_x) \cap U_y = \phi$. Therefore, $y \notin BcCl(\{x\})$. Consequently, $BcCl(\{x\}) \subseteq U$ and (X, τ) is Bc-R_0. Next, we show that (X, τ) is Bc-R_1. Suppose that $BcCl(\{x\}) \neq BcCl(\{y\})$. Then, we can assume that there exists $z \in BcCl(\{x\})$ such that $z \notin BcCl(\{y\})$. There exist Bc-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \phi$. Since $z \in BcCl(\{x\})$, $x \in V_z$. Since (X, τ) is Bc-R_0, we obtain $BcCl(\{x\}) \subseteq V_z$, $BcCl(\{y\}) \subseteq V_y$ and $V_z \cap V_y = \phi$. This shows that (X, τ) is Bc-R_1.

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